Middle School Mathematics Teachers' Reasoning about Students' Nonstandard Strategies: Division of Fractions

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Although students' nonstandard strategies has great importance in understanding students' thinking and creating effective mathematics classrooms, much remains unexplored in the literature. This study investigated 22 middle school teachers' reasoning about a student's nonstandard strategy for division of fractions. The data were collected through semi-structured interviews and a task consisting of a student's nonstandard strategy within a classroom excerpt which simulate how mathematical work emerges in the context of teaching. Six categories of layers were formed based on their reasoning about the validity, generalizability and efficiency of the nonstandard strategy. These layers were categorized as surface, intermediate and deep level of reasoning. It was found that while half of the teachers had surface level of reasoning, only one third of teachers are at deep layer of reasoning. On the other hand, teachers' reasoning approaches of how and when the nonstandard strategy works for all problems was determined as equating the answer, equating the process, being multiples of each other and equating the denominators. The results and implications are discussed, and recommendations are presented in accordance with the findings of the study.

In mathematics teaching, one of the main tasks of teachers is to attend, interpret and respond to students' thinking. Teachers are supported and encouraged to understand students' thinking as the analysis of students' thinking is viewed as an important tool for teachers to make instructional decisions to improve students' learning in mathematics classrooms (National Council of Teachers of Mathematics [NCTM], 2014). Such mathematics classrooms may create perception in students that their teacher respects their thinking, and students may gain confidence in doing mathematics (Carpenter, Fennema, & Franke, 1992; Davis, 1996). However, understanding students' thinking is not an easy issue for teachers. They need to spend substantial amount of time and noteworthy effort to analyze students' thinking, especially if their thinking is not the same as the standard mathematics (NCTM, 2000; Son, 2016a). It was stated that in order to understand students' thinking, one of the competencies that teachers need to have is adequate knowledge of mathematics (Casey, Lesseig, Monson, & Krupa, 2018).

Researchers have claimed that both the quality of the mathematics teaching and student learning depends on teachers' knowledge (Ball, Hill & Bass, 2005) due to the fact that it takes place in practice (Ball, Lubienski, & Mewborn, 2001). Especially, knowledge of students allows teachers to make principal changes in their practice and these changes are reflected in students' learning. One way of making changes in practice is to engage with students' strategies and use these strategies in teaching (Ebby, Hulbert, & Fletcher, 2019). Even and Tirosh (2002) claimed that teachers are required to make sense of students' nonstandard strategies in addition to standard strategies before, during, and after the lessons. Fuson et al. (1997) stated that nonstandard strategies are mostly conceptual rather than procedural since the students generate them by making relationships between numbers, operations and their previous knowledge (Son, 2016b). Consequently, to attend and interpret students' nonstandard strategies, teachers need to have conceptual knowledge of mathematics.

Theoretical Background of Teacher Knowledge

Teacher knowledge was firstly introduced to the education community by Shulman (1987) with the study of Knowledge Growth in Teaching. He categorized teacher knowledge into three components: subject matter knowledge (SMK), pedagogical content knowledge (PCK), and curricular knowledge (CK). While SMK is focused on teachers' knowledge of the facts, truths, concepts, reasons for knowing the facts, truths and concepts and the relationships between these within or without the discipline; PCK deals with knowledge of the students' learning process including what extent students comprehend the topic, which topics are difficult for them and which materials/teaching methods are the most efficient way to overcome them. As the last component, CK is related to knowledge of topics or issues, that are being studied at the same time in other subject areas and that were taught in the preceding year, have been taught at the same year, and will be taught in later years. Grossman (1990) categorized teacher knowledge into four general areas: a) general pedagogical knowledge, b) subject matter knowledge, c) pedagogical content knowledge, and d) knowledge of context. She expanded Shulman's definition of PCK and categorized it into four components: 1) the knowledge and beliefs about the purposes for teaching a particular subject at a particular grade level, 2) knowledge of students' understanding, 3) curricular knowledge, and 4) knowledge of instructional strategies. Later years, a framework related to teachers' knowledge for mathematics teaching was developed by Ball, Thames, and Phelps (2008) based on and expanding Shulman's PCK notion. Ball and her colleagues emphasized that while Shulman's SMK was classified as specialized content knowledge (SCK), common content knowledge (CCK) and horizon knowledge, PCK was specialized into knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC). Similar to Ball and her colleagues, Rowland, Huckstep, and Thwaites (2005) proposed a framework special to mathematics teaching. The framework, Knowledge Quartet, consists of four categories, foundation, transformation, connection, and contingency focusing on Mathematical Knowledge in Teaching. All the researchers agreed that in general all categories of teacher knowledge have great importance for mathematics teaching, in particular, teacher knowledge of student plays critical key role in orchestrating effective mathematics teaching environment and enhancing students' learning. From this point of view, it would be significant to investigate teacher knowledge of students, discuss its impact on mathematics teaching and learning, and make recommendations and implications about teacher knowledge of students.

Teacher's knowledge of students. Knowledge of students is defined as knowledge related to the characteristics of the specific group of students and planning a mathematics instruction based on the needs of these students. In other words, knowledge of students comprises knowledge of students' conceptual and procedural knowledge, knowledge of students' difficulties and misconceptions, knowledge of teaching strategies which are the most appropriate for students' learning, and knowledge of students' thinking process (Fennema & Franke, 1992; Grossman, 1990). Ball and her colleagues emphasized that knowledge of students is regarded as one of the parts of the KCS which is the combination of knowledge about students and knowledge about mathematics. The KCS requires knowing the topics which the students find easy, difficult or confusing, identifying the students' preconceptions and errors/difficulties/misconceptions, knowing the reasons for these errors/difficulties/misconceptions, and knowing the ways of responding them. Additionally, An, Kulm, and Wu (2004) presented a model which consists of four aspects of students' thinking. "Building on students' math ideas, addressing students' misconceptions, engaging students in math learning, promoting student thinking mathematics" are the aspects of knowing students' thinking (p. 147). These four aspects include the view of teaching with understanding which comprises attending students' thinking, orchestrating an instruction that is consistent with students' understanding, that meets students' needs, and that address students' difficulties/misconceptions with applying particular strategies (An et al., 2004).

There have been several research studies in relevant literature investigating both prospective and in-service teachers' knowledge of students in different subject areas (Chick, Baker, Pham & Cheng, 2006; Kilic, 2011; Morris, Hiebert & Spitzer, 2009; Yesildere & Turnuklu, 2007). Although they emphasized that teachers' knowledge affects their attempts to help and enhance student learning conceptually, they reported that teachers' knowledge of students was either incomplete or inadequate in general. More specifically, they stated that many teachers had difficulty in understanding the reasoning under the students' responses, could not notice the lack of students' conceptual knowledge and failed to address students' difficulties/misconceptions. However, for effective mathematics teaching, teachers should not only have knowledge of students such as the students' difficulties/misconceptions, their lack of knowledge, their previous knowledge, but also they should have knowledge of students such as their variety of thinking, the reasoning behind these thinking and their nonstandard strategies (NCTM, 2000). Due to the fact that most of the studies focused on prospective teachers' knowledge of students from the point of students' misconceptions/difficulties, it is necessary to explore teachers' knowledge of students on different mathematics subject area. The aim of this research study was to explore teachers' knowledge of students in the context of division of fractions.

Teacher's knowledge of students on division of fractions. Research has shown that many prospective teachers and in-service mathematics teachers have difficulties in fractions (Ball, 2000; Jansen & Hohensee, 2016; Olanoff, Lo, & Tobias, 2014; Tirosh, 2000). Although the teachers could decide whether the students' response was correct or not, could perform division of fraction algorithms correctly, and knew the underlying principle of invert and multiply strategy, they could not explain this algorithm to the students conceptually when the students asked the reason for why they have to invert and multiply. Moreover, they could not understand the meaning of algorithm for division of fractions, could not generate an appropriate representation for division of fraction tasks and could not specify students' misconceptions/errors (Ball, 2000; Olanoff et al., 2014; Tirosh, 2000). For instance, Tirosh (2000) and Isik, Ocal, and Kar (2013) found that teachers had little knowledge of students' difficulties/errors, and they themselves made mistakes while explaining students' errors. Moreover, Isiksal and Cakiroglu (2008) conducted a study to investigate prospective mathematics teachers' knowledge concerning students' misconceptions/difficulties in the division of fractions. They concluded that prospective teachers specified four misconceptions/difficulties that the students might have while dividing the fractions. Also, they proposed various strategies such as using concrete materials to overcome students' misconceptions/difficulties.

Based on the literature, researchers agree that teachers' knowledge of mathematics has a significant role in effective mathematics teaching as well as their knowledge of students (Ball et al., 2008; Shulman, 1987). Such knowledge could allow teachers to create an effective classroom environment including performing appropriate instructional activities, designing instruction to overcome students' misconceptions/difficulties, evaluating students' understanding, addressing students' needs, and designing tasks to further student understanding (An et al., 2004; Ball et al., 2008; Johnson & Larsen, 2012; Peterson, Fennema, Carpenter, & Loef, 1989). In order to create this kind of classroom environment, one of the requirements is teachers' knowledge of students, that is, knowing students' misconceptions, knowing students' prior knowledge and knowing the strategies invented by the students (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). Among these issues, students invented strategies which allowed teachers to enter the students' mind and understand students' thinking which is considered as a basis for effective mathematics instruction (Franke, Kazemi, & Battey, 2007; NCTM, 2014). For this, the teachers need to comprehend the underlying reasoning behind students' thinking in addition to deciding whether the students' response is correct or not. From this perspective, this study focused on teacher knowledge of students from the point of their reasoning of students' nonstandard strategies for division of fractions.

Teachers' reasoning is one of the five interrelated strands of mathematical proficiency which is the ability to understand the relationship among concepts and situations logically (Kilpatrick, Swafford, & Findell, 2001). In other words, it requires giving explanation to provide enough reason for their work (Kilpatrick et al., 2001). In the current study, teachers' reasoning means teachers' attending to students' nonstandard strategy, explanations of students' understanding, justifications of mathematical arguments, and providing rational for the nonstandard strategy.

Students' Nonstandard and Standard Strategies for Division of Fractions

There is no doubt that an attempt to explore teachers' knowledge of students about a certain mathematical topic requires an analysis of the body of knowledge on students' ways of thinking about that topic. This section provides a short summary of the research studies on students' ways of thinking about division of fractions.

Division of fraction is one of the most difficult topics for students and one of the least understood topics in elementary school (Fendel, 1987; Tirosh, 2000). One of the reasons for this might be that the invert and multiply strategy, which is the least understood standard algorithm, for division of fractions are still available in the mathematics curricula (NCTM, 2000). However, significant number of research has shown that students are able to invent nonstandard strategies based on their understanding of division of fractions and their interpretations of the problems (Campbell, Rowan, & Suarez, 1998; Schifter, Bastable, & Russel, 1999). The nonstandard strategies that students used to divide fractions are given below.

Common denominator strategy is one of the informal strategies on the basis of knowledge of whole number division and of equivalent fractions and built on the interpretation of division as measurement (Warrington, 1997).

Repeated subtraction strategy is formed from students' knowledge of the division of whole numbers through removing the same amount from the dividend. Schifter et al. (1999) stated that this strategy has emerged in accordance with the interpretation of the division as measurement.

Decimal strategy, similar to common denominator and repeated subtraction strategies, is related to the interpretation of the division as measurement and regarded as informal strategy. This strategy depends on students' knowledge of fractions, decimals and decimal operations (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981).

Applying distributive law strategy is constructed on the basis of the interpretation of division as the inverse of multiplication and emerged from using traditional algorithms for dividing fractions and the distributive law (Wearne & Hiebert, 1988).

Dividing the numerator and denominators strategy includes the interpretation of division as the inverse of a Cartesian product which Sinicrope, Mick, and Kolb (2002) explained that finding the length of unknown side of rectangle whose area and one of the lengths of its sides are known.

Apart from the informal strategies, invert and multiply strategy, and using a unit rate strategy are regarded as formal strategies.

Invert and multiply strategy emerged from the interpretation of division as the inverse of multiplication like applying distributive law strategy. NCTM (2000) stated that invert and multiply strategy is the most commonly used strategy and is constructed through students' knowledge of multiplication and division.

Using a unit rate strategy requires students to determine a unit rate and arises from students' interpretation of division as a determination of a unit rate (Sinicrope et al., 2002).

Previous research shows students are able to invent several strategies for division of fractions based on their reasoning about the meaning of division and interpretations of the division. The analysis of students' invented strategies let the teachers to decide which strategy is noteworthy to discuss in the mathematics classrooms.

The related literature presented in this section provides a general picture of teacher knowledge, teachers' knowledge of students, teachers' knowledge of students on division of fractions, and students' strategies for division of fractions. Based on the results of those studies, it could be concluded that the teachers' knowledge of division in fractions and knowledge of students' thinking were not adequate even though they have a crucial role in teaching and learning mathematics (Ball , Hill, & Bass, 2005; Ma, 1999). It is essential to understand students' thinking to support and enhance students' learning and to make instructional decisions. Investigating teachers' knowledge of students' thinking from the point of their reasoning of students' nonstandard strategy will contribute to the literature and inform teachers, teacher educators, and curriculum developers about how teachers reason about students' nonstandard solution strategies. Furthermore, many research studies investigated prospective teachers' knowledge of students, and not all prospective teachers necessarily have teaching experience. Teaching experience is one of the variables that influence prospective teachers' interpretation of students' thinking, and it provides opportunities for them to analyze student nonstandard strategies (Jacobs, Lamb, & Philipp, 2010; Schoenfeld, 2011). Therefore, in-service teachers might present more in-depth reasoning about student's nonstandard strategies. This study aimed to investigate teachers' understanding of students' nonstandard strategies for division of fractions. The research questions were as follows:

(1) What is the depth of teachers' reasoning of students' nonstandard strategy for division of fractions?
(2) What kind of approaches do middle school mathematics teachers present while reasoning a students' nonstandard strategy for division of fractions?

Method

Design of the Study and Sampling

As Creswell (2007) stated, in order to develop an in-depth description of any case, one of the most appropriate research methods is a qualitative case study design. Among the categorization of the case study, the current study is a single instrumental case study in which the researcher focuses on an issue or concern by selecting one bounded case to explain the issue (Stake, 1995). The issue was middle school mathematics teachers' reasoning about a nonstandard strategy for division of fraction and the particular cases were middle school mathematics teachers whose teaching experience is less than 10 years, and an instrumental case study design was used to support the methodological perspective of the study.

In order to obtain a richer and deeper understanding related to the middle school teachers' reasoning about a nonstandard strategy for division of fraction, the participants should be selected from among the people from whom the most knowledge can be gained, can be accessed easily and with whom the most time can be spent (Merriam, 1998). Therefore, purposive sampling method, one of the most common forms of nonprobability sampling, was appropriate to achieve the purpose of the study. Based on these criteria, 22 middle school mathematics teachers who have been teaching in a public school for less than 10 years were selected as participants. Each participant was coded as T1, T2, T3 and so on instead of using their names.

Data Collection and Analysis

The data collection process had two stages. In the first stage, a task adapted from the literature (Son & Crespo, 2009; Tirosh, 2000) was applied to 22 middle school mathematics teachers individually. The task consists of a student's nonstandard strategy within a classroom excerpt which simulates how mathematical work emerges in the context of teaching. The task was given in Figure 1.

> Pelin is teaching division of fraction to 6th graders. She writes the division operation presented in below and asks the students to solve the question.
>
> $\frac{2}{9} \div \frac{1}{3} = ?$
>
> Bora, one of the students, said that he had solved the question and Pelin asked him to share his strategy with his classmates. He came to the board and wrote:
>
> $\frac{2}{9} \div \frac{1}{3} = \frac{2 \div 1}{9 \div 3} = \frac{2}{3}$
>
> He explained that the division of fractions is similar to the multiplication of fractions. Pelin asked the class that what they thought about Bora's solution and whether they agreed with Bora or not. Irmak, another student, said that she didn't agree with Bora and said "I don't understand this solution. Dividing the fractions is not done in this way. You must invert and multiply".
>
> Imagine that you are Pelin:
>
> A. What do you think about Bora's strategy? What do you think about the correctness of Bora's strategy? Explain your reason.
> B. Do you think that Bora's strategy can be generalized to the division of fractions? If yes, why? If no, why?
> C. How do you respond to Bora and Irmak, and why? Explain it in as much detail as you can.

Figure 1. Division of fraction task

As given in Figure 1, the task consists of only one nonstandard strategy which is dividing the numerator and denominator strategy. The reason for selecting this strategy is that it is an easy strategy to perform but not widespread strategy in the mathematics classes. Moreover, the mathematics curriculum in Turkey does not contain any information related to dividing the numerator and denominator strategy. Thus, it is assumed that Turkish mathematics teachers do not have any experience in dividing the fractions using this strategy. From this point of view, their reasoning of this nonstandard strategy would come from their previous knowledge, their content knowledge, and their experiences.

The task consists of three sub-questions to explore teachers' reasoning about Bora's nonstandard strategy. The first question (A) aims to elicit whether this strategy is valid or not. Campbell et al. (1998) stated that if the strategy works in a given problem, then this strategy could be regarded as valid. While the participants were explaining the questions, "What do you think about Bora's strategy? What do you think about the correctness of Bora's strategy?", teachers' reasoning about whether this strategy is applicable to solve the given question or not has been revealed. In the second question (B), the aim is to explore teachers' reasoning about whether the strategy is generalizable or not. The teachers' explanation of the second question provides information about their reasoning of whether the strategy can be performed in several questions or not. Lastly, the aim of asking the third question (C) is that while responding Bora and Irmak, they might provide further explanation related to how and when each of these strategies can be used efficiently. Thus, it is expected that this question will serve as an indicator of the teachers' reasoning about the efficiency of Bora's strategy.

In the second stage of the data collection process, semi-structured one-to-one interviews which took approximately 15 minutes were conducted with all participating teachers individually after they responded to the questions presented in the task. The aim of the interviews was to get detailed data regarding their reasoning about the student's nonstandard strategy. The questions asked during the interview were more specific in order to understand the depth of their reasoning about the validity, generalizability and the efficiency of the student's strategy. In accordance with this purpose, the questions such as "whether Bora's strategy is efficient or not?" and "Do you teach this strategy to your students? If yes, why? If no, why?", "Do you think this strategy can be used to solve any question related to division of fraction?" were asked.

In order to determine teachers' reasoning about the student's nonstandard strategy, the data gathered from the task and semi-structured interview were analyzed based on the three criteria articulated by Campbell et al. (1998). They proposed three criteria to evaluate students' invented strategies which are validity, generalizability and efficiency. The teachers' reasoning about whether or not the strategy works in a given problem is related to the *validity* of the strategy. In relation to validity, if the teacher says "Bora's strategy is correct for this question", then the teacher's reasoning about Bora's strategy was coded valid. Teachers' reasoning about whether the strategy works for other problems besides the given problem or not is related to the *generalizability* of the strategy. In other words, if the teacher reasons that the nonstandard strategy is generalizable to other problems related to division of fraction, then his/her reasoning was regarded as generalizable. Lastly, teachers' reasoning of how and when the strategy can be used more efficiently is related to *efficiency* of the strategy. More specifically, if the teacher reasons that Bora's strategy works for all questions of division of fraction and she/he provides the situations in which Bora's strategy can be used more efficiently than the standard strategy, then his/her reasoning was coded as efficient. Otherwise, teachers' reasoning about student's nonstandard strategy was that the nonstandard strategy is not valid, not generalizable, and not efficient.

After analyzing teachers' reasoning about student's nonstandard strategy in terms of validity, generalizability and efficiency, then their responses were categorized into six layers which exhibited the depth of the teachers' reasoning about the given strategies. These layers were adapted from the study of Son and Crespo (2009). Layer 1 reasoning recognized that Bora's strategy is not correct for a given problem. In this case, the teacher's reasoning about Bora's strategy in terms of generalizable and efficiency was coded as not generalizable and not efficient. If a teacher stated that Bora's strategy works only in a given problem, then his/her reasoning was coded as Layer 2. As a consequence, the teacher reasoned that this strategy is not generalizable and not efficient. The teacher's reasoning at Layer 3 is similar to Layer 2 except their reasoning at Level 3 provide further consideration about how and when this strategy did not work for all problems of division of fraction. In addition, Layer 4 reasoning comprised of reasoning about the nonstandard strategy as valid, generalizable and efficient. However, the teacher could not explain how and when Bora's strategy is efficient. If a teacher established both the validity and the generalizability of Bora's strategy and provided some special cases in which Bora's strategy can be used, then his/her reasoning was coded as Layer 5. Lastly, the teacher who provides further evidence why Bora's strategy is generalizable and efficient was Layer 6. Table 1 shows specific characteristics of each layer of reasoning.

Table 1.
Layers of Reasoning about Students' Nonstandard Strategies

Layers	Validity	Generalizability	Efficiency
1	Incorrect	Not generalizable	Not efficient
2	Correct	Not generalizable	Not efficient
3	Correct	Not generalizable	Not efficient (stated reasons for why)
4	Correct	Generalizable	Not efficient
5	Correct	Generalizable	Efficient in some cases
6	Correct	Generalizable	Efficient

After categorizing teachers' reasoning, these layers were labeled as surface, intermediate and deep based on their characteristics (Son, 2016a). The Layer 1 through 3 was considered to represent "surface" reasoning about the nonstandard strategy. As it was indicated in Table 1, the teachers at Layers 1 through 3 reasoned that nonstandard strategy was not generalizable and not efficient. Due to the fact that the teachers' reasoning about the nonstandard strategy was valid and generalizable at the Layer 4, the teachers at this layer represented "intermediate" reasoning. However, Layers 5 and 6 was regarded as "deeper" reasoning since the teacher at one of these layers established both the generalizability and efficiency of Bora's strategy. In such a categorization of teachers' reasoning, addressing the generalizability of a nonstandard strategy is important because it requires an analysis of whether the strategy works for a wider range of solved problems. In addition, establishing how and when the nonstandard strategy is efficient is an important issue because it provides teachers' reasoning about different strategies. Thus, establishing whether Bora's strategy is generalizable or not was considered as an indicator to distinguish between surface and intermediate levels of reasoning. Moreover, addressing both the generalizability and efficiency of Bora's strategy was regarded as a separating line between intermediate and deeper layers of reasoning.

To establish inter-rater reliability, each task was coded by two independent researchers who are experts in mathematics education. The inter-rater reliability was calculated and a 90 percent correlation was found between ratings.

Results and Discussion

The Depth of Teachers' Reasoning about Nonstandard Strategy for Division of Fractions

Based on the analysis of the data according to Campbell et al.'s (1998) criteria, middle school mathematics teachers' reasoning was categorized into six layers to present the depth of teachers' reasoning about the student's nonstandard strategy. The percentages of each layer of reasoning were presented in Table 2.

Table 2.
Distribution of Teachers by Layer of Reasoning

Layers	Validity	Generalizability	Efficiency	f (%)
1	Incorrect	Not generalizable	Not efficient	2 (9.09)
2	Correct	Not generalizable	Not efficient	2 (9.09)
3	Correct	Not generalizable (stated some reasons)	Not efficient	8 (36.36)
4	Correct	Generalizable	Not efficient	3 (13.64)
5	Correct	Generalizable	Efficient in some cases	3 (13.64)
6	Correct	Generalizable	Efficient	4 (18.18)

According to Table 2, two teachers among 22 teachers (9.09%) thought that Bora's strategy was not correct and it was not the way of division of fractions. For this reason, these teachers' reasoning about students' nonstandard strategy was considered as not valid and they were at Layer 1. While the remaining 20 teachers agreed that Bora's strategy was correct, they are in contradiction about the generalizability and the efficiency of the strategy. Among 20 teachers, 10 teachers reported that Bora's strategy could be used to solve the given problem, but it could not be performed for all problems related to division of fractions. Although two teachers (9.09%) (out of 10) stated that Bora's strategy could not work for all problems, they could not explain why it cannot be applied to other problems related to division of fractions. In other words, they did not reason about why Bora's strategy was not efficient. Therefore, these teachers' reasoning level was at Layer 2. Different from the teachers at Layer 2, eight teachers (36.36%) specified the reason why Bora's strategy is not generalizable and not efficient, which was categorized as Layer 3. An example of T9 related to teachers' reasoning at Layer 3 was presented below.

> Bora's strategy is correct for this problem. However, I don't think that he can divide all fractions using this strategy. I wonder that what he will do if the denominator and the numerator of the dividend are not the multiple of these of divisor. In such a case, he could not perform this kind of solution.

The remaining seven teachers at Layer 3 presented the same justification for Bora's strategy is not generalizable and efficient. Due to the fact that although the teachers at Layer

2 and Layer 3 addressed the generalizability of Bora's strategy, they thought that the strategy was not efficient; their level of reasoning was considered as surface level.

As it can be seen from Table 2, only three teachers (13.64%) exhibited intermediate level of reasoning, namely three teachers were at Layer 4, by establishing both the validity and generalizability of Bora's strategy. Related example of T7 was as follows:

> Bora's strategy is correct and it can be used for different problems of division of fractions. But I think that using this strategy for all problems may cause some difficulties.

Similar to T7, other two teachers (T2 and T16) stated that Bora's strategy can be performed in some cases; however, they did not explain these cases. In other words, they did not reason how and when Bora's strategy can be used. During the interviews, it was asked them to give more detail about these cases; they only gave the examples similar to the given problem such as $\frac{6}{21} \div \frac{3}{7}$. However, they could not justify the reasons for choosing these numerators and denominators for both dividend and divisor. By virtue of not being address why Bora's strategy works for any problem, their reasoning level was regarded as intermediate.

Seven teachers among 11 teachers had deep level of reasoning; three (13,64%) were at the Layer 5 and four (18.18%) at Layer 6. The main difference between the teachers who were at Layer 5 and Layer 6 was that the teachers at Layer 5 recorded that Bora's strategy was efficient in some cases. Also, the main difference between at Layer 4 and Layer 5 was explaining in which cases Bora's strategy is efficient. The explanation of the T8 at Layer 5 during the interview was presented as an example.

> Bora's strategy is different but it is correct. I have not thought this kind of solution before. In this problem, the denominator and numerator of dividend are the multiple of those of divisor. I am not sure that it can be used for all problems related to the division of fractions. Hmmmm..I think so. If the denominator and numerator of dividend are the multiple of those of divisor, then it can be applied to the other problems. Otherwise, it cannot be applied.

The other teachers at Layer 5 provided similar explanation as T8. On the other hand, four teachers at Layer 6 paid more attention how the Bora's strategy is connected to Irmak's strategy. These teachers explained why Bora's strategy is generalizable and how it is efficient. The related example of T20 related to teachers' reasoning at Layer 6 was presented below.

> I like Bora's strategy. I have not seen this strategy before. This strategy is correct and I am sure that it is generalizable to all problems. In this problem (presented in the task), the denominator and numerator of dividend is divisible by those of the divisor. In such a case, Bora's strategy can be performed. However, if not, we can change both denominators into common denominator.

As it can be realized from T20's explanation, he provided further evidence of how and when Bora's strategy is efficient. Also, he showed that how Bora's strategy can be generalized to all problems. Due to the fact that 11 teachers established reasoning about whether Bora's strategy is generalizable or not and how it is generalizable, they exhibited deeper level of reasoning.

Clearly, more than half of the middle school mathematics teachers had surface level of reasoning about student's nonstandard strategy. However, one third of the teachers' reasoning level was deeper. The teachers at intermediate level were only 13.64 percent. Based on this analysis, it can be concluded that most of the mathematics teachers did not know how any nonstandard strategy can be generalized to all problems and also did not provide the situations in which Bora's strategy can be used more efficiently than the standard strategy.

The Teachers' Reasoning Approaches about Nonstandard Strategy for Division of Fractions

Although it is important to determine the level of teachers' reasoning about students' nonstandard strategies, the depth of their reasoning is also important issue to consider. Teachers' depth of reasoning provides information in relation to how they reason about the generalizability of students' nonstandard strategies (Son, 2016a). Establishing the generalizability of the nonstandard strategy allows the teacher to make decisions about how and when the nonstandard strategy is and is not efficient (Son & Crespo, 2009). Due to the fact that the teachers at deeper level provided justification related to why Bora's strategy is generalizable and efficient and the teachers at surface level of reasoning provided justification why Bora's strategy is not generalizable and efficient, their reasoning approaches about the nonstandard strategy were presented. However, three teachers at level 4 could not give any reasoning about why Bora's strategy could be generalized but why it is not efficient, any approach could be identified. The teachers' reasoning approaches was presented in Table 3.

Table 3.
The Teachers' Reasoning Approaches about Nonstandard Strategy for Division of Fractions

	The teachers' reasoning approaches	f (%)
Surface Reasoning (Layer 1-3)	Equating the answers	3 (18.18)
	Equating the process	1 (4.54)
	Not being multiples of each other	7 (31.82)
Intermediate Reasoning (Layer 4)	No reasoning approach	3 (13.64)
Deep Reasoning (Layer 5-6)	Equating the process	3 (13.64)
	Equating the denominators	4 (18.18)

The teachers who thought that the nonstandard strategy does not work for all problems addressed three approaches for their reasoning: equating the answer, equating the process and not being multiples of each other. Three teachers (13.64%) whose reasoning focused on *equating the answers* inferred the generalizability of Bora's strategy by solving the problem using Irmak's strategy. *Equating the answers* approach is to compare the answers of the problem solved with both standard and nonstandard strategies. If these answers are the same, the teachers reason that the nonstandard strategy is valid for only given problem. The teachers did not reason about the underlying principles behind the nonstandard strategy, contrary, they relied on the procedural aspects of the both strategies. The teachers stated that Bora's strategy was valid however it could not be generalized. Moreover, some teachers claimed that the answer to the problem was found correct coincidentally and it was not possible to present any mathematical justification for Bora's strategy. The teachers at surface level of reasoning (Layers 1-3) commonly used this approach.

The other approach about why Bora's strategy was not generalizable was *equating the process*. This approach consists of establishing some kind of equivalence between the two strategies (Son & Crespo, 2009). In this study, this approach is regarded as making connection between the nonstandard strategy with standard strategy for any division algorithm. Only one teacher (4.54%) at Layer 2 presented this approach and reported that this strategy is meaningless since Bora applied the division algorithm of whole numbers to division algorithm of fraction. On the other hand, the most commonly used reasoning why Bora's strategy cannot be generalized is that the denominator and numerator of dividend and divisor were *not multiples of each other*. Seven teachers (31.82%) at surface level of reasoning (Layer 1-3) asserted that unless the denominator and numerator of divisor is multiple of those of dividend, nobody perform this strategy. They stated that it would be an efficient and easy way to divide the fractions if they were multiple of each other.

On the other hand, the teachers who thought that the nonstandard strategy works for all problems addressed two approaches for their reasoning: equating the process and equating the denominators. As it was presented before, *equating the process* is connecting nonstandard strategy and standard strategy such as connecting division of fractions and division of whole numbers. In relation to this, three teachers (13.64%), who had deeper level of reasoning, stated that 2:1 is the same as $\frac{2}{1}$ and 9:3 is the same as $\frac{9}{3}$. From this point of view, their reasoning was as follows:

$$\frac{2}{9} \div \frac{1}{3} = \frac{2 \div 1}{9 \div 3} = \frac{\frac{2}{1}}{\frac{9}{3}} = \frac{2}{3}$$

Clearly, they connected Bora's strategy with the strategy for dividing the whole numbers. During the interview, it was asked them if the denominator and numerator of dividend was not divisible by those of divisor, how Bora's strategy could be performed. The teachers explained that division algorithms of decimals can be used in such a case.

Another approach related to teachers' reasoning about efficiency of nonstandard strategy is related to common denominator strategy. This approach was used by teachers coded in the deeper layers of reasoning. Four teachers (18.18%) attempted to demonstrate the efficiency of Bora's strategy by *equating the denominators*. As it was presented, some teachers emphasized that in order to apply Bora's strategy, the denominator and numerator of dividend should be divisible by those of divisor. Otherwise, it does not work for all problems. However, four teachers (out of 22) stated that we can equate the denominators. The explanation of their reasoning is presented below:

$$\frac{2}{9} \div \frac{1}{3} = \frac{2}{9} \div \frac{1x3}{3x3}$$

$$\frac{2}{9} \div \frac{3}{9} = \frac{2:3}{9:9} = \frac{2}{3}$$

The teachers' reasoning at the surface level and deeper level was presented according to when and how Bora's strategy was efficient. These teachers addressed some justifications for how and when Bora's strategy works for all problems. On the other hand, teachers at Layer 4 intermediate level of reasoning, did not make any explanation related to how and when Bora's strategy is used more efficiently even though they stated that it can be generalized to all problems. Therefore, their reasoning was not analyzed in terms of the depth of reasoning.

Discussion

This study investigated the depth of middle school mathematics teachers' reasoning about a nonstandard strategy and their reasoning approaches. Teachers' reasoning about nonstandard strategy was analyzed based on three criteria, validity, generalizability, and efficiency, articulated by Campbell et al. (1998). According to these criteria, teachers' reasoning was categorized into six layers of reasoning, making a distinction between surface, intermediate and deeper layers. Furthermore, the depth of middle school mathematics teachers' reasoning of a nonstandard strategy was examined on the basis of how and when the nonstandard strategy was and was not efficient. The teachers who thought that the nonstandard strategy does not work for all problems addressed three approaches: equating the answer, equating the process and not being multiples of each other. On the other hand, the teachers who thought that the nonstandard strategy works for all problems addressed two approaches: equating the process and equating the denominators.

The results showed that half of the middle school mathematics teachers' reasoning about students' nonstandard strategy is at surface level. In other words, these teachers stated that the nonstandard strategy does not work for all problems without explaining its reasons. Similarly, in the study of Tirosh (2000) and Ball (2000), the teachers could not explain why the student's nonstandard strategy is not correct. Also, the teachers claimed that the most appropriate strategy to divide the fractions is invert and multiply strategy and dividing the denominators and numerators in itself is not a correct way for division of fractions. However, the teachers could not explain the reasoning behind invert and multiply strategy. This result of the current study is consistent with the study of Olanoff et al. (2014) in which they concluded that the teachers had difficulty in justifying the meaning of the procedures or why and when the procedures work. On the other hand, the result of this study is in contradiction to the result of Borko et al.'s (1992) study. Borko et al. specified that the prospective teacher knew the underlying principle of invert and multiply strategy while dividing fractions, but they could not explain this algorithm to the elementary students conceptually when the students asked the reason for why they have to invert and multiply.

As stated, the teachers need to have both conceptual knowledge of mathematics to understand the relationship between numbers and operations (Fuson et al., 1997) and knowledge of students to address students' prerequisite knowledge that the students use to invent the strategy (Ball et al., 2008; Shulman, 1987). These are essential for reasoning of students' nonstandard strategies since while inventing strategies, the students establish relationship between numbers and operations and connect their previous knowledge and new knowledge (Son, 2016b). As a result, it could be concluded that the teachers' knowledge of students is insufficient due to the fact that they could not interpret how and when the nonstandard strategy of student does work best. The similar results were revealed in previous studies that the teachers have inadequate knowledge of students (Even & Tirosh, 1995; Ma, 1999). However, teachers' inadequate knowledge of students affects teachers' practice and students' mathematics learning significantly. As Carpenter et al. (1999) stated, teachers make principal changes in their practice and decide how to respond the students depends on their knowledge. Moreover, in order to make decisions about in which ways and to what extent teachers guide the students while they are constructing nonstandard strategies, requires an understanding of students' thinking (Even & Tirosh, 1995). Understanding students' nonstandard strategies is a challenging issue for teachers and they need to spend a great amount of time and significant effort to analyze students. Fortunately, one-third of the mathematics teachers were at deeper level of reasoning about students' nonstandard strategy. This result is consistent with the result of the study of Son and Crespo (2009) in which one-third of the teachers were categorized into the deeper layers (4–6) of reasoning. The teachers at deeper level of reasoning had conceptual understanding of division of fraction as did Chinese teachers in Ma's study. The important issue of reasoning about students' nonstandard strategies is to enter the students' mind and understand students' thinking which is considered as a basis for effective mathematics instruction (Franke, Kazemi, & Battey, 2007; NCTM, 2014). Only one-third of teachers had enough knowledge to enter students' mind, understand their thinking and create classroom environment to support and enlarge students' thinking.

Another important issue that was focused on was the depth of teachers' reasoning approach. Son (2016) stated that teachers' depth of reasoning provides information in relation to how they reason about the generalizability of students' nonstandard strategies. It is not surprisingly that the teachers at Layer 1-3 believed that student's nonstandard strategy could not be generalized to all problems. They provided three reasons for that: equating the answer, equating the process and not being multiples of each other. The teachers, who reasoned the non-generalizability of the student's strategy through equating the answer and equating the process, did not explain the underlying principles of their responses. Their justification related to equating the answers approach depends on the procedural knowledge such as comparing the answer of the problem by applying invert and multiplies strategy. Also, the teachers whose reasoning approach was equating the process likened the division in whole numbers to division in fractions. They did not address any meaningful explanation how two kinds of division is similar and why the student's nonstandard strategy did not work for other problems. This could be interpreted as although the teachers had not enough knowledge of students' understanding, they had no conceptual knowledge related to division of fractions. Therefore, they asserted that the nonstandard algorithm cannot be applied to all problems even though it can be applied. Different from equating the answer and equating the process approach, teachers present some superficial justifications related to not being multiples of each other approach. They proved their justification by stating an example which includes division of fractions whose denominator and numerator of dividend are not the multiples of those of divisor As it can be realized their reasoning about this approach does not verify non-generalizability of the student's nonstandard strategy. In contrast to the teachers at Layer 1-3, the teachers at Layer 5-6 present meaningful justifications of the generalizability of the student's nonstandard strategy. Their reasoning depends on connecting the division of fractions with division of whole numbers and equating the denominators. These are meaningful and stated in the previous studies (Tirosh, 2000; Warrington, 1999). While connecting the division in fractions with division in whole numbers, the students apply their knowledge of division in whole numbers and also deal with the properties of the denominators and numerators to be the multiples of each other. Therefore, the teachers address the reasoning behind student's nonstandard strategy and understand student's thinking. Another significant reasoning approach about the generalizability of the student's nonstandard strategy was equating the denominators, namely common denominator. If the denominators and numerators of dividend are not divisible by those of divisor, then the student may think to make the denominator 1 by dividing the denominator of dividend to the denominator of divisor. In order to do this, s/he may consider equating the denominators. In this case, the generalizability of nonstandard strategy may be proved.

As a result, this study has contribution to the continuing effort to improve mathematics learning for all students by focusing on knowledge of students' understanding that the teachers need to have to teach effectively. The study emphasizes the importance of teachers' reasoning of division of fractions which requires how and why the nonstandard strategy works. However, 12 middle school teachers out of 22 did not recognize Bora's strategy as an effective way for division of fractions. This indicates that middle school mathematics teachers have limited mathematical understanding as well as their difficulty in justifying why this nonstandard strategy work. From this point of view, the present study stresses the crucial role of the teacher educators and teacher education programs to expose middle school teachers, while they were studying at the teacher education programs, to students' nonstandard strategies and to determine why these strategies are working in the field of division of fractions. Also, in accordance with Campbell et al.'s (1998) criteria, the framework of this study for determining the layers of reasoning about nonstandard strategies (validity, generalizability, efficiency) might be beneficial for teacher educators to plan and carry out similar tasks in their classrooms. Thus, the prospective teachers, who are the teachers of future, might engage in this kind of tasks which they should take into account the validity, generalizability, and efficiency of the students' nonstandard strategies.

Another implication of the study is for curriculum developers and textbook writers. Curriculum developers and textbook writers might include the students' nonstandard strategies to the curriculum and textbooks to make teachers familiar with them. In this way, the teachers learn the reasoning behind these strategies before they teach the subject, consider how and when these strategies work well and connect the standard and nonstandard strategies. Then, they may plan how they will teach such strategies to their students meaningfully.

Lastly, this study highlights the prominence of teachers' reasoning about students' nonstandard strategies. Therefore, it would be valuable for mathematics education research to continue to investigate prospective and also in-service teachers' reasoning about students' thinking. When the importance of students' nonstandard strategies is considered, researchers whose aim is investigating other possible nonstandard strategies in this and in other mathematical topics will make great contribution to the literature. In addition to the teachers' reasoning, studies on students' reasoning are necessary to learn more about how students think and understand, and what non-standard strategies they have invented.

References

An, S., Kulm, G., & Wu, Z. (2004). The pedagogical content knowledge of middle school, mathematics teachers in China and the US. *Journal of Mathematics Teacher Education, 7*, 145-172

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education, 51*, 241–247.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, *29*(3), 14–17, 20–22, 43–46.

Ball, D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special?. *Journal of Teacher Education*, *59*(5), 389-407.

Ball, D. L., Lubienski, S. T., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (pp. 433–456). New York: Macmillan.

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily?. *Journal for Research in Mathematics Education, 23*, 194–222.

Campbell, P. F., Rowan, T. E., & Suarez, A. R. (1998). What criteria for student-invented algorithms? In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in School Mathematics: 1998 yearbook of the National Council of Teachers of Mathematics*. Reston, VA: NCTM

Carpenter, T. P., Corbitt, M. K., Kepner, H. S., Lindquist, M. M., & Reys, R. E. (1981). Results and implications of the second NEAP mathematics assessments: Elementary school. *The Arithmetic Teacher,28*(8), 34–37.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C. P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal*, *26*(4), 499-531.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1992). *Cognitively guided instruction: Building the primary mathematics curriculum on children's informal mathematical knowledge.* Paper presented at the annual meeting of the American Educational Research Association.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction.* Postmourth, NH: Heinemann.

Casey, S., Lesseig, K., Monson, D., & Krupa, E. E. (2018). Examining preservice secondary mathematics teachers' responses to student work to solve linear equations. *Mathematics Teacher Education and Development*, *20*(1), 132-153.

Chick, H., Baker, M., Pham,T., & Cheng, H. (2006). Aspects of teachers' pedagogical content knowledge for decimals, In Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education:* Vol. 2 (pp. 297-304), Prague, Czech Republic: Charles University

Creswell, J. W. (2007). *Qualitative inquiry and research design: Choosing among five approaches (2nd ed.).* Thousand Oaks, California: Sage Publications, Inc.

Davis, B. (1996). *Teaching mathematics: Toward a sound alternative.* New York: Garland Publishing.

Ebby, C. B., Hulbert, E. T., & Fletcher, N. (2019). What can we learn from correct answers? *Teaching Children Mathematics*, *25*(6), 346-353.

Even, R., & Tirosh, D. (1995). Subject-matter knowledge and knowledge about students as sources of teacher presentations of the subject-matter. *Educational Studies in Mathematics*, *29*(1), 1-20.

Even, R., & Tirosh, D. (2002). Teacher knowledge and understanding of students' mathematical learning. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 219–240). Mahwah, NJ: Erlbaum.

Franke, M. L., Kazemi, E., & Battey, D. (2007). Mathematics teaching and classroom practice. In F. K. Lester (Ed.), *The second handbook of research on mathematics teaching and learning* (pp.225-256). Char1otte, NC: Information Age Publishing.

Fendel, D. M. (1987). *Understanding the structure of elementary school mathematics*. Newton, MA: Allyn an

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147–164). New York: Macmillan

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*(4), 403–434.

Fuson, K. C., Wearne, D., Hiebert, J., Murray, H., Human, P., Oliver, A.,....,Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28*, 130-162.

Grossman, P. (1990). *The making of a teacher*. New York: Teachers College Press.

Isik, C., Ocal, T., & Kar, T. (2013). *Analysis of pre-service elementary teachers' pedagogical content knowledge in the context of problem posing.* Paper presented at the meeting of Eighth Congress of European Research in Mathematics Education (CERME 8), Antalya, Turkey.

Isiksal, M., & Cakiroglu, E. (2008). Pre-service teachers' knowledge of students' cognitive processes about the division of fractions. *Hacettepe University Journal of Education, 35*, 175-185.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, 169-202.

Jansen, A., & Hohensee, C. (2016). Examining and elaborating upon the nature of elementary prospective teachers' conceptions of partitive division with fractions. *Journal of Mathematics Teacher Education*, *19*(6), 503-522.

Johnson, E. M., & Larsen, S. P. (2012). Teacher listening: The role of knowledge of content and students. *The Journal of Mathematical Behavior*, *31*(1), 117-129.

Kilic, H. (2011). Preservice secondary mathematics teachers' knowledge of students. Turkish Online Journal of Qualitative Inquiry, 2(2), 17–35.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum.

Merriam, S. B. (1998). *Qualitative research and case study applications in education n (Rev. ed.*). San Francisco, CA: Jossey-Bass.

Morris, A. K., Hiebert, J., & Spitzer, S. M. (2009). Mathematical knowledge for teaching in planning and evaluating instruction: What can preservice teachers learn? *Journal for Research in Mathematics Education*, 491-529.

National Council of Teacher of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.* Reston, VA: Author.

Olanoff, D., Lo, J. J., & Tobias, J. (2014). Mathematical content knowledge for teaching elementary mathematics: A focus on fractions. *The Mathematics Enthusiast*, *11*(2), 267-310.

Peterson, P. L., Fennema, E., Carpenter, T. P., & Loef, M. (1989). Teacher's pedagogical content beliefs in mathematics. *Cognition and Instruction*, *6*(1), 1-40.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, *8*(3), 255-281.

Schifter, D., Bastable, V., & Russel, S. J. (1999). *Number and operations, part 2: Making meaning for operations*. Parsippany, NJ: Dale Seymour.

Schoenfeld, A. H. (2011). Toward professional development for teachers grounded in a theory of decision making. *ZDM Mathematics Education*, *43*(4), 457-469.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1–22.

Sinicrope, R., Mick, H., & Kolb, J. (2002). Fraction division interpretations. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, rations, and proportions: 2002 Yearbook* (pp. 153–161). Reston, VA: National Council of Teachers of Mathematics

Son, J. W. (2016a). Moving beyond a traditional algorithm in whole number subtraction: Preservice teachers' responses to a student's invented strategy. *Educational Studies in Mathematics*, *93*(1), 105-129.

Son, J. W. (2016b). Preservice teachers' response and feedback type to correct and incorrect student-invented strategies for subtracting whole numbers. *The Journal of Mathematical Behavior*, *42*, 49-68.

Son, J., & Crespo, S. (2009). Prospective teachers' reasoning about students' non-traditional strategies when dividing fractions. *Journal of Mathematics Teacher Education, 12*(4), 236–261.

Stake, E. (1995). *The art of case study research*. Thousand Oaks, CA: Sage.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*(1), 5–25.

Warrington, M. (1997). How children think about division with fractions. *Mathematics Teaching in the Middle School, 2*(6), 390–397.

Wearne, D., & Hiebert, J. (1988). A cognitive approach to meaningful mathematics instruction: Testing a local theory using decimal numbers. *Journal for Research in Mathematics Education, 19*, 371–384.

Yesildere, S., & Turnuklu, E. B. (2007). Examination of students' mathematical thinking and reasoning processes. *Ankara University, Journal of Faculty of Educational Sciences*, *40*(1), 181-213.