10 Equations

10.12B Finding Algebraically the Points of Intersection of a Circle and a Straight Line Graph

In Section 13.11B, we saw how to use graphical methods to find the points where a straight line intersects a circle. In this section, we use algebraic methods to find such points.

Worked Example 1

(a) Solve algebraically the simultaneous equations

$$x^2 + y^2 = 10 \qquad (1)$$

$$y = x + 2 \qquad (2)$$

(b) Write down the coordinates of the points of intersection of the circle $x^2 + y^2 = 10$ and the straight line $y = x + 2$.

Solution

(a) We use the linear equation (2) to substitute for y^2 into equation (1), the equation for the circle.

The square of equation (2) is

$$y^2 = (x + 2)^2$$

so $$y^2 = x^2 + 4x + 4$$

Substituting into equation (1) now gives

$$x^2 + \left(x^2 + 4x + 4\right) = 10$$

which simplifies to $$2x^2 + 4x - 6 = 0$$

which, dividing by 2, simplifies again to

$$x^2 + 2x - 3 = 0$$

We can now solve this equation for x by factorising

$$(x + 3)(x - 1) = 0$$

so $$x = -3 \text{ or } x = 1$$

Substituting these values into the linear equation $y = x + 2$ gives corresponding values $y = -1$ and $y = 3$.

Therefore the solutions of the two equations are

$$x = -3,\ y = -1 \text{ and when } x = 1,\ y = 3$$

(b) The points of intersection of the circle $x^2 + y^2 = 10$ and the straight line $y = x + 2$ are $(-3, -1)$ and $(1, 3)$.

N.B. In Worked Example 1, the substituting of the linear equation led to a quadratic equation in x that could be solved by factorisation. In some cases, the quadratic does not factorise and the values of x have to be found by using the quadratic equation formula or by completing the square.

Exercises

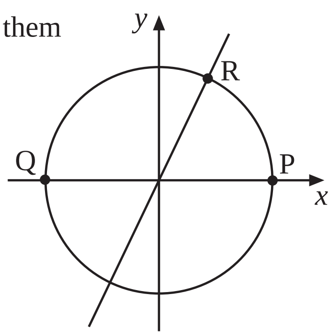
1. Solve the following pairs of simultaneous equations. Where appropriate, give your solutions correct to 2 decimal places.

(a) $x^2 + y^2 = 25$, $y = 3x - 5$ (b) $x^2 + y^2 = 45$, $y = 2x$ (c) $x^2 + y^2 = 8$, $y = x + 4$

(d) $x^2 + y^2 = 68$, $y = 3x + 2$ (e) $x^2 + y^2 = 6$, $y = x - 3$ (f) $x^2 + y^2 = 20$, $y = 3 - 2x$

2. Find the coordinates of the points where the line $y = 4 - x$ intersects the circle $x^2 + y^2 = 22$.

3. *In this question you should use surds for exactness.*



(a) Draw the centrally-placed x- and y-axes and scale them using the ranges $-12 \le x \le 12$, $-12 \le y \le 12$.

(b) Accurately construct the locus $x^2 + y^2 = 100$.

(c) On the same graph, draw the line $y = 3x$.

(d) Write down the coordinates of the points P and Q (as labelled in the diagram) where the x-axis meets the circle. PQ is a diameter of the circle.

(e) By solving the simultaneous equations

$$x^2 + y^2 = 100$$

$$y = 3x$$

find the x- and y-coordinates of the point, R, where the line meets the circle in the first quadrant.

(f) Calculate the area of ΔPQR.

(g) Show that the gradient of the line segment QR is $\dfrac{3}{\sqrt{10} + 1}$.

(h) Find the gradient of the line segment PR.

(i) Show that PR is perpendicular to QR.

N.B This illustrates the fact that the angle in a semicircle is a right angle.

Answers

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1. (a) $x = 3,\ y = 4$ and $x = 0,\ y = -5$

   (b) $x = 3,\ y = 6$ and $x = -3,\ y = -6$

   (c) $x = -2,\ y = 2$ (There is only one solution here which means that the line is actually a tangent to the circle.)

   (d) $x = 2,\ y = 8$ and $x = -3.2,\ y = -7.6$

   (e) $x = 2.37,\ y = -0.63$ and $x = 0.63,\ y = -2.37$

   (f) $x = 3.11,\ y = -3.22$ and $x = -0.71,\ y = 4.42$

2. $(4.65,\ -0.65)$ and $(-0.65,\ 4.65)$

3. (d) P (10, 0) and Q (–10, 0)

   (e) R $\left(\sqrt{10},\ 3\sqrt{10}\right)$

   (f) $30\sqrt{10}$ units^2

   (h) $\dfrac{3}{1 - \sqrt{10}}$