On Singapore prospective secondary school teachers' mathematical content knowledge

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This paper reports the performance of one entire cohort of Singapore prospective secondary school mathematics teachers in a mathematics proficiency test. The prospective teachers were admitted to the teacher education program specializing in teaching secondary school mathematics. The strengths of the prospective teachers' content knowledge, their errors and misconceptions in school mathematical content knowledge are discussed.

Keywords

Prospective teachers, mathematical content knowledge, mathematics proficiency test

Introduction

It is a worldwide phenomenon that more students who might not have the recommended tertiary mathematics qualification to teach secondary school mathematics are seeking admission to secondary teacher education programs and specializing in teaching mathematics (see for example McKenzie, Kos, Walker, & Hong, 2008, p. 27; Toh, Chua & Yap, 2007). Thus, the mathematics attainment level of these prospective teachers is therefore of interest to the administrators and the researchers in the respective teacher training institutes.

Why is there a need to be concerned with prospective teachers' knowledge of school mathematics, since they must already have acquired mathematical content knowledge of school mathematics when they were students, even though they might not have university specialization in mathematics? According to Jaworski and Gellert (2003), prospective teachers' knowledge of school mathematics is largely limited because it is mainly based on their *experience as students*. Students who have learnt mathematics as mathematically immature learners are unlikely to have experienced it with the richness and perspective required for teaching (Stacey, 2008). Moreover, literature on prospective teachers' mathematical content knowledge shows that prospective teachers enter teacher education programs with rather narrow conceptions of mathematics as a set of rules and conventions (for example, Ball, 1990; Cooney, 1999; Taylor, 2002; Wilson & Ball, 1996).

Compared to the numerous studies on elementary school teachers' mathematical content knowledge (for example, Browning, Edson, Kimani & Aslan_Tutak, 2014; Livy & Vale, 2011; Ma, Millman & Wells, 2008; Muir & Livy, 2012; Norton, 2010; Ryan & McCrae, 2005/2006; Thanheiser, Whitacre & Roy, 2014), there are relatively few similar studies on secondary school teachers in the literature. This paper aims to add to the relatively less explored area of secondary school teachers' mathematical content knowledge.

Toh, Kaur and Koay (2013) reported a preliminary study of Singapore secondary school mathematics teachers' mathematical content knowledge as part of an international comparative study on mathematics teacher training. However, the instrument used in that study was of the form of an objective testing, in which the teachers only needed to state the correct answers without showing any mathematical reasoning. Hence, little information could be revealed about the prospective teachers' understanding of the individual mathematics questions. The study reported in this paper is based on a mathematics proficiency test, in which detailed solution for the individual question was expected of the prospective teachers.

Recruitment of Singapore mathematics teachers

Singapore's expansion in recruitment of mathematics teachers in recent years has resulted in more students seeking admission to the teacher education program. Many of these candidates who are prepared to teach secondary school mathematics come from diverse educational backgrounds, for example, a rough estimate of about 65% of the prospective teachers are from mathematical sciences and engineering backgrounds, and the rest of the prospective teachers come from other disciplines such as computer engineering, chemistry, business, humanities and others. Even among the former 65%, a relatively low proportion come from a background of mathematics specialization and a significantly higher proportion are from various engineering backgrounds. The wide range of credentials before admission made it difficult to assess their proficiency in school mathematics.

To ensure that prospective teachers selected for the secondary school mathematics teacher education programs are competent in mathematics knowledge, they must meet the criteria of sufficient exposure to university mathematics during their undergraduate education, with at least one module of undergraduate calculus and one module of linear algebra, or equivalent. It should be noted that in the 1990s and earlier, most of the prospective teachers for secondary school mathematics were from the background of pure mathematics.

Teachers' mathematical content knowledge

Content knowledge describes a teacher's understanding of the structures of his or her subject, or "a deep understanding of the domain itself" (Shulman, 1986; 1987). In the model of knowledge bases of the expert teacher identified by Turner-Bisset (1999), content knowledge comprises three of the list of knowledge bases for teaching. According to Shulman (1986), "the teacher need not only understand that something is so, the teacher must further understand why it is so" (p. 9). Shulman (1985) identified content knowledge as one of the three dimensions of teachers' professional knowledge, together with pedagogical content knowledge and general pedagogical knowledge. In the United States, the Glenn commission (U.S. Department of Education, 2000) asserted that "high quality teaching requires that teachers have a deep knowledge of subject matter" (U.S. Department of Education, 2000).

There are different views among mathematics educators on what teachers need to know about teaching mathematics proficiently (Cooney, 1999; Graeber, 1999; Kilpatrick, 2001). Despite this, most agree that good mathematical content knowledge is one aspect that teachers need to have in order to be an efficient teacher. Good mathematical content knowledge is necessary (but not sufficient) for good teaching in the mathematics classroom (e.g., Chapman, 2005).

Studies have shown that the quality and the rigor of the mathematics curriculum are correlated to the mathematical content knowledge of teachers (Schmidt, 2002). Li, Huang and Shin (2008) reported that both China and Korea, whose students generally perform very well in mathematics based on international standards, place great emphasis in developing their mathematics teachers with sound mathematical content knowledge. Ma (1999), in her comparative study across the United States and China, concluded that the level of mathematical content knowledge influences the ability of teachers to analyze students' errors or to develop conceptual understanding in mathematics. There are examples of teachers who wanted to do a good job in teaching mathematics but faced many problems that were largely due to their limited mathematical content knowledge (e.g., Hutchinson, 1997).

What exactly constitutes the mathematical content knowledge that teachers need to know? Krauss, Baumert and Blum (2008) identified this mathematical content knowledge as lying between "the school-level mathematics knowledge that good school students have", and "the university-level mathematics knowledge that does not overlap with the content of the school curriculum" (p. 876). Stacey (2008) stated that it is crucial "that teachers should have strengthened their mathematics knowledge and skills beyond what is needed at the school level, that they should know more than their students, and that they should have some perspective on where school mathematics lead" (p. 93).

The mathematical content knowledge that teachers need to know is classified under two main subcategories: (1) common content knowledge; and (2) specialized content knowledge (Ball, Thames & Phelps, 2008; Hill, Schiling & Ball, 2004). Common content knowledge refers to the mathematical content knowledge and skills that not only teachers but other adults must know. Specialized content knowledge refers to mathematical content knowledge and skills that are specific to teaching. It includes not only knowing how and why, but the knowledge in identifying and analyzing student errors or assessing the workability of a nonstandard approach (Ball et al, 2008; Hill, Ball & Schilling, 2008).

Having sound mathematical content knowledge is so important that Cooney and Wiegel (2006, p. 806) recommended that prospective teachers should explicitly study and reflect on school mathematics. This was stated as one of the three key principles stipulated for prospective mathematics teachers (p. 806).

As the mathematics curriculum worldwide evolves in the twenty-first century, more are expected of school teachers. For example, the Australia National Curriculum Board (2008) states that teachers are expected to be able to "use a variety of mathematics task types including those that give students choice of approach and for which there is an optimal strategy…." (p. 14). All these expectations of teachers are feasible only if teachers are competent in mathematical content knowledge.

Research has shown that teachers' content knowledge is an important prerequisite for high-quality teaching, and that it contributes significantly to the quality of classroom instruction (Fauskanger, 2015; Hill & Charalambous, 2012). This is likely because teachers with strong mathematical content knowledge are able to accurately launch the curriculum tasks better, compared with teachers with lower mathematical content knowledge.

Mathematics Proficiency Test

This article reports an exploratory study of an entire cohort of prospective teachers' mathematical content knowledge based on their performance on a school mathematics proficiency test. A mathematics proficiency test is administered for every cohort of prospective teachers after they have been admitted to the teacher education program in

Singapore (Toh, Chua & Yap, 2007). The main objective of the test is to identify the prospective teachers' strengths, weaknesses, errors and misconceptions in school mathematics, so that remedial steps can be taken during the one-year teacher education program in order to help them improve on their secondary school mathematical content knowledge. The prospective teachers' errors and misconceptions in school mathematics knowledge as surfaced from their performance in the test, can serve as positive indicators of learning opportunities for them. The prospective teachers "must come to value and engage with their own errors and misconceptions" (McCrae, 2005/2006, p. 73). This section reports the content, format and design of the test items of the mathematics proficiency test. The prospective teachers were expected to complete the test within two hours. The items used in the test focus on teachers' common content knowledge (Ball et al, 2008; Hill, Ball & Schilling, 2008) as it does not require the prospective teachers to evaluate different types of students explanations of mathematical concepts.

Content and Format

The content of the test was based on the main strands of the Singapore mathematics syllabus: (a) Algebra and Arithmetic; (b) Geometry, Trigonometry and Measurement; (c) Functions and Graphs; (d) Probability and Statistics; (e) Higher Algebra; (f) Higher Geometry and Trigonometry; and (g) Calculus (Ministry of Education, 2007a; 2007b). The test consisted of sixteen questions requiring a mixture of both long and short answers.

Design of the Test Items

Extending the works of Ball et al (2008) and Hill et al (2008), Tchoshanov (2011) and Fauskanger (2015) classified the mathematical content knowledge under two categories: (1) knowledge of facts and procedures (*type 1 knowledge*); and (2) knowledge of concepts and connections (*type 2 knowledge*). Tchoshanov (2011) concluded that type 2 knowledge (concepts and connections) is a good predictor of teaching that could have a positive impact on students' achievement. Incidentally, many questions in the mathematics mastery test matched the items that assess pre-service teachers' type 2 knowledge.

About 50% of the questions were modeled after the secondary school national examination questions. The other questions were designed based on concepts that were identified to be important for teachers but pre-service teachers usually have difficulty grappling with these concepts. This latter category of questions assessed the type 2 knowledge identified by Tchoshanov (2011) described in the preceding paragraph. In this article, only questions that were designed to assess prospective teachers' type 2 knowledge involving mathematical concepts and connections are presented for discussion in this section.

The Number System (Algebra and Arithmetic)

Teachers must be familiar with the number system (Lee, 2008, pp. 10 – 12), hence Questions 1 and 2 (Table 1) were designed. The concept tested in 1(b) and 2(b) occurs as common errors and misconceptions among practicing classroom teachers (Toh, 2007). In order to be able to teach estimation and rounding numbers sensibly, and lead students to appreciate the practical uses of numbers (New Zealand Ministry of Education, 1992, pp. 48 - 50), teachers must be competent with concepts involving significant figures and standard forms. Thus, 2(a) was designed. The comments by the author for each part of the question are shown in Table 1 below. We define the *success rate* of the prospective teachers' attempt of each part of the question as $\frac{\text{Number of correct answers}}{\text{Total number of candidates}} \times 100\%$. The success rate of each part of the question is also shown in Table 1 and all the subsequent tables displaying the test items.

Table 1
Test items on the real number system

No.	Question	Key Concepts/ Comments	Success Rate (%)
1(b)	$0.\dot{4}12\dot{3}$ is irrational. True or False?	Recurring decimals represent rational numbers	58.6%
2(a)	How many significant figures can the number 564 000 possibly represent? Write down all the possible answers.	Concept: Significant figure. The number could possibly have 3, 4, 5 or 6 significant figures	54.0%
	Write the number 564 000 in the standard form to denote it as having 3 significant figures.	Concept: Standard form Candidate should express the number as 5.64×10^5 to represent 3 sig fig.	
2(b)	Is it possible to find a circle for which the circumference and diameters are exactly integers? Explain your answer.	Concept: π is irrational. Applying the irrationality of π, it is impossible to find such a circle.	82.9%

Teachers need to be familiar with the entire number system, including irrational numbers (Toh, 2006, p. 10). They must also be able to connect the irrational number π with the ratio of the circumference to the diameter of a circle. Question 2(b) was designed to assess them on this aspect.

Algebraic Manipulation and Equation (Algebra and Arithmetic)

The teaching of algebra in schools focuses on procedural manipulation at the expense of students' conceptual understanding (Yeap, 2008). To assess the prospective teachers' conceptual understanding of algebra, Questions 3 was designed (Table 2). Question 3(a) assesses the prospective teachers' ability to derive meaningful formula for compound interest, which was compounded once every six months. Question 3(b) involves application of routine factorization procedure followed by deducing the property of integers from the result of factorization.

Table 2
Test item on algebraic manipulation

No.	Question	Key Concepts/Comments	Success Rate (%)
3(a)	A bank offers compound interest at a rate of 3% per annum compounded once every six months. I deposit a sum of $ 5000 in this bank. After one year, how much will I have in the bank?	Concept: Compound interest Recognize that the interest is compounded once every six months, so twice per year. It is not the usual formula for compound interest compounded annually. Candidate may use calculator for computation throughout this paper.	53.0%
3(b)	Factorize $n^3 – n$ completely. Hence, explain why $n^3 – n$ is always divisible by 6 for all integers n.	Concept: factorization n(n-1)(n+1) . Candidates will need to identify that the factorization is a product of three consecutive integers, hence divisible by 6. Alternatively, candidate can split into several cases to check that $n^3 – n$ is both even and divisible by 3.	72.0%

The solution of some algebraic equations requires more than just standard procedural knowledge (Toh, 2006; 2007). These questions can be seen as higher order tasks involving problem solving, as mere procedural fluency is not sufficient to solve these algebraic equations (Toh, Quek & Tay, 2008a; 2008b). An algebraic equation of this type was designed in the test as Question 4 (see Table 3).

Table 3
Test item on solution of an algebraic equation

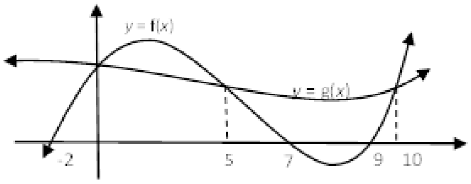
No.	Question	Key Concepts/Comments	Success Rate (%)
4	Solve the equation $\lvert x+y-3\rvert + \sqrt{2x-y} = 0$.	Concept: Algebraic equation Observe that since each of the two terms on the left hand side of the equation is non-negative, each term must attain the value of 0, i.e. it is equivalent to solving the simultaneous equations $x + y – 3 = 0$ and $2x – y = 0$.	50.0%

Graphic Heuristics in Solving Inequalities and Equations

Tsamir, Tirosh and Almog (1998), and Sackur (2004) reported that teachers generally do not apply graphic heuristics to solve inequality. They reported that teachers invariably performed algebraic transformations on inequalities without taking much care of the constraints of inequalities. It was also observed by Boero and Bazzini (2004) that inequalities are usually taught in a purely algorithmic manner that “trivializes” the subject.

Question 5 (see Table 4) was designed to assess the prospective teachers' ability to apply graphic heuristics to solve inequalities and equations.

Table 4
Test item on solution of inequality

No.	Question	Key Concepts/Comments	Success Rate (%)
5(a)	Write down the solution set of the inequality f(x) > g(x) where the graph of f and g are shown below. 	Concept: Graphical solution of inequality. The solution of the above inequality can directly be obtained by examining the graph: $0 < x < 5$ or $x > 10$.	91.0%
5(b)	Find the solution set of the equation $\|x\| = x$.	Concept: Graphical solution of equation. By interpreting the absolute value function, this equation is true for all $x > 0$.	80.0%

Definite Integral as Area under Graph

Calculus is usually taught in schools procedurally at the expense of conceptual understanding (Blum 2000; Anatoli, 2008). In particular, definite integrals are usually evaluated algorithmically using Fundamental Theorem of Calculus without much relational understanding (Toh, 2009). Thus, students may not associate definite integral with area under the graph (Anatoli, 2008; Orton, 1983; Sealey, 2006; Thomas & Hong, 1996). Most assessment items in national examinations on integration focus on techniques in evaluating or manipulating algebraically the anti-derivatives (Toh, 2009). Question 16(b) (see Table 5) was thus designed to assess the prospective teachers' ability to connect a definite integral with area under the graph.

Table 5
Test item on evaluating a definite integral.

No.	Question	Key Concepts/Comments	Success Rate (%)
16(b)	Evaluate the definite integral $\int_0^a \sqrt{a^2 - x^2}\,dx$ by using a graphical approach.	Concept: Definite integral as area under the graph. Recognize that the integral represents a quadrant of radius a. Hence, it has the value of $\frac{\pi a^2}{4}$.	82.1%

Results and Discussion

In tables 1 to 5, we have defined the *success rate* (SR) of each part of the questions using the formula $\frac{\text{Number of correct answers}}{\text{Total number of candidates}} \times 100\%,$ based on the performance of all the prospective teachers who participated in the test. Only those questions with the highest success rates (more than 80%) and the lowest success rates (less than 60%) are identified for discussion (Table 1). In this paper, we take the items with the highest success rate as indicators of the types of questions which the prospective teachers have the highest fluency, and the lowest success rate as the types of questions which they demonstrated a lack of proficiency. Except for Question 16(b) (which has a success rate of more than 80%), which is on calculus, interestingly all the other questions which had the highest and the lowest success rates were from the Algebra and Arithmetic strand. A summary table is shown as Table 6.

Table 6
Summary statistics of the performance in the questions with the highest and the lowest Success Rates (SR) (N = 152)

Question Numbers	**Topic**	**Number of correct Responses**	**Highest SR (%)**	**Lowest SR (%)**
1(b)	Number system	89		58.6
2(a)	Number system	50		54.0
2(b)	Number system	131	82.9	
3(a)	Algebraic manipulation	78		53.0
4	Algebraic equation	78		50.0
5(a)	Algebraic inequality	117	91.0	
5(b)	Algebraic equation	107	80.0	
6(a)	Algebraic manipulation	115	85.0	
16(b)	Definite integral	112	82.1	

Prospective Teachers' Strengths in Mathematical content Knowledge

Graphic Heuristics (Table 4)

Contrary to the findings of Tsamir, Tirosh and Almog (1998), the Singapore prospective teachers were able to use graphic heuristics to solve the inequality of 5(a) (SR = 91.0% in Table 6). The prospective teachers' ability to use graphic heuristics is further shown in solving the equation of 5(b) (SR = 80.0%). We believe that the high success rate of this question could be attributed to the fact that the prospective teachers had been exposed to the use of graphing calculators at the pre-university level; hence they were able to solve an inequality both algebraically and graphically.

Irrationality of π and Property of Circle (Table 1)

The prospective teachers applied the knowledge that π is an irrational number to consider the ratio of the circumference to the diameter of the circle with high accuracy in Question 2(b) (SR = 82.9% in Table 6). This suggests that the much feared misconception among mathematics educators (Toh, 2007) of erroneously regarding $\pi = \frac{22}{7}$ is not founded among our prospective teachers. Furthermore, they were able to connect the number π to the ratio of the circumference, to the diameter of a circle.

Simplification of Algebraic Fractions

The prospective teachers were able to solve Question 6(a) (See Figure 1) accurately (SR = 85.0%, in Table 6). This is consistent with anecdotal evidence from Singapore classrooms that teachers and students are highly competent and efficient in applying algorithmic procedures in mathematics.

Simplify the expression $\frac{1}{x-1} + \frac{1}{x^2 + x - 2}$, leaving your answer in a form as simple as possible.

Figure 1. Test item Question 6(a) on algorithmic algebraic manipulation

Knowledge of the Definite Integral (Table 5)

Question 16(b) (SR = 82.1%, Table 6) shows that the prospective teachers were generally able to identify the definite integral as representing the area of the quadrant of a circle. Although school students do not usually acquire appropriate comprehension of the definite integral concept as representing the area under the graph (Anatoli, 2008; Orton, 1983; Sealey, 2006; Thomas & Hong, 1996), most prospective teachers in this study apparently did not have the difficulty with interpreting definite integrals.

Prospective Teachers' Errors and Misconceptions Solution of Algebraic Equation by Interpretation (Table 3)

There were 84 incorrect answers for Question 4, which included 61 incorrect responses using an algebraic approach (SR = 50.0%, Table 6).

Solve the equation $|x + y - 3| + \sqrt{2x - y} = 0$.

Figure 2. Test item Question 4 on solving algebraic equation

A total of 61 prospective teachers attempted to solve the equation procedurally by algebraic means without obtaining the correct solution. There were another eight responses from prospective teachers who attempted to apply graphic heuristics to obtain the solution, but did not formulate the problem into one that can be solved graphically. Generally, the prospective teachers who attempted this question tried to rearrange the equation into the form $|x+y-3| = -\sqrt{2x-y}$ and then squared both sides of the equation. However, they did not proceed further as the equation became too complicated for simplification. In addition, 15 prospective teachers did not respond to this question at all.

The prospective teachers had been exposed to mathematics problem solving and Polya's model of problem solving (Polya, 1945). Their responses to Question 4 (Figure 2) show little evidence of applying "problem solving heuristics" in solving a non-routine problem, or of exercising “control” (Schoenfeld, 1985; 1992), or of switching to another method at the dead end when a particular method failed. Teaching prospective teachers about problem solving is insufficient; the prospective teachers must experience mathematical investigations and problem solving as processes during their teacher education program (Stacey, 2008). We believe that Question 4 (Figure 2) is an example of such a question which is useful for problem solving. Question 3(a) on compound interest is another problem that is suitable for problem solving (since the use of the standard formula for compound interest is inadequate and would yield an incorrect answer).

Compound Interest (Table 2)

There were 74 incorrect responses for Question 3(a) (SR = 53.0%, Table 4).

A bank offers compound interest at a rate of 3% per annum compounded once every six months. I deposit a sum of \$ 5000 in this bank. After one year, how much will I have in the bank?

Figure 3. Test item Question 3(a) on compound interest

The errors made by the prospective teachers can be classified under three categories: (1) 49 candidates used the formulae for interest compounded annually; (2) 19 candidates had correct consideration of interest compounded twice annually, but did not divide the rate per annum (for compound interest) by two; and (3) 6 candidates used the correct compound interest rate but erroneously wrote 1.5% as 0.15 in computation. Case (3) was likely a slip on the part of the prospective teachers during the test.

Students do not normally examine the meaning of the formulae they apply. According to Harel (2001), students do not usually treat the mathematics symbols and notations as entities in a coherent reality; rather, they always perform operations on symbols “as if they possess a life of their own” (Harel, 2001). It seems that many prospective teachers in this proficiency test also fell short of examining the formulae they applied – exhibiting the same behavior of students handling algebraic entities.

Significance of the Digit Zero (Table 1)

How many significant figures can the number 564 000 possibly represent? Write down all the possible answers.
Write the number 564 000 in the standard form to denote it as having 3 significant figures.

Figure 4. Test item Question 2(a) on Significant figures

This question has one of the lowest SR (SR = 54.0%) in which 102 candidates indicated that 564 000 in Question 2(a) has six significant figures, without considering other possibilities of the significance. 32 candidates did not give the correct response of 5.64×10^5 as the standard form to denote the unambiguous way of representing it as having three significant figures. Anecdotal evidence from the Singapore mathematics classrooms suggest, that many students had difficulty with significant figures and how to identify the number of significant numbers of a number represented in the standard scientific notation. Most probably they did not have deep understanding of the basic structure of numbers, although they have learnt much algorithmic manipulation.

It is highly probable that significant figures and scientific notation of numbers are usually not presented in meaningful contexts in mathematics classrooms. Anecdotal evidence in the mathematics classrooms shows that other than the procedural knowledge of determining the number of significant figures of a number or expressing a number in the standard form, students generally do not have a deep understanding of the structures of numbers, especially in relation to real-world applications. Here we observe a similar situation with the prospective teachers in their understanding of significant figures. However, there is insufficient evidence in the existing data of the prospective teachers' answer scripts that could give a definite explanation of this.

Decimal Representation of Rational and Irrational Numbers (Table 1).

Question 1(b) had 63 incorrect responses (SR = 58.6%, Table 4).

$0.\dot{4}12\dot{3}$ is irrational. True or False?

Figure 5. Test item Question 1(b) on rationality of a number

The errors made by the prospective teachers can be classified under three main categories: (1) misconception that a recurring decimal is not rational; or a rational number has finite number of digits in its decimal representation; (2) misconstrued relation with other groups of numbers such as real number and transcendental number; and (3) correct concept of rational number with recurring decimal representation but inability to convert a recurring decimal to fractional form. Many of the prospective teachers had difficulty with identifying the decimal representation of rational and irrational numbers.

Two Other Questions in the Test

The prospective teachers' performance in two other questions of the mathematics proficiency test is reported in this section: Questions 3(b) (Figure 6) and 1(a) (Figure 7).

Interpretation of the Result of Algebraic Manipulation

Factorize $n^3 – n$ completely. Hence, explain why $n^3 – n$ is always divisible by 6 for all integers n.

Figure 6. Test item Question 3(b) on factorization and its interpretation.

The factorization of the expression $n^3 – n$ into $n (n – 1) (n + 1)$ is not difficult for most of the candidates. In this question, they needed to recognize that the factorized product is the product of three consecutive integers, hence it must always be even and divisible by 3, hence divisible by 6. There are many ways to present this second part of the argument – for example, by considering different cases (consider, for example, cases when n is even and when n is odd); mathematical induction, or even number theory arguments.

In this question, 151 candidates factorized $n^3 – n$ correctly, a clear indication of the teachers' procedural fluency. 97 candidates did not correctly explain why the algebraic term $n^3 – n$ is divisible by 6. This latter part of Question 3(b) entails the prospective teachers' ability to associate the algebraic expressions as representing generalized numbers

– which is difficult for most students and even teachers. The presentation of the solution to this part of the question requires sophistication in presenting a logical argument, which is very much dealt with in the pre-university or university mathematics courses. A prospective teacher's presentation of the solution to this part of the question is a reflection of his or her mastery of mathematical reasoning. Thus, the categories of incorrect responses are analyzed in greater detail in Table 7 compared to other questions.

Table 7
Breakdown of the incorrect response to Question 3(b)

No.	**Description of Incorrect Responses**	**No. of responses**
1	Checked that for $n = 2, 3, 4$ etc. that $n^3 - n$ is divisible by 6 and made a general conclusion that $n^3 - n$ is divisible by 6 for all integers n.	22
2	Checked that for some groups of three numbers {1, 2, 3}, {3, 4, 5} etc. that their product is divisible by 6, and made a general conclusion here.	2
3	Concluded by observation.	1
4	Considered $n = 2m$ and $n = 2m + 1$; and $n = 3m$, $n = 3m + 1$ and $n = 3m + 2$ but failed to get correct conclusion that for all cases the expression $n^3 - n$ is divisible by both 2 and 3.	13
5	Considered $n = 6k$, $n = 6k + 1$,……, $n = 6k + 5$ but failed to get conclusion that it is divisible by 6.	5
6	No responses	41
7	Other incorrect responses which had not mathematical basis	13

Studies have shown that teachers have difficulties with mathematics proofs (Larsen & Zandiah, 2008). In fact, many fully qualified mathematics teachers have little understanding of mathematics proofs (Stacey, 2008). The prospective teachers' performance in Question 3(b) (Figure 6) shows they generally lacked a deep understanding of mathematics proofs. The error items 1 to 3 show the prospective teachers' mistaken belief of inductive reasoning as acceptable mathematical proofs. In error items 4 and 5 in Table 7, the prospective teachers attempted to apply their knowledge of university mathematics (number theory) but lacked the mastery of mathematics proofs associated with number theory.

Teachers who have mastered mathematics proofs are able to give their students rich mathematical experiences (Stylianides, Stylianides & Philippou, 2007). However, in reality, many might not have mastered mathematics proofs, though they may have knowledge of university mathematics. The university courses have apparently equipped the prospective teachers with knowledge or "resources" (Schoenfeld, 1985, 1992) of higher level mathematics, but probably had not helped them understand secondary school mathematics. Incorrect responses items 1 to 5 in Table 7 are good examples of a lack of mastery of higher level mathematics, but which also provide evidence that these prospective teachers had been exposed to these higher level mathematical arguments and proofs.

The Radical Symbol ($\sqrt{\ }$).

$\sqrt{4} = \pm 2.$ True or False?

Figure 7. Item Question 1(a) on Radicals

In Question 1(a), 41 candidates agreed that $\sqrt{4} = \pm 2.$ In the Singapore mathematics curriculum, the radical symbol $\sqrt{\ }$ denotes the positive square root *function*. The prospective teachers had either forgotten this convention of the school mathematics curriculum, or could possibly have suffered interference from their exposure to some undergraduate mathematics, where the radical symbol could mean any of the square roots. This question could serve as a timely reminder to the prospective teachers regarding the convention of the Singapore mathematics curriculum, and also for the teacher educators to bring this point up during the teacher education program.

The prospective teachers' errors for their learning

The prospective teachers who had not cleared the mathematics proficiency test (with 75% of the total score) were required to attend a series of remedial lectures on school mathematical content knowing. They were encouraged to re-learn the key mathematics concepts rather than be item-specific or to learn only from their mistakes. It is a university policy that the original answer scripts should not be returned or revealed to the students after any university test or examination. Although the original test paper was not returned to them, a series of lectures was conducted by the tutor-in-charge for the prospective teachers to discuss the general misconceptions of the prospective teachers' mistakes in the test in general

Limitations

This article reports the performance in a mathematics proficiency test for an entire cohort of prospective secondary mathematics teachers in Singapore. As the prospective teachers completed the items within a typical university test setting, the author was not able to determine with certainty whether they had understood the concepts. Due to the regulation of the university, interview or further interaction with the prospective teachers about their performance in specific test items was strictly prohibited. The prospective teachers' ability to answer a question is not necessarily synonymous to their understanding of the related concepts, as it measured only their *performance* and not *competence*. Thus, we present in our article as reporting the *performance* of the prospective teachers in the various types of questions. Hopefully, this article will spur further interest in conducting research to identify prospective teachers' understanding of various school mathematics concepts.

Conclusion

We have identified the areas which Singapore prospective teachers performed well (highly efficient in algebraic manipulation, competent use of graphic heuristics to solve algebraic inequalities, ability to interpret definite integral as area under graphs) in comparison with studies worldwide (Toh, Kaur and Koay, 2013). The areas of school mathematics which the prospective teachers had not performed well (decimal representation of rational and irrational numbers, significant figures, meaning of the radical symbol) are also identified in this study. The information identified in this study is used both for the personal development of the prospective teachers and collective treatment during pre-service teacher education and in-service courses.

Traditionally, mathematics educators might have assumed in designing and conducting pedagogical courses in teacher education programs that the prospective teachers admitted into the program, are equipped with the essential mathematics content knowledge for teaching. This preliminary study challenges this assumption. In designing mathematics pedagogical courses, from the experience we have in Singapore, it is crucial to examine how school mathematics content might be infused into these courses in order to ensure a more complete professional development of our mathematics prospective teachers.

We do envision that the next step forward would be to establish the essential mathematical content knowledge that prospective teachers for secondary school mathematics must acquire. However, to define this knowledge is not easy and will be a challenge for any prospective teacher education system (Linsell & Anakin, 2012).

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