13 Graphs

13.5B Gradients of Perpendicular Lines

In this section we explore the relationship between the gradients of perpendicular lines and line segments.

Worked Example 1

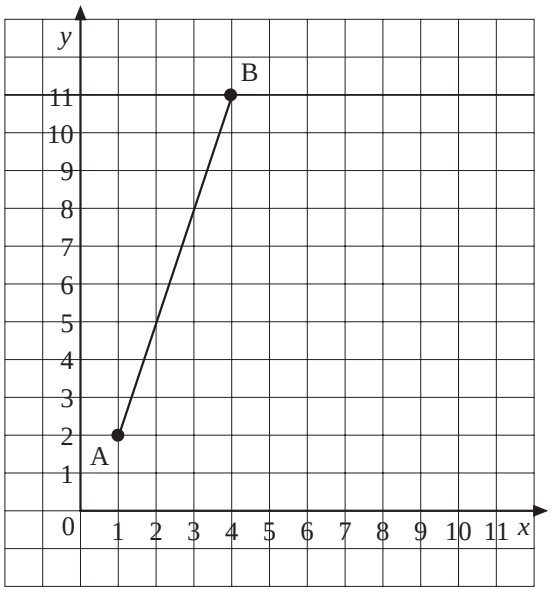
(a) Plot the points A (1, 2) and B (4, 11), join them to form the line AB and then calculate the gradient of AB.

(b) On the same set of axes, plot the points P (5, 4) and Q (8, 3), join them to form the line PQ and then calculate the gradient of PQ.

(c) Measure the angle between the lines AB and PQ. What do you notice about the two gradients?

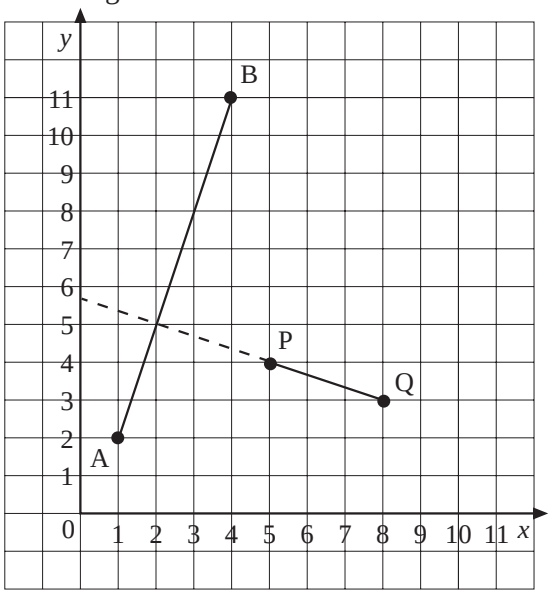
Solution

(a) The points are shown in the diagram.



$$\text{Gradient of AB} = \frac{11-2}{4-1} = \frac{9}{3} = 3$$

(b) The points P and Q can now be added to the diagram as shown below.



$$\text{Gradient of PQ} = \frac{3-4}{8-5} = \frac{-1}{3}$$

(c) The line PQ has been extended on the diagram, so that the angle between the two lines can be measured.

The angle is $90°$, a right angle.

In this case,

the gradient of AB $= 3$

the gradient of PQ $= \dfrac{-1}{3}$

and the gradients multiply to give $3 \times \dfrac{-1}{3} = -1$

Note

The product of the gradients of two perpendicular lines will always be –1, unless one of the lines is horizontal and the other is vertical.

In the example above,

gradient of AB = 3

gradient of PQ $= \dfrac{-1}{3}$

$3 \times \dfrac{-1}{3} = -1$

Note

Gradient of PQ $= \dfrac{-1}{\text{Gradient AB}}$.

If the gradient of a line is *m*, and $m \neq 0$, then the gradient of a perpendicular line will be $\dfrac{-1}{m}$.

Worked Example 2

Show that the line segment joining the points A (3, 2) and B (5, 7) is perpendicular to the line segment joining the points P (2, 5) and Q (7, 3).

Solution

Gradient of AB $= \dfrac{7-2}{5-3} = \dfrac{5}{2}$

Gradient of PQ $= \dfrac{3-5}{7-2} = \dfrac{-2}{5}$

Gradient of AB $\times$ Gradient of PQ $= \frac{5}{2} \times \frac{-2}{5} = \frac{-10}{10} = -1.$

So the line segments AB and PQ are perpendicular.

Exercises

1. (a) On a set of axes, draw the lines AB and PQ where the coordinates of these points are

   A (1, 2)

   B (10, 6)

   P (1, 9)

   Q (5, 0)

   (b) Are the lines perpendicular?

   (c) Calculate the gradient of AB.

   (d) Calculate the gradient of PQ.

   (e) Check that the product of these gradients is –1.

2. In each case, decide whether the lines AB and PQ are parallel, perpendicular or neither.

(a)	A (4, 3)	B (8, 4)	P (7, 1)	Q (6, 5)
(b)	A (–2, 0)	B (1, 9)	P (2, 5)	Q (6, 17)
(c)	A (8, –5)	B (11, –3)	P (1, 1)	Q (–3, 7)
(d)	A (3, 1)	B (7, 3)	P (–3, 2)	Q (1, 0)

3. The points P (–3, 1), Q (1, 2), R (0, –1) and S (–4, –2) are the vertices of a quadrilateral.

   (a) Calculate the gradient of each side of the quadrilateral.

   (b) Is the quadrilateral a parallelogram?

   (c) Is the quadrilateral a rectangle?

4. A triangle has vertices A (3, 1), B (7, 5) and C (1, 3). Show that the triangle is a right-angled triangle.

5. Show that the triangle with vertices A (4, 7), B (8, 2) and C (7, 3) is *not* a right-angled triangle.

6. The coordinates of the point A, B, C and D are listed below.

   A (3, 0) B (0, 1)

   C (1, 4) D (4, 3)

   Show that ABCD is a square.

7. The points A (3, 2), B (6, 0), C (5, 4) and D (2, 6) are the vertices of a quadrilateral.

   (a) Show that this is *not* a rectangle.

   (b) Show that this is a parallelogram.

8. The lines AB and PQ are perpendicular. The coordinates of the points are

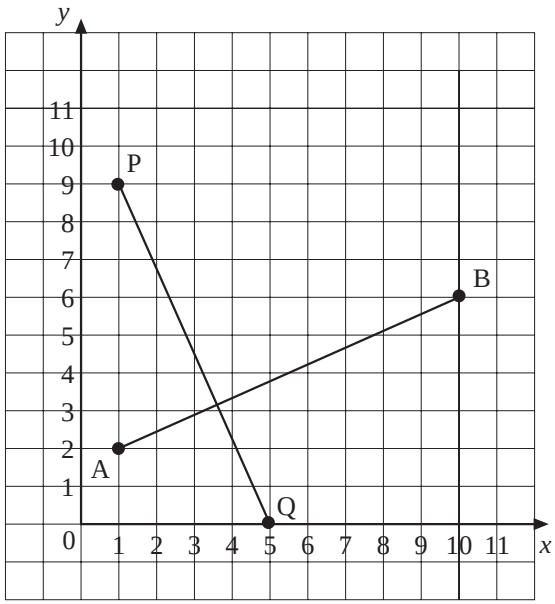
   A (3, 2) B (7, 4) P (3, 7) and Q (6, q)

   Determine the value of q.

Answers

13.5B Gradients of Perpendicular Lines

1. (a) Diagram



(b) Yes, the lines are perpendicular.

(c) Gradient AB $= \frac{4}{9}$

(d) Gradient PQ $= -\frac{9}{4}$

(e) Gradient of AB $\times$ Gradient of PQ $= \frac{4}{9} \times \left(-\frac{9}{4}\right) = -1$

2. (a) Gradient of AB $= \frac{1}{4}$ Gradient of PQ $= -4$

The lines are perpendicular.

(b) Gradient of AB $= 3$ Gradient of PQ $= 3$

The lines are parallel.

(c) Gradient of AB $= \frac{2}{3}$ Gradient of PQ $= -\frac{3}{2}$

The lines are perpendicular.

(d) Gradient of AB $= \frac{1}{2}$ Gradient of PQ $= -\frac{1}{2}$

The lines are neither parallel nor perpendicular.

3. (a) Gradient PQ $= \frac{1}{4}$; Gradient QR $= 3$; Gradient RS $= \frac{1}{4}$; Gradient SP $= 3$

(b) Yes (c) No

4. Gradient of AB $= 1$ Gradient of BC $= \frac{1}{3}$ Gradient of AC $= -1$

Gradient of AB $\times$ Gradient of AC $= 1 \times -1 = -1$.

$\therefore$ AB is perpendicular to AC $\therefore$ Δ ABC is right-angled $(\angle \text{BAC} = 90°)$

5. Gradient of AB $= \frac{-5}{4}$ Gradient of BC $= -1$ Gradient of AC $= \frac{-4}{3}$

Gradient of AB $\times$ Gradient of BC $= \frac{5}{4} \neq -1$ $\therefore$ AB is not perpendicular to BC.

Gradient of AB $\times$ Gradient of AC $= \frac{5}{3} \neq -1$ $\therefore$ AB is not perpendicular to AC.

Gradient of AC $\times$ Gradient of BC $= \frac{4}{3} \neq -1$ $\therefore$ AC is not perpendicular to BC.

$\therefore$ Δ ABC is not right-angled.

6. Gradient of AB $= \frac{-1}{3}$ Gradient of BC $= 3$

Gradient of CD $= \frac{-1}{3}$ Gradient of DA $= 3$

$\therefore$ AB is perpendicular to BC, BC is perpendicular to CD and CD is perpendicular to DA.

$\therefore$ ABCD is a rectangle.

AB $= \sqrt{(0-3)^2 + (1-0)^2} = \sqrt{10}$ BC $= \sqrt{(1-0)^2 + (4-1)^2} = \sqrt{10}$

CD $= \sqrt{(4-1)^2 + (3-4)^2} = \sqrt{10}$ DA $= \sqrt{(3-4)^2 + (0-3)^2} = \sqrt{10}$

$\therefore$ AB = BC = CD = DA

$\therefore$ ABCD is a square.

7. Gradient of AB $= \frac{-2}{3}$ Gradient of BC $= -4$

Gradient of CD $= \frac{-2}{3}$ Gradient of DA $= -4$

(a) Gradient of AB $\times$ Gradient of BC $= \frac{8}{3} \neq -1$ $\therefore$ AB is not perpendicular to BC.

$\therefore$ ABCD is not a rectangle.

(b) Gradient of AB $=$ Gradient of CD $\therefore$ AB is parallel to CD.

Gradient of BC $=$ Gradient of DA $\therefore$ BC is parallel to DA.

$\therefore$ ABCD is a parallelogram.

8. $q = 1$