Pre-Service Teachers' Concept Images on Fractal Dimension

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Abstract

The analysis of pre-service teachers' concept images can provide information about their mental schema of fractal dimension. There is limited research on students' understanding of fractal and fractal dimension. Therefore, this study aimed to investigate the pre-service teachers' understandings of fractal dimension based on concept image. The descriptive method was used in this study. The sample under investigation comprised of 78 (26 males and 52 females) freshmen pre-service elementary mathematics teachers at Afyon Kocatepe University in Turkey. Data was collected by an open-ended questionnaire and clinical interview. The questionnaire was developed by the researcher based on previous studies in the areas of teaching and learning fractals. Data was categorized by semantic content analysis and analyzed using descriptive methods. Analyzing the data gathered from clinical interviews, themes determined in the questionnaire were taken into consideration. The findings showed that pre-service teachers had difficulty in understanding fractal dimension and in making sense of it. Moreover, pre-service teachers could calculate fractal dimension, but they were not aware of what was being calculated completely.

Keywords: Dimension, fractal dimension, concept images, fractal

Literature review

In learning a mathematical concept, definitions, examples, counter examples and experiences faced by students related to the concept are important. These definitions, examples, counter examples and experiences create an image in students' minds for a concept. Concordantly, one of the most important elements in determining a student's understanding about a concept is the concept image formed in his/her mind.

Fractals and fractal dimensions are rather new concepts for students. To explain the concept of fractal dimension, it is necessary to understand what we mean by dimension. Many people accept dimension as one of an object's characteristics just like length, width and height. The concept of dimension is perceived as mostly intuitive depending on spatial relations. Although the importance of the concept of dimension is emphasized in school mathematics, mathematics teachers rarely give adequate explanations for it (Skordoulis, Vitsas, Dafermos & Koleza, 2009). For this reason, the concept of dimension is not being fully understood by students at all levels (Paksu, Musan, İymen & Pakmak, 2012; Skordoulis, et al., 2009). Therefore, it is necessary to make changes in activities intended for dimension in mathematics curriculum. Devaney (1995) uses dimension concept in an analytical sense when he explains the fractal dimension. There is a main direction that can be followed on a line. There are two main directions on a square. In the Sierpinski triangle, there are more directions than a line but less directions than a square. This kind of thought can be helpful to reflect that the Sierpinski triangle dimension is intuitively between 1 and 2. Using a similar approach to other fractals, an intuitive understanding can be formed in students' minds about the existence of fractional dimension and its meaning. When we look at the emergence of fractals historically, we can see that the first fractal examples are seen in the 1870. But their unusualness made them mathematical monsters. In the 1970s, Mandelbrot dealt with these mathematical monsters again using computer technology. In this context, it can be said that fractal geometry is one of the new branches of mathematics. For that reason, integrating the concept's historical development (with their unusualness) in teaching activities can help form characteristics related to concept such as fractal dimension in students'minds more meaningfully. There are a few studies on students' understanding of fractal dimension in the literature. Generally their infinitive structure and complex formation process make fractals difficult for students to understand (Bowers, 1991; Langille, 1996; Murratti & Frame, 2002; Karakuş, 2011; Karakuş, 2013;

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2015). The aim of this study was to determine how students understand fractal dimension by focusing on their concept images of fractal dimension.

What is Fractal Dimension?

How do we define a dimension of a geometric object? It is easy for certain familiar figures like lines or triangles or cubes. To explain the concept of fractal dimension, it is necessary to understand what we mean by dimension. Largely our understanding of dimension is intuitive. Students often say that the line is one-dimensional because it has length; the plane is two-dimensional because it has length and width and a cube is three-dimensional because it has length, width and height. But why is this? We consider the line is one-dimensional since there is exactly one independent direction that we can move along on a line, two independent directions in a square, and three independent directions in a cube (Devaney, 1990). Although, this is intuitively true for the line segment, the square and the cube, it does not work well for the Sierpinski triangle. The Sierpinski triangle (see Figure-1) is strictly self-similar such that any of its parts, in whatever size and location you choose, must contain a replica of the whole (Peitgen, Jürgens, Saupe, Maletsky, Perciante & Yunker, 1991).

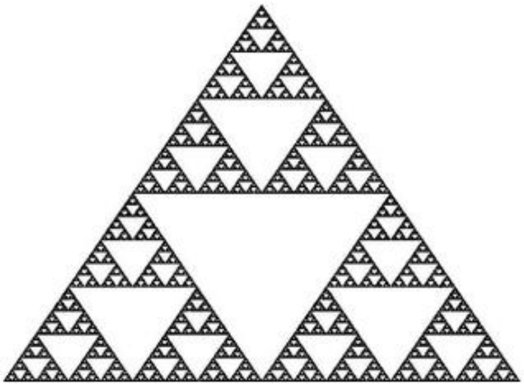

Figure-1. Sierpinski triangle

It can be moved in more than one direction at any point but cannot be moved in every direction (Devaney, 1990). That is, a point on the Sierpinski triangle has more freedom of direction than a line segment but less freedom than a square. So, intuitively the Sierpinski triangle has a dimension somewhere between 1 and 2. One of the most important features of the fractals is the fractal dimension. Fractal dimension is not the common kind of dimension. It is a measure of the thickness or roughness of the fractal (Choate, Devaney & Foster, 1999). Fractals usually have fractional dimension. This means their dimension is not a whole number, but instead is a fraction. The formula for the fractal dimension of an object is

$$n = m^d$$

Where $n =$ number of self-similar pieces, $m =$ magnification factor and $d =$ dimension.

With the help of this formula, the fractal dimension of the Sierpinski triangle given in Figure 1 is calculated as following. The Sierpinski triangle may be broken into three self-similar pieces ($n = 3$), each of which may be magnified by a factor of 2 ($m = 2$). So, using $n = m^d$, we have $3 = 2^d$. Taking the log of both sides, we have, $d = \frac{log3}{log2} \approx 1.585$. So the dimension of the Sierpinski triangle is approximately 1.585, and it is not an integer.

Studies on fractal dimension

Studies about teaching and learning fractals have mostly been focused on difficulties in the teaching and learning of fractals and their features. For example, Bowers (1991) determined that students have difficulties in three subjects in learning of fractals. The first difficulty is learning fractal dimension, the second is determining the scaling factor in the self-similar parts, and the third is the construction of a fractal. Bowers (1991) stated that students believed dimension was a unique characteristic of an object, and they had some difficulties in understanding the concept of fractional dimension. Namely, does an object with 1.5 dimension have length and half width? Students perform mental queries such as how does an object have length and a little bit of width? On the other hand, they understand the formula used for calculating fractal dimension and they can use formula in calculating different fractal dimensions. Likewise, Langille (1996) conducted a study about integration of fractal geometry into the 12^{th} grade mathematics curriculum. He determined that students have difficulties in determining the basic features of a fractal: self-similarity and fractal dimension. Langille (1996) stated that students have difficulty in understanding self-similarity dimension and he also added that students perceived fractional dimension as confounding. He related one reason for this to Euclidean dimension schemas. He indicated that existing dimension schema and learned knowledge related to fractal dimension were in conflict, and for this reason, it is difficult to learn fractal dimensions. As another reason, he indicated that students learned fractal dimension mostly in an operational way. Bowers (1991) and Langille (1996) stated that students also used the formula which enables students to calculate fractal dimension. Yet neither of the students understood exactly what it was they were calculating. Karakuş and Kösa (2010) examined pre-service teachers' experiments and understandings about fractal dimension. They emphasized that pre-service teachers could calculate fractal dimension using a formula but the majority had difficulty in understanding fractional dimension and they also had difficulty in explaining the characteristics of a shape with fractional dimension. Furthermore, they also stated that few students could explain what was calculated with formula and could understand what fractional dimension meant. Karakuş (2011) also determined that pre-service teachers generally had difficulties in understanding fractal dimension. He stated that pre-service teachers had some difficulties in determining self-similar parts and the magnification factor in calculating the fractal dimension of a fractal. In addition, Karakuş (2011) expressed that students learned fractal dimension mostly in an operational way and none of them understood exactly what they were calculating similar to the result of Langille (1996) and Bowers (1991).

Fractals in Turkish Mathematics Curriculum

In the Turkish educational system, fractal teaching begins with an introduction to fractals at the age of 13-14 years in Grade 8. The Grade 8 mathematics curriculum includes a goal about fractals which is *"to build patterns from line, polygon and circle models, to draw them and to determine fractals from these patterns"* (MEB, 2008a). The goal was prepared in terms of building fractal patterns using figures in Euclid geometry or deciding whether given patterns are fractals or not. In that grade, fractals are taught by drawing activities and finding fractal patterns. When the textbooks are examined, the definition of fractal is written as "*the patterns which were built proportionally with the magnified and reduced of a part of a shape"* (MEB, 2008b). The definition emphasizes the important properties of fractals as iteration and self-similarity. Self-similarity is defined as a part of the whole closely resembling the whole (Lornell & Westerberg, 1999). Imagine taking a fern and breaking off a chunk. That chunk looks like the original. The other characteristic of a fractal is iteration, defined as the same operation carried out repeatedly with the output of one iteration being the input for the next one (Peitgen, Jürgens & Saupe, 1992). In brief, definitions, examples and explanations focus on two fractal properties: self-similarity and iteration in the textbooks and mathematics curriculum at Grade 8.

The Grade 10 geometry curriculum includes two goals about fractals. The first goal is *"to build fractals with segments, to explain them and to compute the length of the fractal in a particular step"*, and the second is *"to build fractals with triangles, to explain them and to compute the area of fractal image in a particular step"* (MEB, 2010). The goals were prepared in terms of calculating fractals' perimeter and area in a particular step. However, there is not any goal related to fractal dimension in Grade 8-12 geometry curriculum. In conclusion, in Turkish mathematics and geometry curriculums, fractals are studied as follows; recognition of fractals in Grade 8, finding patterns and mathematical operations (perimeter and area) with fractals in Grade 10.

Students' Learning of Mathematical Concepts

An individual creates right or wrong schemas in his mind during the process of learning a mathematical concept. Tall and Vinner (1981) gave the name “concept image” to these concepts. *Concept image* is defined as *“describing the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and process.”* (Tall & Vinner, 1981: p.152). Students’ experiences are very important in the formation of the concept image for a concept. For example, a student who observes that as the perimeter of a rectangle increases, the area increases and can later recognize that if the perimeter of a rectangle increases then the area always increases. For such a student, this observation is a part of his concept image and may cause problems when he meets a situation in which the perimeter increases, but the area reduces or remains fixed. In learning a mathematical concept, the definition of a concept is also important. A *concept definition* is defined as *“to be a form of words used to specify that concept.”* (Tall & Vinner, 1981: p.152). But concept definition does not guarantee understanding of the concept (Vinner, 2002). To understand the concept means to have a concept image about this concept. The definition of a concept helps to form the concept image. Vinner (2002) claims that in learning a concept, if you have an existing concept image and if you encounter a definition related to the concept, in this case there may be three situations. These situations are;

- With the help of the definition, one changes his own current concept image.
- Accepting the definition short term (mostly by memorizing) yet, loss of definition occurs over time and the existing image does not change.
- Protecting both concept definition and concept image together

These three definitons as described above are valid for the process of an individual learning a new concept. Concept images will be formed by means of the concept definition or will be formed by experience during the process of concept formation. In particular, examples, counter-examples and experiences are very important for the formation of students' concept images (Wilson, 1990). A student who gives the formal definition perfectly really does not show an understanding of the concept (Edwards & Ward, 2008). In the context of determining the students’ understanding of a concept, concept images are very important.

Fractal and fractal dimension are a new and unclear topic for students and teachers. Unlike conventional beliefs that the dimension should be an integer such as 1, 2 and 3, the fractal dimension can be a rational number. Because of this, teachers at various grade levels may encounter difficulties in conveying the concept of fractal and fractal dimension to their students. Therefore, in order to provide students with a strong conceptual understanding about fractal dimension, it is important to determine teachers’ and pre-service teachers’ understandings and concept images about fractal dimension. Although there have been some studies about teaching and learning fractals, there are a few studies determining pre-service teachers’ concept image about fractal dimension.

The purpose of the study

Determining the students’ concept images about fractal dimension can provide information about their mental schema. There is a limited research on the students’ understanding and knowledge of fractal dimension. Therefore, this study sought to investigate this gap in the literature and focused on students’ concept image about fractal dimension. The problem of the study is “what are pre-service mathematics teachers’ concept images about fractal dimension?”

Method

Participants

The participants for this study were 78 (26 males and 52 females) freshmen pre-service elementary mathematics teachers at Afyon Kocatepe University in Turkey. These students took the geometry course during the 2014-2015 spring academic semester. These students have experienced fractals for the first time at Grade 8. They learned concepts of forming fractal structures, iteration and self similarity at this level. Moreover, besides self similarity, they learned the perimeter and area of a fractal at Grade 10. In this context, it can be said that these students have prior knowledge about fractals but none of them have prior knowledge about fractal dimensions.

Content and Process of the Course

The subject of fractal dimension was given during geometry course in the spring semester. The geometry course is a course of 3 hours per week for 14 weeks. In this course, Euclidean geometry subjects were studied intensively. In the last 4 weeks, non-Euclidean geometries were introduced. During the last four weeks, students were introduced to fractal geometry and especially with the subject of fractal dimension. The course content is given below.

Table 1. The content of the course based on fractal dimension

Week	**Course content**
1st week	Basic characteristics of fractals, iteration and self-similarity
2nd week	Definition of fractal dimension
3rd week	Practices about fractal dimension
4th week	Practices about fractal dimension

Before introducing the fractal dimension, some reminders were given about self-similarity and iteration and students were asked to determine some self-similar parts of fractals and their magnification factor. Then, for preparation of fractal dimension, a discussion was initiated by crumpling a paper up into a ball. What is the dimension of the ball? And when you carefully re-open the ball of paper, what dimension has it become? The aim was to make students discuss the dimensions of crumpled paper, which existed between two dimensional and three dimensional. Next, using self-similar Euclid shapes, the fractal dimension formula was obtained for calculating fractal dimension. With the help of the formula, students were asked to compare the appearances of fractals by calculating several different fractal dimensions. Thus it was expected that students would realize the relationship between fractal dimension and the complexity of a shape.

Instrument and Data Collection

Asking direct or indirect questions about the subject is one of the ways to determine the schema formed by an individual for a concept. Vinner (2002) emphasizes that it is important to ask indirect questions in order to detect one's own concept image, for that reason an open-ended questionnaire was used in order to detect the students'own concept images created in their minds for fractal dimension. While preparing the questionnaire, interview questions of Karakuş (2011) were used. Moreover, to increase the validity and reliability of the instrument, the questionnaire was discussed with a mathematician and a mathematics educator. These two experts reached a consensus on the content validity and reliability of the instrument. In the first question of the questionnaire, students were asked a question which was *"What is dimension? Can you define it?"*. The aim of the first question was to reveal the students' concept definitions for dimension. In the second question of the questionnaire, students were asked a question which was *"What is fractal dimension? Can you define it?"*. The aim of the question was to reveal the students' concept definitions for fractal dimension. In the third question of the questionnaire, students were asked to draw two fractals; one of them has a 1.12 fractal dimension and the other has a 2.71 fractal dimension. The aim of the question was to reveal the students' concept images for fractal dimension. In the last question of the questionnaire, students were asked to calculate the fractal dimensions of some fractals. The questionnaire was carried out in 40 minutes. The names of the participants were kept anonymous to ensure the confidentiality. For this reason, codes such as S1, S2, S3… for students were used. Moreover, clinical interviews were conducted with 6 students in determined themes and they answered the last two questions (draw two fractals one of them with a 1.12 fractal dimension and the other with a 2.71 fractal dimension and calculate the fractal dimensions of following fractals) of the questionnaire. The basic aim of the clinical interview is to discover one's wealth of thinking and to determine his/her cognitive skills by revealing which concepts he/she knows and what relationships exist among these concepts (Hunting, 1997; Goldin, 2000; Zazkis & Hazzan, 1998). During interviews, students were asked to explain their answers and drawings. To this end, students were asked the following questions. How did you draw the fractal with a 1.12 dimension? To which aspects did you pay attention while drawing? Why did you draw this shape? How did you think of it? Clinical interviews were conducted in a comfortable and an appropriate atmosphere by the author. The interviews were audio taped with the permission of each interviewee. Each interview lasted about 20-25 minutes. R symbolizes researcher, S1, S2 and so on, symbolize students in the text while presenting the interview.

Data Analysis

In analyzing students' answers to the questionnaire, Miles and Huberman's (1994) three-stage qualitative data-analyzing method was used. These stages are dealt as "data condensation", "data display" and "conclusion drawing/verification" (Miles and Huberman, 1994). In the data display stage, the researcher determined some key sentences (length, width, height, self-similarity, and magnification factor) by examining data gathered from the questionnaire and the researcher also did some elimination out of raw data. During the data display stage, themes and categories that reflect the main idea of the key sentences were determined (definition of dimension in Euclidean sense, measure of complexity, formal definition, fractional dimension, and so on) and they were organized by a researcher with the help of tables. In conclusion, the drawing/verification stage, deductions were made related to the analysis and then these deductions were compared among themselves, and they were also compared with the results of previous studies. This part was included in the discussion section. While analysing the data gathered from clinical interviews, themes determined in the questionnaire were taken into consideration. Quotations made from students own answers and then students' explanations were interpreted together.

Reliability of study

In qualitative studies, triangulation increases the accuracy of the study since it enables the evaluation of common and different findings by grouping, and it also enables the comparison of findings acquired from different data sources (Creswell, 2012). In this study, triangulation was ensured with data gathered from clinical interviews and with the questionnaire consisting of open-ended questions. In order to provide the reliability of the data collected from the questionnaire, categories organized by the researcher were given to a different researcher, an expert in fractal. The researcher matched coded every student's answer into the suitable category. Then expert's matchings were compared with the researcher's matchings. Considering the comparisions, agreement and disagreement numbers were calculated according to Miles and Huberman's (1994) reliability formula, and the correspondence percentage was found as 83%. Since this rate (83%) is greater than .70, the classifications of the researchers were reliable (Miles & Huberman, 1994). Moreover, in order to categorize the data obtained from the clinical interviews, the interview transcripts of the students were read more than once. The words used by the students were exactly written down on the paper without making any changes. Then these written texts were given to the students for their approval which enabled the member to check the reliability of the data. According to the Creswell (2012), member checking is a process in which the researchers check their findings with participants in the study to determine if their findings are accurate.

Findings

Students' definition of dimension

Students' definitions of dimension were given in Table 2.

Table 2. Frequencies and percentages related to students' definition of dimension.

Themes	Codes	f	%	Examples
Students' definition of Euclidean dimension	Dimension indicates length, width and height of an object.	66	84.6	Dimension indicates an object's length, width and height (S1). When dimension is mentioned, length, width and height comes to mind (S3).
	Dimension is related to the concepts of length, area and volume.	3	3.9	If we are calculating area of object, it is two dimensional, if we are calculating its volume, it is three dimensional (S6). Dimension is a concept that gives rise to

				concepts such as volume and area that are related to shape's length, width and height (S11).
Students' definition of analytic dimension	Dimension is extension of an object in any direction.	9	11.5	Dimension is a coordination number used for determining objects. We can say that it is an extension of an object in any direction (S51). Dimension is the objects's directions that occupy a place in space (S8). It is the extension of objects at any direction (S27).

When Table 2 was analyzed, it can be seen that most of the students (88.5%) had the definition of Euclidean dimension. Most of them (84.6%) associate dimension with an object's length, width and height. As a consequence of this, students perceive dimension as a unique characteristic of an object. Moreover, it is clear that some of the students (11.5%) try to define dimension in an analytical sense by associating it with dimension.

Students' definition of fractal dimension

Students' definitions related to fractal dimension are shown in Table 3.

Table 3. Frequencies and percentages of students' definitions related to fractal dimension

Themes	**Codes**	**f**	**%**	**Examples**
Formal fractal definition	The relationships between self-similar pieces and the magnification factor.	52	66.7	Fractal dimension can be found out by dividing the number of an object's self-similar pieces to a magnification factor (S3). Fractal dimension is connected with magnification factor and self-similar part number (S7). Fractal dimension is calculated with formula $n=m^d$, m which means magnification factor and n which means the number of self similar parts (S14). Fractal dimension is a logarithmic division of the number of self-similar parts to magnification factor (S54).
Measure of complexity	The measure of self-similar object's complexity.	5	6.4	Fractal dimension serves for measuring how complex a self-similar object is (S4). Fractal dimension identifies how dense self-similar parts are depending on certain iteration. For instance, Sierpiski triangle's dimension is bigger than the Koch curve, that is to say Sierpiski triangle is more complex than the Koch curve (S34). It is a statement showing how complex a self-similar object is (S76).
	The degree of complexity between two objects consisting of endless points.	9	11.5	Fractal dimension gives information about shapes of objects consisting of endless points, for instance, line and plane comprise of endless points. Fractal dimension gives information about which of these two sets comprised of endless points (S13). Fractal dimension enables the comparison of two sets possessing endless points (S49). Fractal dimension roughly measures how many points exist in an endless set. So that we can find which set (between two sets) includes more points (S74).
Fractional dimension	Dimension of	9	11.5	Fractal dimension is a dimension of objects

	objects whose dimension is not an integer.			which are not integer such as 1.21 and 1.56 and are continuing in accordance with specific rules (S10). Fractal dimension is not always calculated as integer and it is a characteristic of fractals (S26). Since fractal image is formed by continuously adding a part to a shape or omitting a part from the whole shape or from a certain area of a shape, its dimensions are always between 1 and 2 or between 2 and 3.That is to say, it is integer (S46). Because fractals are formed with a continuous iteration, their areas and perimeter are not constant. That results in dissimilarity of length, width and height each time. For this reason, fractals have no dimension in integer value (S59).
Blank		3	3.9	

Table 3 showed that most students (66.7%) preferred to use the formal definition in a classroom environment when defining fractal dimension. The students conceived fractal dimension as a relation between the logarithm of an object's self-similar part number and logarithm of magnification factor. Moreover, it was also determined that 17.9% of students related fractal dimension to the complexity of shape. Some of these students stated that fractal dimension was a degree of complexity of a shape. On the other hand, other students claimed that among two shapes with endless points, fractal dimension helped deciding which one had more points. Additionally, 11.5% of the students expressed fractal dimensions were always fractional. These students related the unusualness of fractals' area and perimeter to their dimensions. 3.9% of students did not answer this question.

Students' fractal drawings

Categories determined depending on students drawings about fractal shapes along with given dimensions and their frequencies and percentages are given in Table 4.

Table 4. Students drawing categories for shapes with given dimensions.

Themes	**Codes**	**f**	**%**
Students who draw a shape	Students who use the dimension of Euclid shapes and fractal formation process	22	28.2
	Students paying attention to the shape's being zigzagged and curved	15	19.2
	Students drawing using fractal dimension definition	11	14.1
	Students relating dimension to perimeter/area and volume	7	9.0
Students who did not draw a shape	Students who left it blank	14	18.0
	Students not drawing a shape but giving explanations about how it would look like	9	11.5

When Table 4 was analyzed, it is defined that students mostly (28.2%) used fractal's creation process while drawing fractals with given dimension values. They took Euclidean shapes' dimension into consideration such as line, square and triangle. These students tried to form a fractal using the methods of removing parts (as in the Sierpinski triangle) or adding parts to a shape (as in Koch's snowflake). Especially, it is specified that students have an understanding as, adding part to a Euclid shape increases its dimension and removing a part from it decreases its dimension. For instance, most students think that when some parts are removed from a three dimensional cube, its dimension will be between 2 and 3. When a part is added to a one dimensional line, its dimension will be between 1 and 2. In their drawings, these students did not pay much attention to the number of self-similar parts and magnification factor. They mostly drew fractals intuitionally. For instance, the drawing of student S12 with the dimension of 1.12 was shown in Figure 2.

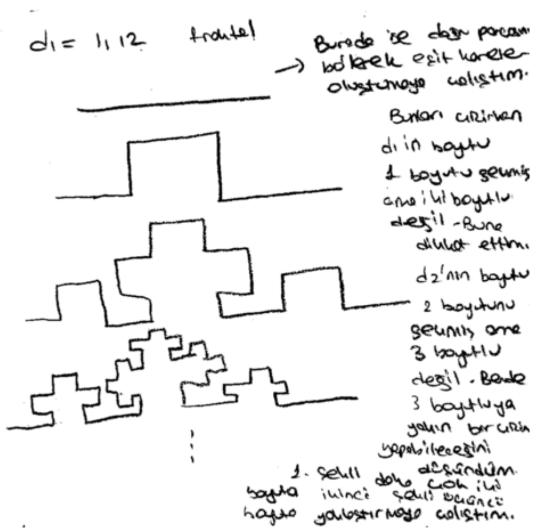

Figure 2. The fractal drawing of student S12 with the dimension of 1.12

While drawing a fractal which has a dimension of 1.12, student S12 first handled a piece of line and he considered it as one-dimensional then he tried to form a fractal shape as it is in the Koch curve.

A clinical interview conducted with this student showed that he tried to form a fractal similar to the Koch curve.

R: How did you draw fractal with a 1.12 dimension?
S12: I thought fractal with a 1.12 dimension must be somewhere between 1 and 2. For that reason, I got a piece of segment. Then I remembered the Koch curve. Its dimension was between 1 and 2. I draw a shape similar to the Koch curve.
R: You drew a fractal similar to the Koch curve. But when we look at the shape you drew, it looks like a curly curve. Why would its dimension be 1.12?
S12: Since I formed a fractal and my fractal is going on continuously. For that reason, when you start with a piece of line and form a fractal in this way, its dimension is between 1 and 2.
R: Are you sure that the dimension of your shape is 1.12? How do you know that it is 1.12?
S12: I don't know exactly if it is going on continuously. So its dimension would be neither 1, nor 2. It is between two of them.

The explanations of student S12 showed that he tried to form a fractal which had a dimension between 1 and 2, instead of forming a fractal with the given dimension. When the same student's drawing related to fractal with the dimension of 2.71 was examined, he tried to change dimension of a cube by removing some parts of it. This shows that he has an understanding as adding some parts to a line increases its dimension and removing some parts from a cube decreases its dimension.

According to Table 4, 19.2% of the students paid attention to appearance by drawing a curly shape with zigzags. Also, these students used the dimension of Euclid shapes while drawing fractals. They established a relationship between dimensions and being curly that was given in the definition of fractal. Students did not pay attention to the relationship between magnification factor and the number of self-similar parts, instead they tried to change the dimension of a Euclid shape by adding zigzags. For instance, the drawing of student S4 with the dimension of 2.71 was presented in Figure 3.

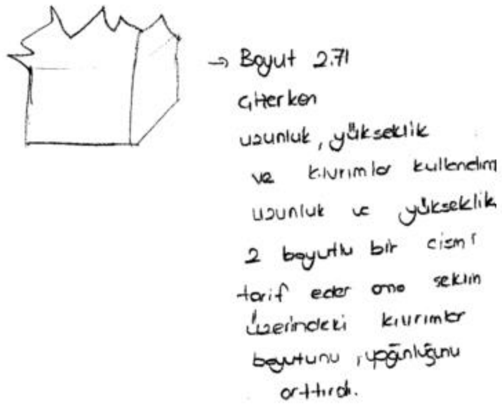

Figure 3. The fractal drawing of student S4 with the dimension of 2.71

Considering the student's answers, it can be said that he thinks adding curls or zigzags to a shape increases its dimension. The clinical interview conducted with this student was presented below.

R: How did you draw the shape with a 2.71 dimension?
S4: This time I drew a height then I added zigzags on it. The length and width of a shape describe a shape but zigzags on it increase its dimension.
R: Does drawing zigzags on a shape increase its dimension?
S4: It could be. While drawing fractals, it was just like that. That is to say, while we were forming fractal from a line, when it began to be a curly fractal, at the beginning of a curly shape, the dimension was between 1 and 2.
R: Is it necessary for a shape with fractional dimension to have an appearance with curls and zigzags?
S4: I don't know it exactly, but it looks like that.

The explanations of the student S4 showed that he established a relationship between fractal dimension and the curly or zigzag appearance of a shape.

According to Table 4, 14.1% of students tried to make use of the definition of fractal dimension in their drawings. It is determined that these students draw their shapes by considering the relationship between self-similar part number and magnification factor. For example, the drawing of student S32 was presented in Figure 4.

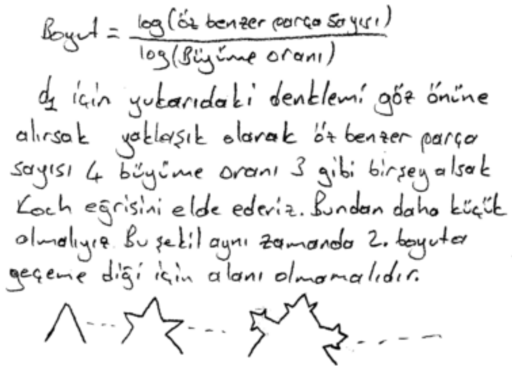

Figure 4. The fractal drawing of student S32 with the dimension of 1.12

It can be seen that while drawing fractal with 1.12 dimension, this student used Koch's curve (approximate dimension is 1.26) which she already knew. A clinical interview conducted with this student was presented below.

R: How did you draw the shape with a 1.12 dimension?
S32: If we think self-similar part number as 4 and magnification factor as 3 we get Koch curve. Its dimension was something similar to 1.20. We can consider it to be less than this.
R: So what did you do?

S32: We must decrease the self-similar piece number. In the Koch curve there were four self-similar pieces. I made it 2 for this fractal.
R: Which pieces are the ones you mentioned?
32: (By showing 2 lines in the first shape) these two.
R: Considering this what is the dimension?
S32: I didn't calculate dimension by using logarithmic.

The explanations of student S32 indicated that he used a fractal dimension formula while drawing shape with 1.12 dimension. The student who knew that the dimension of the Koch curve was more than 1.12, stated that the number of self-similar parts with a dimension of 1.12 should be less. So, he tried to decrease self-similar parts of the Koch curve. He drew two self-similar parts in the first iteration stage however, in the next iteration stage he did not pay much attention to this rule and used the Koch curve forming rule.

According to Table 4, 9% of students drew their fractals by relating dimension to the perimeter, area and volume of a fractal. These students paid attention to some unusual characteristics of fractals' perimeter, area and volume such as in an endless perimeter having limited area (as the Koch snowflake) or having zero area and endless perimeter (as the Sierpinski triangle). For instance, the student S7 stated that perimeter of a fractal with a dimension between 1 and 2 increases endlessly and it does not have an area and so he drew the following shape. In his drawing, he tried to show that the perimeter diminishing in a proportion of one half all the time will have endless length, yet the area will increase very little.

The clinical interview conducted with this student was presented below.

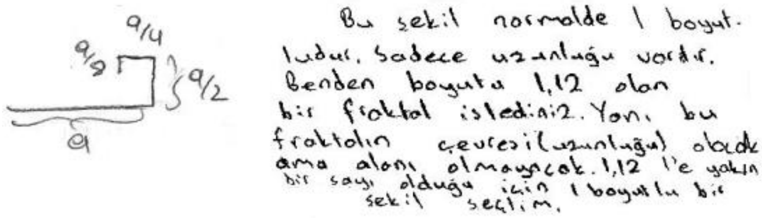

Figure 5. The student S7's drawing related to fractal with 1.12 dimension.

R: How did you draw this shape?
S7: Since you asked me to draw a shape with a 1.12 dimension, this fractal will have a circumference but it will not have an area. So, I took a one-dimensional shape. It has length only. Then, I iterated it as it will go to infinity.
R: The shape you drew looks like a curve with zigzags, why is its dimension 1.12?
S7: In each following step, it always continues diminishing in a proportion of one half. For that reason, its length increases constantly. That is to say, its dimension increases yet it doesn't have an area, its dimension would be between 1 and 2.
R: Unless it has a constantly increasing length, would its dimension still be 1. 12?
S7: No, it wouldn't be.

The student's explanations indicated that shape's length constantly increases, its dimension exceeds 1, and on the other hand, since it does not have an area it does not reach a dimension of 2. Namely, as in the Sierpinski triangle, the student paid attention to some extraordinary situations of fractals such as zero or limited area in an endless perimeter.

According to Table 4, 18% of students did not draw and not give any explanation for two fractals with the dimensions of 1.12 and 2.71. Moreover, 11.5% of students did not draw any fractal, but they gave some explanations about how these fractals would look like. For example,

S1: Shapes with 1.12 dimension would look like a one-dimensional shape. Namely, a line has length but it doesn't have an area, this shape will have a bigger area.
S19: While drawing shapes with 1.12 and 2.71 dimensions, the shape with 1.12 dimension will look like a two dimensional shape, the other one will look like a three dimensional shape.

S26: While drawing a shape with 1.12 dimension I must draw a shape with a perimeter and without an area. I would use a 2-dimensional shape. While drawing the other one, I must draw a geometric shape with area, perimeter length but without a volume. So I must take of a 3-dimensional geometric shape.

These explanations indicated that students tried to explain fractal dimensions by using dimensions of Euclid's shape. It is clear that students perceive dimension as a basic characteristic of shape**.**

Calculating fractal dimension

The frequencies and percentages related to students' success in calculating fractal dimensions were presented in Table5.

Table5. Frequencies and percentages related to students' success in calculating a given fractal dimension.

Categories	Sub-Categories	1. Fractal		2. Fractal		3. Fractal	
		f	%	F	%	f	%
Correct		64	82.1	56	71.8	24	30.8
Wrong	The ones who calculated the number of self-similar parts incorrectly.	10	12.8	22	28.2	45	57.7
	The ones who calculated the magnification factor incorrectly.	10	12.8	13	16.7	19	24.4
Blank						5	6.4

(While calculating, some sudents made mistakes both in self similar part number and in magnification factor, for that reason, their total percentage was not out of 100)

Considering Table 5, it was clear that most students calculated the first and second fractals' dimension correctly (in order the first one is 82.1% and the second one is 71.8%). However, in calculating third fractal dimension, achievement was quite low (30.8%). For the students who calculated each fractal's dimension wrong, common errors were the number of the self similar-parts and the magnification factor. In calculating the first fractal dimension, 12.8% of students made mistakes in both determining the self-similar part number and the magnification factor. In calculating the second fractal dimension, the percentages are 28.2% in determining the self-similar part number and 16.7% in determining the magnification factor. In calculating the third fractal dimension, 57.7% had difficulty in determining the self-similar part number, 24.4% had difficulty in calculating the magnification factor. This showed that students had difficulty mostly with self-similar part number while calculating fractal dimension. According to Table 5, 6.4% of students did not do any calculation and they left blank the calculation of the third fractal's dimension. However, they did some calculations intended to the first and second fractals dimension. In this context, students found it difficult to calculate fractal dimension between 2 and 3.

All six students attending the clinical interview calculated the first fractal's dimension correctly. The students coded as S4, S7, S12, S32, S42 calculated the second fractal's dimension correctly, yet the student S78 calculated the second fractal's dimension incorrect. The clinical interview conducted with this student was presented below.

R: How did you calculate the second fractal's dimension?
S78: I used the dimension formula which we learned during our course, 3 is the number of similar shapes, 2 is the ratio, I calculated depending on it.
R: Can you point a self-similar part?
S78: (thinking for a while) Like these ones, I guess no, they are not. I confused it with Sierpinski triangle that we learned during our course.
R: Now, can you show self similar parts?

S78: I confused it in the exam also. I can not decide. Could we say that these ones are self-similar parts? I don't know.
R: Why did you say 2 for a magnification factor?
S78: I guess I confused it because it looks like Sierpinski triangle.
R: Would you say it now? Look at it?
S78: I can not decide which one I will look. I don't know.

This student's explanations showed that he confused the second fractal and the Sierpinski triangle. We can see that the student can not determine self-similar piece numbers and a magnification factor.

In the question of calculating the third fractal's dimension, the student S78 left the question blank, the students coded as S4, S7, S12 and S32 answered it incorrectly and S42 coded student answered it correctly. The student S78 stated that he had difficulty in answering this question because in a 3-dimensional shape it is difficult to find self-similar parts. The students coded as S4, S12 and S32 made a mistake about determining self-similar part numbers despite the fact that they determined magnification factor correctly in calculating the third fractal's dimension. For example, the student S32's clinical interview is presented below.

R: How did you calculate this fractal's dimension?
S32: Its magnification factor is 3 and its self similar part numbers are 27, so the answer is 3.
R: Why is magnification factor 3?
S32: (By pointing pieces on face of Menger Sponge) the edges of the cube are divided into 3 and it goes on like this here. So it is 3.
R: Why is its self-similar part number 27?
S32: When we divide the edges of a cube into 3 equal pieces, we get 27 equal cubes. For that reason, we have 27 self-similar pieces.

The student S7 made mistakes both in determining the magnification factor and calculating the self-similar part number. The clinical interview conducted with this student is presented below.

R: How did you calculate this fractal's dimension?
S7: I had difficulty in calculating its dimension. I multiplied to calculate its dimension. Namely, the magnification factor for a face is 3, considering this ratio; there are 8 self-similar parts, so we have 6x8 self-similar parts and magnification factor of 6x3.
R: That is to say, do we have to find the magnifation factor and self-similar part number for each face?
S7: Yes, because shapes occuring in this face can be seen on other faces, to calculate the total number we have to do that.

This student's explanations indicated that despite the fact that he determined magnification factor at the beginning correctly, he thinks the magnification factor of each face must be multiplied in order to find the magnification factor. It is identified that he used a similar method in calculating a self-similar part number.

Discussion and Conclusions

The findings of the study indicated that most students had Euclidean dimension understanding. According to their concept image, students mostly thought that dimension was an object's characteristic such as length, width, and height. Moreover, it was determined that some students related dimension to a coordination system and they perceived it as the number of parameters used in determining an object's location. In the formation of both definitions, students' experiences in real life (concrete objects, book surface, movements up-down or movements left-right) with course books and activities that took place in curriculums were rather effective (Peker & Karakuş, 2013). Similarly, in their studies Skordoulis, et al. (2009) examined whether pre-service teachers could determine an object's dimension in Euclidean plane and in the coordinate plane. Results showed that the Euclidean plane and coordinate plane were effective in students'determining the process of given objects' dimensions. Especially for students who made mistakes in determining dimensions of objects given in coordination plane and who did not have problems with the determination of given objects dimension in Euclidean plane. Similarly, in his study, Ural (2011) stated that preservice teachers generally used some criteria such as length-width and height, area-volume and plane-space while determining given object's dimensions.

It is specified that students mostly used the definition of fractal dimension, which was stated as ratio of logarithm of self-similar piece number to logarithm of magnification factor, while defining a fractal dimension. Moreover, it was determined that some students claimed that fractal dimension had a relation with the complexity of a given shape. On the contrary, students did not use the formal fractal dimension definition much while drawing objects with given fractal dimensions. Similarly Rösken and Rolka (2007) also stated that concept images were more effective in students' conceptual learning rather than formal definition. While drawing a fractal shape with a given fractal dimension, students used some Euclidean shapes such as triangle, cube and square; and they used the fractal formation process instead of using formal dimension definition. While drawing a fractal shape with given dimension, these students tried to create a fractal by adding parts to a shape or removing parts from a shape as in the fractal formation process. It was determined that pre-service teachers thought that adding parts (such as a triangle, square, and cube) to a shape increased its dimension and removing parts within it decreased its dimension. For example, students think that when parts were removed from a cube in accordance with the rules, its dimension decreased to somewhere between 2 and 3. According to students' fractal dimension just like in the Euclidean dimension, one reason for this can be a characteristic of an object. Similarly, Bowers (1991) also related students' having difficulty in learning fractional dimensions to their perception of dimension as a characteristic of an object. For instance, a student considers an object with a 1.5 dimension as an object that has a width and half length. At this point, the problem is how can a shape have half length? What does a shape with half length look like? Students can not understand it completely. Another reason for that can be students'current dimension schemas, since students have not met this kind of fractional dimensions before. In our world we meet many one-dimensional (just as electric wires that could be accepted as one-dimensional, however they are three-dimensional in appearance), two-dimensional (book pages, computer screen) and three dimensional (e.g. table, apple) objects. Moreover, many explanations and activities existing in school curriculum include objects with integer dimensions. This results in the creation of an understanding that objects' dimensions are to be integer. Langille (1996) also stated that students'existing Euclid schemas prevented students' understanding of fractal dimensions.

It was determined that some students paid attention to its appearance being curved and having zigzags while drawing a shape with given dimensions. These students tried to liken the shape to a fractal by adding zigzags on it. While drawing a shape with 1.12 dimension, they first took one-dimensional shape (such as a piece of line). Then they added curves on it so they thought its dimension would have been somewhere between 1 and 2. In this respect, it can be said that they think adding zigzags on a shape increases its dimension. The word fractal comes from Latin word "fractus" which means "irregular, complex, and broken". Natural fractals (such as ferns and sharelines) have zigzag shapes. Students' experiences in secondary school and high school may be the reason for this perception. Karakuş (2011) also stated that students related fractal dimension to their zigzag appearance. In this study, students declared that an object with a 2.5 dimension must have more zigzags than an object with dimension 2.

Among the students who participated in this study, 14.1% of them used formal definition (d=logn/logm, here n=self-similar piece number, m=magnification factor, d=dimension) while drawing a fractal shape within the given dimension. This case shows that the formal definition does not have a strong impact on the formation of a concept image. Furthermore, it is determined that few students relate fractal dimension to the unusualness of the perimeter, area, and volume (for example, in the Sierpinski triangle, while its perimeter increases endlessly, its area approaches to zero). Karakuş (2011) states that explaining the unusual relationship between a fractal's perimeter, area and volume by relating them to fractal dimensions helps students understand this unusualness better. It is determined that approximately one third of students are not successful in drawing fractals with given dimensions. The explanations of these students show that they have real difficulty in understanding fractal dimension. Moreover, it can be said that students perceive dimension as one of the characteristics of an object just like length, width, and height. Students' explanations show the variety of fractional dimension with their Euclidean dimension schemas about dimension.

The findings of this study indicated that most students were rather successful in calculating the dimensions of given fractals. In some studies (Bowers, 1991; Langille, 1996; Karakuş & Kösa, 2010; Karakuş, 2011), it was pointed out that students could calculate the dimensions of given fractals using the fractal dimension definition. Yet, students' correct calculation ratio of dimension value decreased when a fractal shapes's complexity increased. In calculating dimensions, students had difficulty mostly in determining self similar part number. In similar studies (Bowers, 1991; Langille, 1996; Karakuş, 2011; Karakuş, 2013; Karakuş & Karataş, 2014; Karakuş, 2015), it was stated that students and pre-service teachers had difficulty in determining an object's self-similar part numbers. Moreover, another concept that students had difficulty in calculating fractal dimension was the magnification factor. In the creation of fractals, some students had difficulty in calculating the value which enabled us to determine at what rate self-similar parts increased or decreased. Most students were successful in calculating dimensions of given fractals, however, they were not completely aware of what they were calculating. This situation showed that students learned dimension mostly as operational. This finding was emphasised in the studies of Bowers, 1991; Langille, 1996; Karakuş, 2011; Karakuş, 2013; Karakuş & Karataş, 2014; Karakuş, 2015. To sum up, students'concept images related to fractal dimension showed that they perceived fractal dimension as one of its characteristics just like length, width and height. It can be seen that more than half of students were successful in giving fractal's formal definition, however, a few students use formal definition in concept images related to fractal dimension. Students had difficulty in making sense of how a shape would appear with a given fractal dimension.

In some studies (e.g. Bowers, 1991; Langille, 1996; Devaney, 1995; Karakuş & Kösa, 2010; Karakuş, 2011), it was stated that students had difficulty in understanding fractals. Especially, it can be said that students could calculate a given fractals' dimension yet they were not aware of what was being calculated. This showed that they had more operational knowledge about fractal dimension. Moreover, another reason for their concept images related to fractal dimension was their current dimension images. Devaney (1995) stated that an insuffiency of students'dimension definition resulted in difficulty in making sense of fractal dimension.

Overall, the results of this study showed that a significant proportion of the students thought that dimension was an object's characteristic such as length, width and height. Furthermore, it was revealed that this kind of thought made fractal dimension difficult to understand. Therefore, some changes to the content of the mathematics courses in secondary school mathematics curriculums should be made. The present study also raises the need for further research. One direction for further study would be an examination of integration of fractal dimension into the secondary school mathematics curriculum. In addition, it would be interesting to identify how secondary school students are being trained in fractal dimension. Moreover, secondary school students' perceptions about fractal dimensions would be determined.

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