Pre-Service Elementary Teachers' Proof and Counterexample Conceptions

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Abstract

This study aimed to document pre-service elementary teachers' (PSTs'), who are going to teach K-5, ability to construct as well as evaluate justifications varied in terms of levels of sophistication. Additionally, this study aimed to investigate how PSTs' knowledge of content in which they were being asked to prove influenced their ability to construct and evaluate mathematical justifications. Participants were a sample of PSTs who enrolled in one section of a geometry and measurement course and one section of a mathematics methods course at a large Mid-western University in the U.S. The participants were selected based on their responses to a written questionnaire, which consists of various open-ended proof questions. Then,task-based interviews were conducted at the beginning as well as near the end of the semester in order to detect whether there were any differences in PSTs' conceptions of proof as a result of taking these courses. The results of this study showed differences in what constitutes as proof in the eyes of the PSTs who participated in this study. Furthermore, this study also demonstrated that the restriction of PSTs' construction and evaluation of proofs/counterexamples was indeed a result of their limited conceptions of the concepts in which they were asked to prove.

Keywords: counterexamples, pre-service elementary teachers, proof, reasoning, refutations.

Introduction

Proof is a mathematical method of convincing one self and others that an idea is absolutely true for all cases in the domain of the statement (Switala, 2013). Proof plays a fundamental role in mathematical inquiry (Lakatos, 1976) and should be considered as a vehicle to enhance students' understanding of mathematical concepts (Hanna, 2000). So, it should be at the core of doing and understanding mathematics and be appreciated as an important component of students' mathematical education (NCTM, 2000)

Traditionally, the concept of proof has been associated with secondary mathematics courses on Euclidean geometry (Dreyfus, 1999). Given that proof is an essential aspect of mathematics and should be at the core of mathematics classrooms, students are deprived of experience with an essential element of mathematical thinking and sense making until high school (Stylianides & Stylianides, 2009). Harel and Sowder (1998) also argue against delaying students' experiences with proof until the secondary-school geometry course. Dreyfus (1999) points out that it is unreasonable to expect students to develop an appreciation for proofs if the study of proofs starts later in students' mathematics education.

To rectify this, both the National Council of Teachers of Mathematics (NCTM) Principles and Standards (2000) and the National Governors Association Center for Best Practices & Council of Chief State School Officers (NGA/CCSSO 2010) recommend that all students learn to construct and evaluate mathematical arguments and to demonstrate a variety of reasoning and proof techniques at all levels from kindergarten through grade 12. More notably, these documents, as well as several studies, advocate incorporating reasoning and proving across the curriculum not as a distinct mathematical topic, but rather as a way of thinking (e.g. Coe & Ruthven, 1994; Knuth, 2002c; Stylianides & Ball, 2008).Therefore, there is growing appreciation of the importance of making proof central to students' mathematical education as early as the elementary grades.

Making proof central to students' experiences as early as elementary grades places significant demands on elementary teachers' knowledge about proof (Knuth, Choppin, & Bieda, 2009; Stylianides & Ball, 2008). Martin and Harel (1989) state that if teachers lead their students to believe that a few well-chosen examples constitute a proof, it is natural to expect that the idea of proof in high school geometry and other courses will be difficult for the students. Furthermore, if an elementary teacher promotes (or tolerates) a conception of proof as an empirical argument, it instills inaccurate mental habits in students, which will be difficult to change later (Stylianides & Stylianides, 2009).Given that teachers' ability to teach mathematics depends on the quality of their subjectmatterknowledge (Hill, Rowan, & Ball, 2005), in order for the aforementioned goal to succeed, elementary teachers should have a solid understanding of proof (Stylianides & Ball, 2008).

Despite the importance of elementary teachers' understanding of proof, relatively little research has been conducted to investigate pre-service elementary teachers', who are just a few steps away from being a teacher, conceptions of proof (Goetting, 1995; Martin & Harel, 1989;Simon & Blume, 1996; Stylianides & Stylianides, 2009). Stylianides and Stylianides (2009) criticizestudies that investigated PSTs' conceptions of proof focused solely on whether or not teachers can distinguish proofs and empirical arguments. Ko (2010) argued that given the importance of a teachers role in the classroom, more research should be done to investigate what factors influence teachers' conceptions of proof.

This study aims to contribute to literature by providing a detailed account of pre-service elementary teachers' (PSTs'), who are going to be licensed to teach K-5 grade level mathematics, understanding of proofs and refutations. This study also aims to contribute to the literature by investigating how PSTs'knowledge of concepts in which they were being asked to prove influences their conceptions of proofs.

Particularly, this study investigates the following questions:

1. What are pre-service elementary teachers' conceptions and/or misconceptions of proof and counterexamples in mathematics classrooms?
2. How does pre-service elementary teachers' knowledge of content in which they are being asked to prove affect their construction as well as evaluation of proofs and counterexamples?

Teachers' Difficulties with Proofs and Refutations

Both high school and college students'difficulty with proof has been well established in the literature (e.g., Balacheff, 1991; Chazan, 1993; Healy &Hoyles, 2000; Knuth et al., 2002; Lin, Yang & Chen, 2004; Moore, 1994; Selden & Selden, 2003; Senk, 1985; Weber, 2001). The majority of students does not appear to recognize the indeterminacy of inductive conclusions and struggle to construct valid mathematical proof (Chazan, 1993; Knuth, Chopin, &Bieda, 2009). Teachers have a responsibility to help students with all of these difficulties, but research (e.g Martin & Harel, 1989; Morris, 2002; Stylianides, Stylianides & Philippou, 2004, 2007; Varghese, 2007)has shown that many teachers exhibit some of the same difficulties as their students.

Knuth (2002b) found that like students, some teachers misunderstood the generality of a completed proof. Martin & Harel (1989) examined the conceptions of proof held by 101 pre-service elementary teachers enrolled in a mathematics course focused on proofs. They found that more than half of the participants accepted either an inductive argument or a false deductive argument as a valid mathematical proof. Morris (2002) confirmed Martin and Harel's (1989) results by finding that pre-service elementary teachers failed to distinguish between inductive and deductive arguments. Goetting (1995) reported similar findings. Stylianidesand Stylianides (2009) focus on a group of 39 prospective elementary teachers who had rich experiences with proof, and examined their ability not only to construct proofs but also to evaluate their own constructions. Even though they stated that focusing on construction-evaluation tasks better illuminate prospective teachers' conceptions, they still documented similar limited understanding of proofs among prospective teachers.

Barkai et al. (2002) reported similar findings for 27 in-service elementary school teachers who enrolled in a 3-year professional development program focused on mathematical topics. They reported that more than half of participants were able to state correctly the true or false condition of all the propositions. However, many experienced teachers claimed that their explanations with supportive examples were mathematical proofs.

Simon and Blume (1996) demonstrated evidence that pre-service elementary teachers also hold a misconception related to providing counterexamples. They documented that PSTs believed providing more than one counterexample was required to refute a statement. In this study, naive understanding of proofs and counterexamples were actually considered as preconceptions as opposed to misconceptions and conceived as essential components to understand PSTs' approaches to mathematical proofs. These preconceptions of proofs were classified along several dimensions, which will be described in detail next.

Proof Levels

A fruitful approach to understanding students' difficulties with proof has been to classify these approaches along several dimensions (Balacheff, 1988; Harel & Sowder, 1998). While many studies (e.g. Chazan, 1993; Martin &Harel, 1989) have focused primarily on distinctions between inductive and deductive justifications, some researchers (e.g. Balacheff, 1988; Harel & Sowder, 1998; Simon & Blume, 1996) have divided inductive and deductive justifications into further subcategories. The same approach was followed in this study.

The taxonomy of proof schemes that was proposed by Harel and Sowder (1998) and later revisited by Harel (2007) is a fundamental framework for research on students' conceptions of proof. According to Harel and Sowder students were found to hold several types of schemes: (1) External Proof Schemes, (2) Empirical Proof Scheme, and (3) Deductive Proof Scheme. External proof schemes employed reasoning processes based on external sources such as knowledgeable authorities, sources or format of the argument (e.g. Harel & Sowder, 1998; Simon & Blume, 1996; Quinn, 2009). These were irrelevant to the inductive and deductive reasoning in proof construction as no mathematical examples nor was logical reasoning considered. Empirical proof schemes employed using relevant mathematical examples, which might be selected randomly (naïve empiricism) or by purposeful selection (crucial empiricism) (e.g. Balacheff, 1988; Simon &Blume, 1996). Deductive proof schemes consisted of transformational proof schemes, which employed purposeful mental operations or axiomatic schemes, which employed deductive based reasoning (e.g. Harel & Sowder, 1998; Simon & Blume, 1996).

However, it is evidenced in the literature that some students may fail to produce a deductive argument even if they start with some deductions (Quinn, 2009). At the naïve reasoning level (see Level 2 in table 1) students' engagement in constructing proofs often is irrelevant and requires minimal inferences (Quinn,2009). As opposed to accepting an argument as a proof solely because it is coming from an external source (i.e. the textbook, teacher or the format of the argument); students at this level try to reproduce a logical argument but fail to proceed. Lee (2016) coded this level as irrelevant engagement in inferences since students' attempted to relate the antecedents and the consequent fail. Additionally it is evidenced in the previous studies that some students may use a particular example – a generic example, to express their deductive reasoning (e.g. Balacheff, 1988; Simon & Blume, 1996). As Lee (2016) describes, the conviction at this level (see Level 4 in table 1) lies at using mathematical properties inferred from generated examples to arrive at general conclusions. Finally, students may produce an argument by primarily using deductive reasoning (see Level 5in table 1), which almost can be marked as a valid proof. However, the argument either has a missing step in making a chain of logical reasoning (Lin, 2005)orsome types of flaws in reasoning which may result in the omission of possible counterexamples (Sandefur et al. 2013).

Since these students do not hold external, empirical, nor fully developed deductive proof schemes, I believe adding these levels (Level 2, Level 4, and Level 5)in order to account for a broader spectrum of proof schemes is important. In this study, these levels were used to analyze PSTs' conceptions of proof.

Table 1

Proof Levels

Categories	Characteristics of Categories	
	Subcategories	Characteristics of Subcategories
Level 1: External Reasoning	Responses appeal to external authority	
	(1) Authoritarian proof	Depends on an authority
	(2) Ritual proof	Depends on the appearance of the argument
	(3) Non-referential symbolic proof	Depends on some symbolic manipulation
Level 2: Naive Reasoning	Responses may appeal to the use of some deduction, something that provers remember hearing often incorrectly, but ends with incorrect conclusion or correct conclusion with incorrect reasons.	
Level 3: Empirical Reasoning	Responses appeal to empirical demonstrations, or rudimentary transformational frame	
	(1) Naïve Empiricism	An assertion is valid from a small number of cases
	(2) Crucial Empiricism	An assertion is valid from strategically chosen cases of examples
Level 4: Reasoning Based on Particular Instances	Responses appeal to using properties inferred from a particular instance	
Level 5: Incomplete Deductive Reasoning	Responses appeal to a chain of inferences that is almost entirely deductive or complete. However, few inferences used in the justification may have some non-deductive reasoning, and may require further justifications or have minor errors.	
Level 6: Deductive Reasoning	Responses appeal to rigorous and logical reasoning	
	(1) Transformational proof scheme	Involves goal-oriented operations on objects
	(2) Axiomatic proof scheme	Involves statements that do not require justification

Counterexample Levels

Potari, Zachariades, and Zaslavsky (2009) argue that refuting conjectures and justifying invalid claims is a complex process that goes beyond the syntactic derivations of deductive proof.Several researchers distinguished three types of counterexamples based on their explanatory power (e.g. Bills et al., 2006; Peled & Zaslavsky, 1997; Selden & Selden, 1998). Similar to the distinction between "proof that proves" and "proofs that explains" (Hanna, 2000),counterexamples could be distinguished as follows: (1) a proof by counterexample that shows only that a proposition is false, (2) a proof by semi general counterexample that provides some ideas about why a proposition is false or how the counterexample contradicts the claim, but does not tell "the whole story" and (3) a proof by a general counterexample provides insight into why a proposition is false and suggests a way to generate not only one counterexample, but an entire counterexample space. These three levels were used to code PSTs' constructions of counterexamples.

Table 2
Counterexample Levels

Categories	Characteristics of Categories
Level 1: Irrelevant Counterexample	Responses appeal to the use of counterexamples to falsify the statement. However, the example(s) provided would be irrelevant to deduce the falsity of the statement.
Level 2: Specific Counterexample	Responses appeal to the use of counterexample(s) that show the falsity of the statement
Level 3: Semi-general Counterexample	Responses appeal to the use of counterexample(s) that show the falsity of the statement and some ideas for why the statement is false
Level 4: General Counterexample	Responses appeal to generating whole space of counterexamples

Method

This study focused on investigating the process of PSTs' determination of the truth or falsity of a statement, producing a proof or a counterexample, evaluating various arguments that span from informal to formal, and verification of given arguments as valid proofs or appropriate counterexamples. Additionally, this study also focused on investigating how PSTs'prerequisite knowledge about underlying concepts of the statements, which they were being asked to prove or refute, influenced their construction-evaluation process. Thus, this study uses a qualitative approach, mainly participants' responses to a written questionnaire, task-based interviews, and their class work analysis including homework, midterm, or final exam responses, which will be described next.

Participants

To select participants representing a broad spectrum in terms of knowledge and beliefs about proof, a proof questionnaire with open-ended questions was developed and administered to all students in one section of a geometry and measurement course and one section of a mathematics methods course at the beginning of the semester.

After administering the questionnaire, a sample of five PSTs from the geometry and measurement course and a sample of seven PSTs from the mathematics methods course were selected. Participants were selected based on their responses in order to assure:

1- To select participants who were anticipated to display various proof schemes
2- To select participants who demonstrated strong as well as limited knowledge of contents— particularly in the concepts of quadrilaterals and area and perimeter.

Data Sources

Questionnaire. A written questionnaire with sixteen open-ended questions was developed to administer to one section of geometry and measurement course and one section of a mathematics methods course to gain background information with respect to PSTs' conceptions of proof and counterexamples, as well as their content knowledge. The questions for the questionnaire were designed to assess pre-service teachers' abilities to prove/refute mathematical statements, to evaluate the correctness of presented arguments (one incorrect and one correct) and counterexamples, and to assess PSTs' knowledge of content, in particular in the concepts of quadrilaterals and area-perimeter. The questions were adopted and modified from existing literature (Knuth, 2002a; Kotelawala, 2009) and from high school geometry textbooks (see Appendix A for sample questionnaire questions).

Task-based interviews. Each PST was interviewed individually for about an hour in a semi-structured manner using an interview script consisting of the two phases described below at the beginning as well as near the end of the semester in order to detect any changes in PSTs' conceptions of proof and counterexamples, as well as their knowledge of contents during the semester.

Phase 1. During this phase of the interview, each PST was presented with the written tasks A, B, C, D and E described in the following section. They were asked to decide whether the statements were true or false. And then, they were asked to justify in cases where they believed the statements to be true or to refute in cases where they believed the statements to be false. Then, they were asked to evaluate their own constructions whether it could be counted as a valid proof/refutation.

Phase 2. After letting the PSTs try to justify the statements in task A,B and C by themselves first, they were presented with four brief arguments (five arguments for the post-task based interviews[1]), varying in terms of levels of justification, one after the other, and asked to judge the correctness, and say to what extent each argument was convincing. After letting PSTs try to develop a method to calculate the area of the shape in task D, PSTs were presented with a method and asked to justify or refute whether the method would work to calculate the area of the shape. No argument or method was presented after task E.

Interview Tasks

Task A	**Task B**
A kite is a quadrilateral with two distinct pairs of adjacent sides that are equal in length. Given the definition, justify whether or not the following statement is true. "*In a kite, one pair of opposite angles is the same*."	Justify whether or not the statement is true: "In a triangle, the sum of the interior angles is 180 degrees." This task was adopted from Knuth (2002b).
Task C Justify whether or not the statement is true: "In any triangle, a segment joining the midpoints of any two sides will be parallel to the third side." This task was adopted from Chazan (1993).	**Task D** Find the area of the figure below.  This task was adopted from Simon and Blume (1996).
Task E "At least one diagonal cuts the area of a quadrilateral in half."Prove whether the statement is true or false. This task was adopted from Ball, Hoyles, Jahnke, and Movshovitz-Hadar (2002).	

Figure 1. Interview tasks.

In this paper, PSTs' responses to Task A and Task D will be used to demonstrate characteristics of proof and counterexample levels.

Arguments for interview tasks. The proof levels (see Table 1) governed the choice of arguments included in tasks A, B, and C. Arguments for those tasks were characterized as empirical, and subdivided as naïve empiricist with a small number of cases (Argument 1) and crucial empiricism with an extended number of cases including non-particular cases (Argument 2), argument requiring concrete demonstration with a generalizable explanation (Argument 3), and a deductive proof, written in a formal style (Argument 4). In order to explore the ritualistic aspect of proof, during post-interviews an

[1] An incorrect justification which was written in two-column proof format was included for tasks A, B, and C during the post interviews in order to further investigate how ritualistic aspect of arguments affected participants' evaluation of proof.

incorrect argument— Argument 4-B—that was written in a two-column proof format was also included (see Appendix B for presented arguments for Task A).

For task D, the following method was presented to the participants:

Justify whether or not the method will work to find the area of the figure: "*If you take a piece of string and measure the whole outside of the area and then pull that into a shape like a rectangle, you can easily calculate the area of the figure*."

*Figure 2.*Presented method for task D.

Data Analysis

The data analysis was grounded in an analytical inductive method in which PSTs' responses were coded with external and internal codes. Each interview was transcribed, and the interview transcripts and participants' written responses were carefully read, initial impressions summarized, and interesting issues regarding the participants' proof and counterexamples validations highlighted, statement verifications, and proof and counterexample productions as well as evaluations. Coding of the data began with a set of external codes that were derived from the proof and counterexample levels (see Table 1 and Table 2). The levels proposed above were used as a priori coding scheme. Such external coding schemes provided a lens with which to examine the data.

By examining the data and reviewing the transcripts again, emerging themes of PSTs' comments on proof and counterexample validations, statement verifications, and proof and counterexample productions were developed. An inductive, grounded-theory approach (Strauss & Corbin, 1990) guided the analysis of the data. After proposing these internal (data-grounded) codes, each transcribed interview was reexamined and recorded to incorporate these new codes.

Second coders, who were very familiar with the literature that addresses mathematical reasoning and proof, were asked to read and code a sample of the interview transcripts in order to ensure the reliability of the coding process. The coded samples were then compared and disagreements were discussed until the problems were resolved.

Results

Participants from the geometry and measurement course - Sara, Scott, Kelly, Elizabeth, and Chloe (all pseudonyms), and the participants from the mathematics methods course - Mary, Daisy, Laura, Casey, Jack, Miranda, and Elaine (all pseudonyms), demonstrated a wide range of thinking and reasoning abilities regarding mathematical proofs and refutations in the questionnaire and during the individual interviews. Table 3 below summarizes the results of task-based interviews. Each participant's proof levels based on the similarities and differences in terms of the reasoning skills that they demonstrated during the individual interviews will be described next.

Table 3
Pre-service Teachers' Proof Levels

Categories		Number of Participants at This Category			
		Pre-Interviews		Post-Interviews	
		Geometry & Measurement	Mathematics Methods	Geometry & Measurement	Mathematics Methods
External Proof Scheme		-	-	Elizabeth	-
Naïve Reasoning		-	Elaine Laura Daisy	-	Elaine Laura
Empirical Reasoning	Naïve Empiricism	Elizabeth Chloe Sara	Casey Laura Daisy	Sara	Casey Laura Daisy
	Crucial Empiricism	Kelly			
Incomplete Deductive Reasoning				Chloe Kelly	
Deductive Reasoning		Scott	Jake Miranda Mary	Scott	Jake Miranda Mary

Pre-service Teachers' Proof Constructions-Evaluations

External reasoning. Elizabeth, from the geometry and measurement class, demonstrated empirical reasoning during the pre-interviews; however, her reasoning better aligned with external proof scheme during the post-interview. When presented a deductive argument, she argued that it was a valid proof. However, the reason that she classified the argument as proof was solely based on the format of the argument as can be seen in the following excerpt.

> Elizabeth: Ok, I mean this (argument 4 in fig.3) is a really good example of a proof and this is convincing.
> Interviewer: How do you decide that this argument (argument 4) is a proof?
> Elizabeth: Well because we learned in class that this is how you are supposed to

write a proof.

In the following excerpt, Elizabeth classified argument 4-B (see argument 4-B in fig. 3) for task A as a valid proof solely based on the format of the argument.

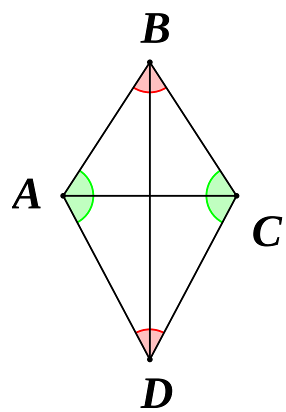
> Elizabeth: I mean it's the same thing; the only thing that's different is that this one (argument 4-B) says that it's congruent by side-angle-side and this one (argument 4) says by side-side-side.
> Interviewer: Does it make any difference?
> Elizabeth: I guess it doesn't really make a difference. I don't really get why they say angle B is congruent to D given by definition. We don't know that yet. It is what the question is asking. But, I think these two arguments (argument 4 and argument 4-B) are basically the same.
> Interviewer: Would you consider argument 4-B as a proof too?
> Elizabeth: Yes.
> Interviewer: And where would you put argument four B in your order?
> Elizabeth: Just in the same as argument four.

In several studies it was noted that students think a proof must be procedural and presented in a particular format, which consists of a set of steps (Healy &Hoyles, 2000; Knuth, 2002b). The results of this study align with those studies. It was evident in the data that Elizabeth perceived proof to be procedural, which apparently consisted of a set of steps, and she used this perceived image of how proofs should look like as the primary source of her reasoning while evaluating presented arguments. Elizabeth recognized the flaw in the reasoning in argument 4-B by saying "*…I don't really get why they say angle B is congruent to D given by definition. We don't know that yet!"* However, she continued to argue that it could be accepted as a valid proof solely because of the format of the argument. Another proof level in which the participants placed an importance on the format of an argument is naïve reasoning, which will be discussed next.

Naïve reasoning. Three participants - Laura, Elaine, and Daisy, showed reasoning that aligned with characteristics of naïve reasoning during the individual interviews. All of these participants placed an important role on the format of the argument and attempted to reproduce the arguments that they remembered (often incorrectly) learning from their previous classes. Even though they all attempted construct logical argument, their engagements in inferences were irrelevant since their attempts to relate the antecedents and the consequent failed.

> Interviewer: Ok, how can you justify that the statement is true? How can you convince someone that it is true?
> Laura: Um, I would do the angle-side-angle, no, trying to think. I would do angle-angle-angle or side-angle-side (emphasized). There you go! I would start out doing that the two sides are equal, like side A is equal to side B and side C is equal to side D. Those would be the two steps. And then, angle A is equal to angle C and then, angle B is equal to angle D and it is proven by the side-angle-side.

$\overline{AB} \cong \overline{BC}$
$\overline{AD} \cong \overline{DC}$
$\overline{BD} \cong \overline{BD}$
$\angle BAD \cong \angle BCD$



SAS

Figure 3. Laura’s argument to convince others for task A.

As can be seen in this excerpt above, Laura started with deductive reasoning—using triangle congruency—to justify the statement. She was able to state the information given by the definition correctly. Even though she was able to state all congruent sides of those triangles (Δ BAD and Δ BCD), she preferred to use side-angle-side instead of side-side-side. Additionally, she did not realize the logical flaw in using side-angle-side nor did she realize that she used statements that required justification (i.e. angle A is equal to angle C and angle B is equal to angle D). Balacheff (1991) argues that when form is seen as paramount, students can create or accept arguments that are illogical or fail to convince. It was evident in this excerpt that Laura was thinking that usage of formal language such as congruency or side-angle-side was paramount in an argument to be considered as a convincing argument.

Empirical reasoning. The majority of the participants (7 PSTs during the pre-interviews and 4 PSTs during the post-interviews) demonstrated empirical reasoning while constructing an argument, as well as while evaluating mathematical arguments presented during the individual interviews. Those participants who demonstrated empirical reasoning failed to demonstrate an understanding of logical necessity to construct a proof to show that the statement holds true for all cases. For instance, when Elaine was presented with an empirical argument for Task A, she believed that she could arrive at a generalization from specific cases.

> Elaine: Based on this one (referring to argument 2 in fig.3), yeah, you would assume that it is true for all kites, because they gave you six concrete examples. I feel like six examples is convincing enough to say that it is true for all kites!Um, whenever I was in geometry, our geometry teacher, she would make us, like if we

> thought something must be true, but she might not believe it until we provide five different examples, so that then she can.

As can be seen in the excerpt above, Elaine was unable to talk about the fact that proofs should prove for all cases. Even though PSTs who reasoned at an empirical level did not recognize that the conclusion that could be derived from empirical arguments could not be generalized to all cases while the conclusion that could be derived from deductive arguments could be generalized to all cases, this criterion held an essential factor when evaluating justifications by PSTs who demonstrated deductive reasoning.

Incomplete deductive reasoning. Kelly and Chloe demonstrated empirical reasoning during the pre-interviews; however, they both were aware of the limitations of empirical arguments during the post interviews. As can be seen in the excerpt below, Kelly attempted to use what was known about kites and marked off the given information (adjacent congruent sides) on the figure. She argued that the angles that are opposite of equal sides were the same. After explaining why the angles marked in the same color should be congruent to each other, Kelly argued that one pair of opposite angles that consisted of one red and one green angle should also be congruent to each other.

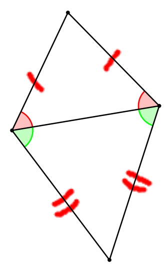


Figure 4. Kelly's argument for task A.

> Kelly: Um, so since you know these two adjacent sides are equal and these two are (marking congruent adjacent sides). Um, (pause) so if you made each side into a triangle (drawing a diagonal from congruent angles in a kite), um, these two sides would be the same, so these two angles (marked in red) would be the same as each other. This angle and this angle (marked in green) would also be the same. Yeah, so, to get the total amount of this interior angle, you would add this angle (angle marked in red) with this angle (angle marked in green) and to get this one you do the same thing and they will equal the same amount.

Kelly attempted to use logical reasoning to justify the statement instead of attempting to use an empirical argument as she did in the pre-interviews. Even though her argument included information that required further justification—the angles opposite of the two sides that are equal in a triangle are equal—her argument primarily focused on deductive reasoning.

Deductive reasoning. Four pre-service teachers - Scott, Jake, Mary, and Miranda, during the pre-interviews and six pre-service teachers - Scott, Jake, Mary, Miranda, Chloe, and Kelly, during the post-interviews were able to reason deductively about the tasks and arguments. These PSTs were not only able to construct deductive arguments to prove true mathematical statements, they all were also able to distinguish between empirical and deductive arguments, unlike the other participants who were coded at different proof schemes.

For instance when Jake was presented with task A, he was able to use only the given information to construct a deductive argument.

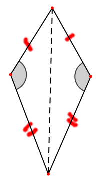


Figure 5. Jake's argument for task A.

> Jake: Ok, well, you can take a kite (see figure 5) and you can divide it in half. These upper sides would be the same, these bottom two sides would be the same (marking adjacent congruent sides). Since these two triangles also shared a side, by the definition of what a congruent triangle is, these two triangles are congruent. Then, these two angles have to be the same (marking the opposite congruent angles).

Furthermore, Jake was able to explain why his argument could be considered as a proof.

> Jake: Yeah, I believe this (his argument) is a proof, using congruent triangles to prove that a kite has two opposite angles that are the same. A kite could also be, um, that can go in where these work like that (see figure 6). These (marked angles) are still the same for the same reason. This is the same with the splitting in the two triangles and these two triangles are congruent to each other.

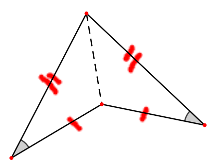


Figure 6. Jake's argument.

As can be seen in the excerpt above, it was clear that Jake was aware of the generality of his argument, which apparently was an important criterion for him to verify that an argument could be considered as a valid proof. Watson and Mason (2005) coin the term example space to refer to the collection of examples (p. 59). It was evident from the excerpt above that not only convex kites, but also concave kites took place in Jake's potential space of examples.

Pre-service Teachers' Counterexample Construction

A general proof, covering all relevant cases, is necessary to validate a true statement while only one counter example is sufficient to refute a false statement. This study showed that the PSTs who were aware of the necessity to cover all possible cases in order to prove true mathematical statements also were aware that providing one counterexample was sufficient to refute a false statement. On the other hand, the majority of the PSTs who were not aware of this generality rule did not always recognize that one counterexample was sufficient to refute the statement or they tended to believe that providing more than one counterexample would make the argument stronger. The table below summarizes PSTs' counterexample levels.

Table 4
Pre-service Teachers' Counterexample Levels

Categories	Number of Participants at This Category			
	Pre-Interviews		Post-Interviews	
	Geometry & Measurement	Mathematics Methods	Geometry & Measurement	Mathematics Methods
Irrelevant Counterexample	Elizabeth Chloe Kelly	Elaine Laura Daisy	Elizabeth	Elaine Laura Daisy
Specific Counterexample	Sara	Casey	Sara Chloe Kelly	Casey
Semi-general Counterexample	Scott	Miranda Jack Mary		Miranda Mary
General Counterexample	-	-	Scott	Jack

Laura demonstrated naive reasoning during the individual interviews. When she was presented with task D, she was able to generate the following argument:

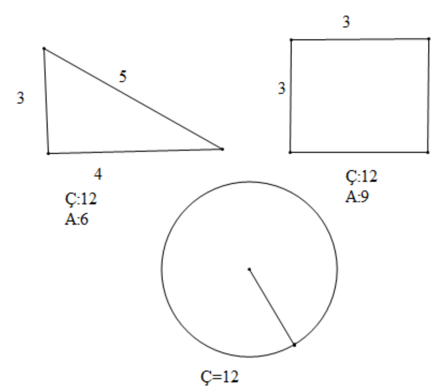


*Figure 7.*Laura's counterexample for task D.

Interviewer: Do you believe the method is true?
Laura: Um. I do not know. I would try with different shapes, like a triangle, square or circle with the same perimeter. Let's say, the perimeter of this triangle is 12 and then I can draw a square with side of 3 which will give me 12as its

perimeter. Then, I can make a circle. If the perimeter is 12 again, then the area would be, um, 2.pi.r is 12, um, I do not know how to do the area, but it will be different. The area for the triangle is 6 and the area of the square is 9. So no this is false.
Interviewer: Ok, so you believe that the method is false?
Laura: Yes
Interviewer: How would you justify it if this were your exam question?
Laura: and then try to find their areas. I mean the more the merrier. I would give more than just these rectangles. But, I think this example shows that does not work.

As can be seen in the excerpt above, Laura was able to recognize that the method would not work and construct specific counterexamples to deduce the validity of the presented method. However, when she was asked to provide a justification, she not only argued that providing more counterexamples would make her justification stronger, she also claimed that providing variety of shapes as counterexamples would add to the strength of the justification. Since the method specifically stated rectangles, using various shapes such as triangles or circles was coded as irrelevant counterexamples for this particular task.

When Scott, on the other hand, who demonstrated deductive reasoning was presented with task D, he was able construct a 2 by 2 square and a 3 by 1 rectangle with perimeter of 8 to refute the method. Even though Scott argued that one counterexample would be enough to refute the method, he searched for further explanation for why the method would not always work as can be seen in the excerpt below.

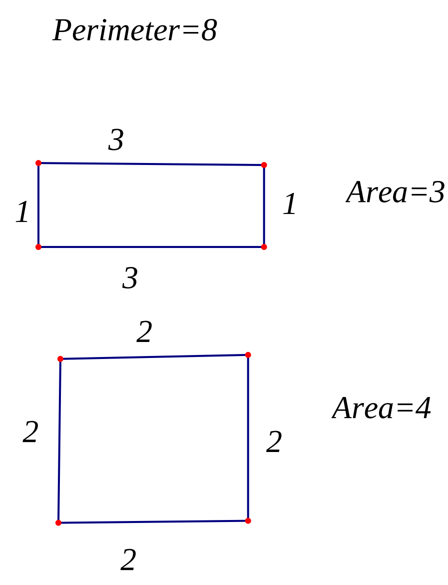


Figure 8. Scott’s refutation for task D.

Scott: Um, this method will not always work. You can make two rectangles with the same perimeter. I did 2 by 2 rectangle, well it is actually a square but squares are rectangles, and 3 by 1 rectangle. They have the same perimeter, 8, but different area. The square has bigger area than the rectangle. It is, um, I think the area will increase if your shape looks more like a square. Like, if the perimeter is

12 then you can make 3 by 3, 2 by 4, and 1 by 5 rectangles (drawing the rectangles in figure 9). And, when the shape is least like the square, the area will be lowest. The area of the square is 9 and the area of the 5 by 1 rectangle is only 5.

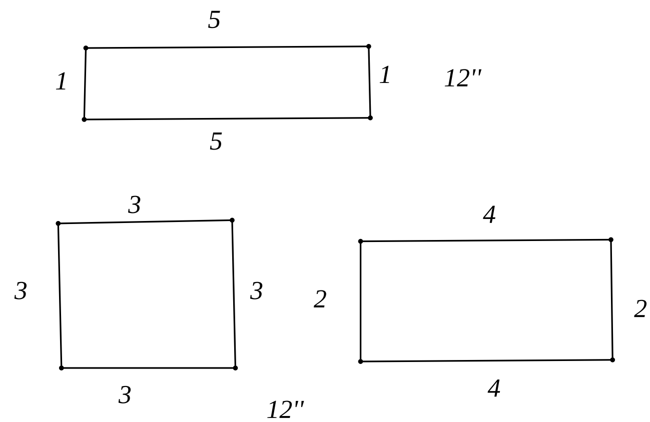


Figure 9. Scott's argument for task D.

Interviewer: Ok, how would you justify that the method would not always work if this were your exam question?
Scott: I would use these two rectangles (rectangles in figure 8). Because if you disprove it once then it disproves it; giving one example is enough to disprove a statement!

Scott knew that one counterexample was enough to show the falsity of a statement. However, he went further by showing a way to generate whole sets of counterexamples by saying "…*It is, um, I think the area will increase if your shape looks more like a square*…" as opposed to finding just one relevant counterexample. Thus, his argument for task D was coded as a general counterexample.

Classwork Analysis Results

PSTs' conception of proof and counterexamples were documented not only through individual interviews, but also through classroom assignments. The conceptions and/or misconceptions that PSTs demonstrated during the individual interviews were evident in their course assignments as well.

Elizabeth demonstrated a limited understanding of proofs and various misconceptions during the interviews. It was evident in the interview data that the source of conviction for Elizabeth was an appeal to external authority, such as the form of arguments or empirical reasoning as described above. In the following question, she was asked to prove that the shape FEHG is a square.

Consider the square ABCD and the line segments connecting the midpoints of its adjacent sides. Use triangle congruence to justify that quadrilateral EFGH is also a square

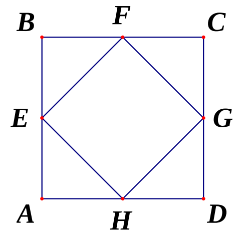


Figure 10. Final exam question.

Elizabeth attempted to construct an argument using steps. As she mentioned during the individual interviews, according to Elizabeth, proof should be written in a certain format, which apparently had steps. She attempted to construct a proof using her concept of image of how proofs should look, as can be seen in the following response:

Step1: Show that EFGH is a square
-Draw a diagonal down the middle from F to G
We know that by SAS

$\overline{EH} \cong \overline{GH}$ $\angle E \cong \angle G$
$\overline{FE} \cong \overline{FG}$

Because we know that all sides are congruent, we know that EFGH is a square by CPCT

Figure 11.Elizabeth's argument for the final exam question.

Classwork analysis also demonstrated the importance of content knowledge on PSTs' construction and evaluation of proofs and counterexamples. It was evident in the pre-interviews that Chloe was reasoning exclusively about quadrilaterals. That is, she did not consider one quadrilateral could possess all the properties of another quadrilateral. It was also evident in the midterm exam that her content knowledge played an essential role in her response to the following question:

Possible or not? For the following statement decide if it is possible or not. If it is possible, write POSSIBLE and draw a picture. If it is not possible, write NOT and give a reason.

A quadrilateral with perpendicular diagonals, which bisect each other, and is NOT a rhombus.

Figure12. Midterm question.

Chloe responded that it was possible and she provided a square as a counterexample.

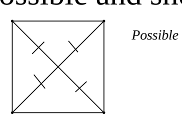


Figure 13. Chloe's response to the Midterm question.

Even though Chloe attempted to falsify the statement by providing a counterexample, which she believed contradicted to the statement, the counterexample she provided was indeed irrelevant to deduce the falsity of the statement. Thus, her counterexample for the midterm question was coded as an irrelevant counterexample.

It should be noted that Chloe demonstrated empirical reasoning during the pre-interviews but she was able to recognize the limit of empirical arguments and the generality of deductive arguments during the post interviews. Additionally, there was a difference in her concept of quadrilaterals as well. The following is one of the questions on the final exam (see figure 14) in the geometry and measurement course and Chloe's response (see figure 15) to this question is displayed below.

Possible or not? For the following statement decide if it is possible or not and justify your answer.

"A rectangle that is NOT a parallelogram."

Figure 14. Final exam question.

It is not possible. All rectangles have opposite sides parallel to each other. All rectangles are parallelograms.

Figure 15. Chloe's response to the final exam question.

As evidenced in her response to this question, Chloe was aware of inclusion relations in quadrilaterals, more specifically invariant and variant properties of these quadrilaterals. She knew that having two sets of parallel sides is an invariant property of a rectangle, which makes it a parallelogram as well. It was evident in the individual interviews as well as class work analysis that PSTs' content knowledge was a factor in the construction and evaluation of arguments. This issue will be discussed in detail next.

Pre-service Teachers' Knowledge of Concepts and Their Influences on Their Conceptions of Proofs

When asked to construct and evaluate mathematical justifications, it was evident that participants' understanding of the concepts, more specifically their possible example spaces constituted an essential factor. As can be seen in the following excerpt, Sara believed that a kite was also a rhombus and she attempted to use a rhombus to construct a justification for a kite statement.

> Sara: I think it is true, because a kite is usually a rhombus and a rhombus requires, um, four congruent sides and four equal angles. Um, I would probably use a ruler, ok, so, um, just say it is 4 cm and then this one; it is 4 cm this way (drawing a rhombus with side of 4 cm.). Um, it is a little off, but, then, I would say that these (sides of the rhombus she drew) are all equal and then these, well it is what I said, and then I would measure these angles (pointing out one pair of opposite angles). So, that's 79 degrees and then, this (one of the opposite angles that are the same) one is the same. It is still the 79 degrees. So, I would say that, these are opposite angles and they are equal to each other.

> Interviewer: How confident are you that your justification is proving that this statement is true—on a 4 to 1 scale, 4 being very confident?
> Sara: I would say 4!

It can be argued from this excerpt that Sara had an understanding of inclusive properties of quadrilaterals, which implies that one quadrilateral possesses all properties of the other.However, her lack of functional understanding of the hierarchical classification of quadrilaterals, which refers to the meaning behind why and how to construct the hierarchical structure of quadrilaterals, hindered Sara's argument.

The second argument for task A included not only the kites with two distinct pairs of congruent adjacent sides, but also special kites—rhombi and squares. Participants who reasoned at an empirical level reacted to the second argument differently depending on their potential example spaces regarding kites, as can be seen in the following excerpts.

> Chloe: I think the difference between them (referring to argument 1 and argument 2) is these are more like, the argument 1 is more of kite shapes, like the way you would think you would see a kite. Um, two adjacent sides, the top two are shorter than the bottom two. That's usually how people think of kites. And this one (argument 2) included rhombuses and squares. I just do not like this argument (argument 2) included these shapes (referring to rhombi and squares in the argument), because I think they have different qualifications than a kite. Because when I think of a kite, like I said, I think of, umm, top two shorter than the bottom two. And then when you think of a square, they are obviously all the sides and all the angles are the same. I do not think they are kites!

PSTs' content knowledge in areas which they were being asked to prove not only influenced their evaluation of presented justifications, but it was also a critical factor that influenced their evaluations of the correctness of mathematical conjectures. For example, when Elizabeth was presented with task D, she struggled with the task, which was indeed a result of her limited knowledge of area and perimeter.

> Elizabeth: I do not know. I think I would not be able to do it (finding the area of the shape). I do not even know how I could measure it (the shape).
> Interviewer: If you were able to use your ruler, protractor or any other measurement tool, would that help to find the area?
> Elizabeth: Maybe. Because of the lines, I do not know how I could measure the lines.
> Interviewer: Would you try to measure the lines of the shape?
> Elizabeth: Yeah, I would try to measure the lines, maybe somewhat the area, not the perfect area.
> Interviewer: What would you do next after you measure the lines?
> Elizabeth: And then times them. No, add them. It would probably not be exact, but it would be an estimate if I add all the lengths of the lines up.

As can be seen in the excerpt, Elizabeth confused area and perimeter. She suggested measuring the lines that surrounded the shape and adding them up to find the area. There is evidence in the literature that pre-service teachers confuse perimeter and area and they believe there is a relationship between perimeter and area (e.g. Baturo&Nason, 1996; Fuller, 1997). Later when the method for task D was presented, she believed that area and perimeter were related so that the method would work.

Unlike Elizabeth, Miranda was both aware of the difference between area and perimeter and that there was not always a relationship between perimeter and area. Thus, she was able to correctly analyze the presented method and recognize the falsity of this method.

> Miranda: (Reading the method) I do not know if this would work (laughs). Because it is assuming that the perimeter of the object is what corresponds with the area, but I do not think that is true.
> Interviewer: How did you decide that it is not true?
> Miranda: Um, well, I drew another shape in my head. It was like, because it has a really long perimeter, but would not have a lot of area, cause it is really skinny, so if you like measured out the lines it would not necessarily, like, correspond with the area.

Conclusion and Discussion

In order to investigate the research questions, participants were engaged in constructing justifications/ refutations for various mathematical tasks and evaluating arguments, ranging in terms of level of sophistication as presented above.Although the participants demonstrated some similarities, they also had unique ways of responding to these tasks and arguments. This study documented pre-service teachers' limited understanding of proofs. The fact that pre-service elementary teachers experience difficulty with proof and counterexamples indeed aligned well with the existing literature on pre-service teachers' capabilities of validating arguments and constructing proofs and counterexamples (e.g. Martin & Harel, 1989; Simon & Blume, 1996).

One of the most common difficulties that the participants demonstrated was to fail to recognize indeterminacy of inductive conclusions and demonstrate an understanding of logical necessity to construct a proof to show that the statement holds true for all cases. Stylianides (2007) argues that learning this fundamental difference between empirical and deductive arguments is essential to move from an empirical understanding of proof to deductive understanding of what constitutes proof.

Although learning the fundamental difference between empirical and deductive arguments may certainly enhance PSTs' conceptions of what constitutes proof, such enhancement may be only a necessary but not a sufficient condition for understanding mathematical proof. Learning to prove is a complex topic, which requires the use of several areas of knowledge and deductive reasoning. In addition to the importance of understanding the nature of proof, this study demonstrated that how PSTs evaluated presented arguments/counterexamples and constructed mathematical justifications/counterexamples differed based on their knowledge of underlying mathematics concepts and what constitutes an example of a case.

This study also documented that content knowledge indeed constitutes an essential factor for concepts of proof. PSTs' limited knowledge of content, specifically regarding to the underlying concepts of the statements that they were asked to prove/refute, inhibited their ability to evaluate the validity of the conjectures as well as presented arguments. Thus, it is necessary to improve pre-service teachers' knowledge regarding to the nature of proof as suggested by Stylianides (2007); however, it is only possible to achieve this if PSTs have a solid understanding of the underlying concepts.

Implications

The findings have implications both for teacher educators working to design mathematics courses for pre-service teachers, as well as for researchers interested in better understanding teacher knowledge. As it has been argued throughout this study as well as in many other studies, learning how to construct viable arguments and critique the reasoning of others as discussed by one of the Common Core State Standards mathematical practices can only happen if students are provided ample opportunities to engage in proving tasks. However, it is only possible to respond to this call if teachers themselves have a solid understanding of mathematical proofs. It is suggested that mathematics teachers' conceptions of proof are usually affected by undergraduate mathematics courses in which they learn the context of proof in mathematics (e.g. Blanton, Stylianou & David, 2009; Knuth, 2002b).Thus; it could be argued that if one wants to improve the proof abilities of students, perhaps the best place would be college courses for pre-service teachers.

This study showed that the fundamental distinction between empirical and deductive arguments blurred among PSTs who had not yet developed deductive reasoning. Stylianides (2007) argues that learning this fundamental distinction is a way to move from empirical reasoning to deductive reasoning. It could be a promising instructional strategy to help pre-service teachers develop an understanding of what is an acceptable proof. This study demonstrated that a proof in the context of analyzing, evaluating, and comparing arguments with various validity levels is likely to help students understand the limitations of an empirical argument and develop an appreciation for proof. Similarly, Knuth (2002c) also recommended that creating learning opportunities for students in which they encounter various types of arguments may not only help them develop deeper understanding of proof and the underlying mathematics, but it may also help students realize the explanatory quality of proofs.

Proof is a complex topic that requires the use of various concepts. It is often stressed that teachers need richer and/or more experiences with proof. However, the struggle the PSTs faced in this study was more a result of their limited knowledge of underlying concepts. This study not only highlighted the importance of solid understanding of underlying concepts for proof constructions and evaluations, but it also documented the fact that having limited understanding of underlying concepts could serve as a limiting factor. Thus, in order to equip pre-service teachers' with robust understanding of proofs, it may be critical to first improve their knowledge of contentprior to improving their proving abilities. As pre-service teachers will play important roles in educating students, much more research is needed on nuanced teaching strategies for fostering PSTs' content knowledge during teacher education programs.

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Appendix A

<table>
<tr><td colspan="2">Sample Questionnaire Questions</td></tr>
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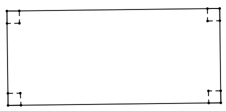
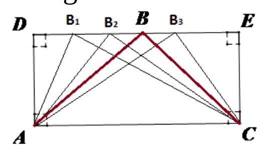
Question: MS. Jones asked the following question to her 5th grade class:
"Possible or Not? A rectangle that is not a parallelogram. Provide an explanation if it is possible or provide a counterexample if it is not possible."
Katie claimed that the statement was not possible and she provided the figure below as her counterexample.

In your opinion, how would Ms. Jones mark Katie's counterexample? Please circle one of the options below and provide your reasons for choosing correct/incorrect.
Correct Incorrect
Your reason(s):

</td><td>

Question: Kellyclaimedthatthe area and perimeter of Δ ABC are going to be the same no matter where B is on segment DE of rectangle ADEC.


Do you agree with Kelly?
If you agree with Kelly: Please prove Kelly's conjecture.
If you disagree with Kelly: Please refute Kelly's conjecture.
If you partially agree with Kelly's conjecture: Please amend Kelly's conjecture and then provide a proof.

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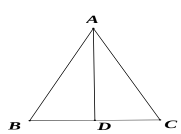
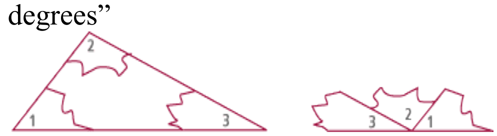
<tr><td>

Question: Jack was asked to justify that in an isosceles triangle ABC, the base angles (∠B and ∠C) are equal. He drew an angle bisector AD from ∠A and showed that AB ≅ AC, ∠BAD ≅∠CAD, and ∠B≅∠C. Therefore, Δ ABD≅ ΔACD by A.S.A. Hence, ∠B ≅∠C


How would you mark Jack's argument? Please circle one of the options below and provide your reasons for choosing correct/incorrect.
Correct Incorrect
Your reason(s):

</td><td>

Question: What would be your reaction to the following justification of the sum of the interior angles of a triangle is 180 degrees:
"I tore up the angles of a triangle and put them together (as shown below), the angles came together as a straight line, which is 180 degrees. Therefore, the sum of the measures of the interior angles of a triangle is equal to 180 degrees"


Is this a proof? Why or Why not?

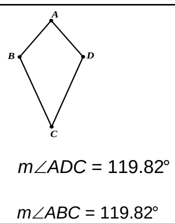
</td></tr>
</table>

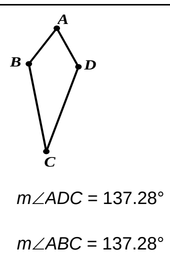
Appendix B

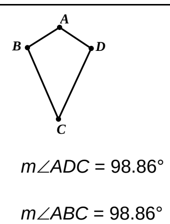
Arguments for Task A

Argument #1

I drew three kites. I labelled each kite ABCD. I measured one pair of opposite angles (∠B and ∠D) in each of these kites and in each case ∠B and ∠D are equal in measurement. Since it worked for these kites, I can be sure that the statement is always true.

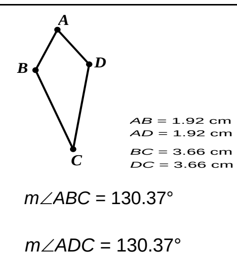


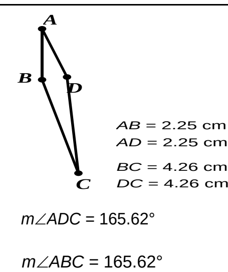


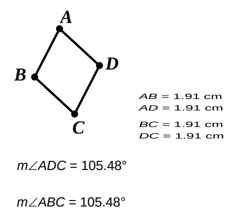


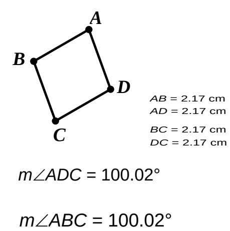
Argument #2

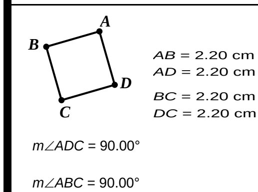
I drew six different kites. I labelled each kite ABCD. I measured one pair of opposite angles (∠B and ∠D) in each of these kites and in each case ∠B and ∠D are equal in measurement. Since I checked different kinds of kites including special kites, namely, rhombus and square, I can be sure that the statement is always true.

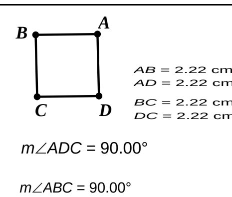




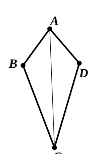






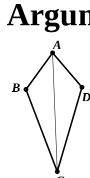


Argument #3



Justification: If we look at this figure ABCD and fold it in half along AC, I can easily show that the two angles (∠ B and ∠D) are the same. The two angles will match up

Argument #4

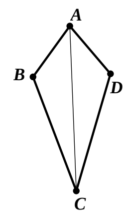


STATEMENT	REASON
AB ≅ AD	Given by definition

because Δ ABC and Δ ADC are congruent triangles since the side lengths, AB ≅AD, BC≅DC, and AC≅AC of these two triangles are the same. Therefore, the statement must be true. Angle B and Angle D are the same!

BC ≅ DC	Given by definition
AC≅AC	Reflexive property
ΔABC≅ΔADC	By S.S.S
∠B≅∠D	By CPCT

Argument #4-B



STATEMENT	REASON
AB ≅ AD	Given by definition
BC ≅ DC	Given by definition
∠B≅∠D	Given by definition
ΔABC≅ΔADC	By S.A.S
∠B≅∠D	By CPCT

Author Information

Zulfiye Zeybek is an assistant professor at Gaziosmanpasa University in Turkey. She has earned her Master's Degree and her PhD. in the field of mathematics education (with a minor in mathematics) from Indiana University in the U.S. She teaches undergraduate and graduate courses at Gaziosmanpasa University in Turkey. Her research focuses on investigating pre-service teachers' mathematical knowledge for teaching, mathematical reasoning and proof skills of students and teachers, geometrical and spatial thinking in K-12. She anticipates continuing to research in these areas.