

14 Loci and Transformations

Additional section to be used *before* Section 14.6.

14.5A Straight Edge and Compasses Constructions

In this section we consider how to draw some shapes and specific angles using only a straight edge and compasses.

We begin by drawing the key constructions.

Construction of a Perpendicular at a Point on a Line



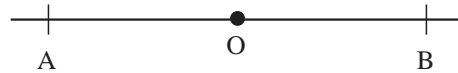
Worked Example 1

Construct a perpendicular at a point, O , on a line.

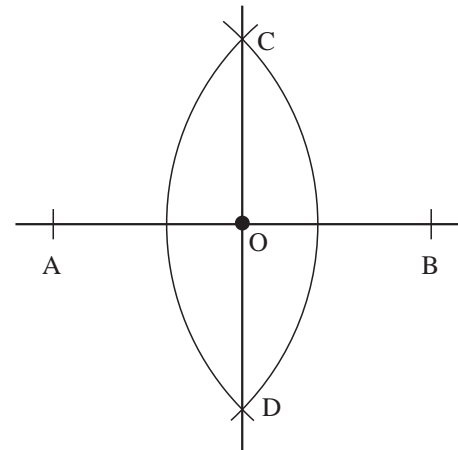


Solution

Use compasses to mark points A and B on the line that are at equal distances from the given point O .



Draw arcs centred at A and B . These two arcs intersect at C and D . Then the required perpendicular line can be drawn through the points C and D .



Construction of the Perpendicular Bisector of a Line Segment



Worked Example 2

Construct the perpendicular bisector of the line AB .



Then label the midpoint of AB , M .

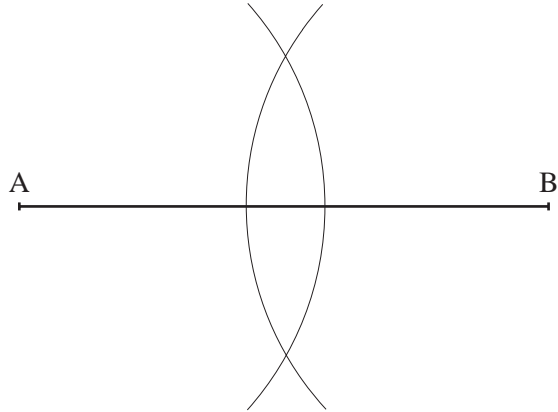


Solution

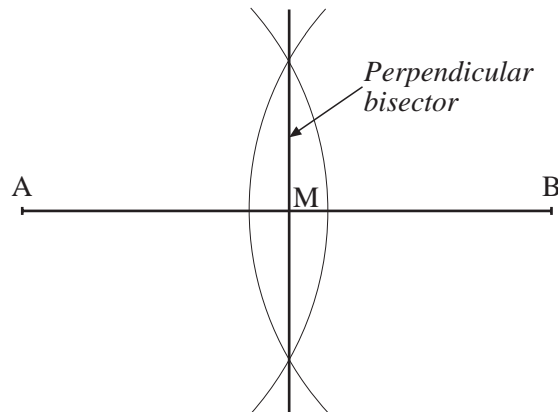
There are many lines that cut AB exactly in half. We have to construct the one that is perpendicular to AB.

We begin by drawing arcs of equal radius, centred on the points A and B, as shown in the diagram.

The radius of these arcs should be roughly $\frac{2}{3}$ to $\frac{3}{4}$ of the length AB.



Then draw a line through the intersection points of the two arcs. This line is the perpendicular bisector of AB.



The point where the perpendicular bisector intersects AB can then be labelled M. M is the midpoint of AB.

Construction of the Perpendicular to a Line from a Point not on the Line



Worked Example 3

The diagram shows the line AB and the point C.

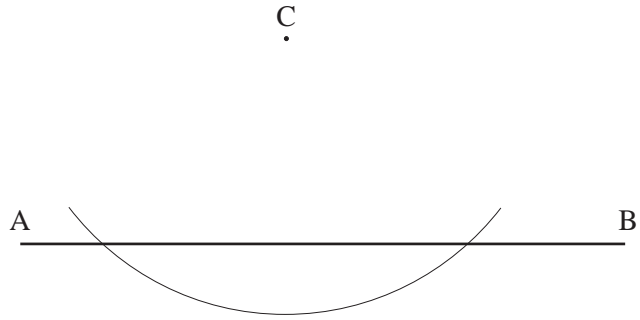
Draw a line through C that is perpendicular to AB.



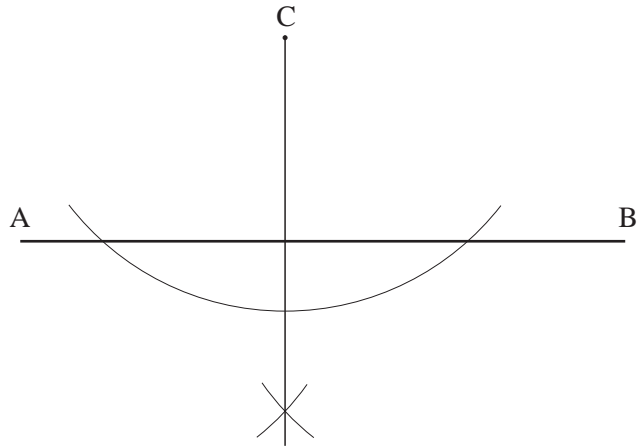


Solution

Using C as the centre, draw an arc as shown.



Then using the intersection points of this arc with the line AB as centres, draw two further arcs with radii of equal length. The perpendicular line can then be drawn from C through the point where these two new arcs cross.

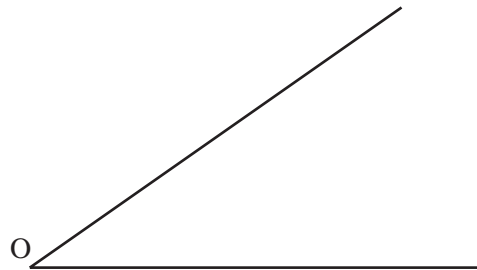


Construction of the Bisector of an Angle



Worked Example 4

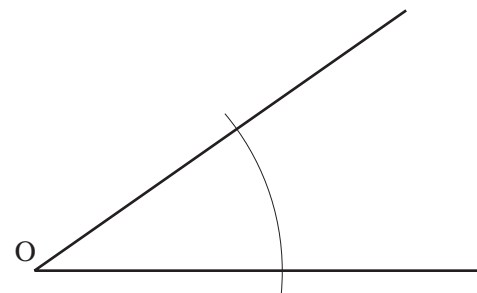
Bisect this angle.



Solution

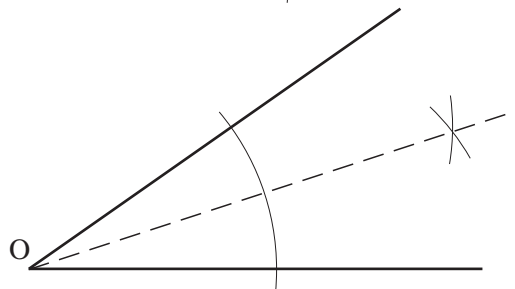
To bisect an angle you need to draw a line that cuts the angle in half.

First draw an arc using O as the centre.



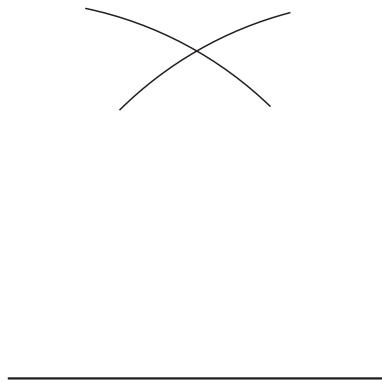
Then draw two further arcs of equal radius, using the points where the arc intersects the lines as the centres.

The bisector can then be drawn from O through the point where these two new arcs cross.

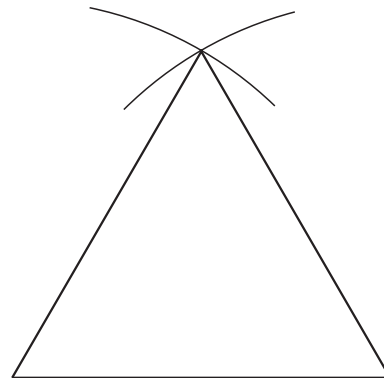


Uses of these constructions

Construction 1: *To draw an equilateral triangle*

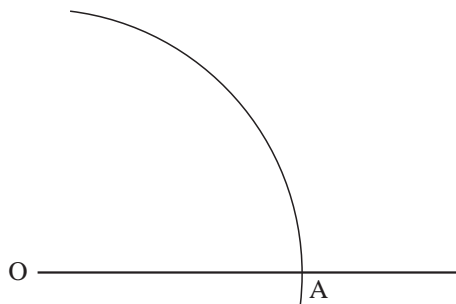


Draw one side. Then draw two arcs, one from each end of this side, with radius the same as the length of the first side of the triangle.

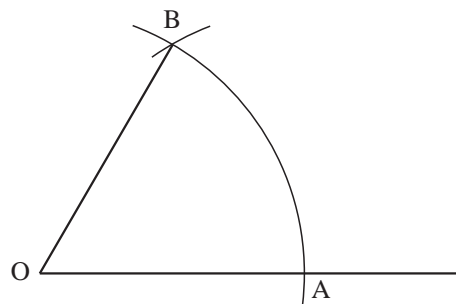


Join the ends of the line to the intersection of the arcs to complete the triangle

Construction 2: *To draw an angle of 60°*



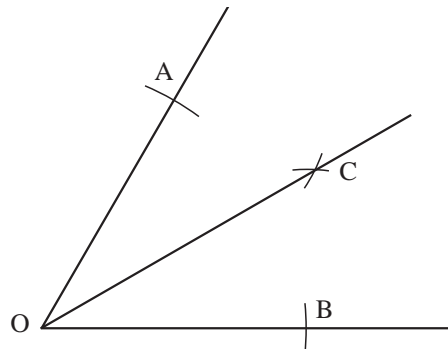
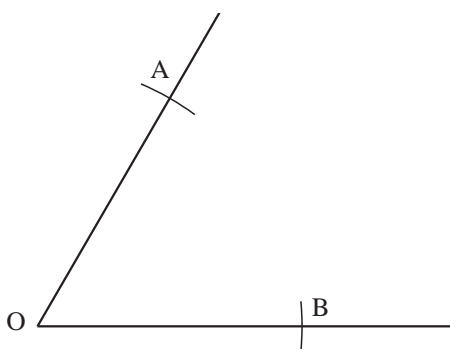
If the angle of 60° is to be drawn at O, draw a line through O that will form one side of the angle. Then draw an arc, with the compass point at O, crossing the drawn side at A, as shown above.



Draw a second arc, of the same radius, with the compass point at A, intersecting the first arc at B. Then join O to B to complete the angle

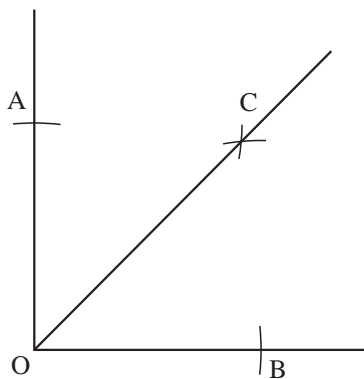
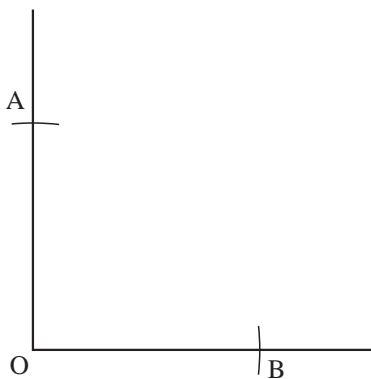
Construction 3: *To draw an angle of 30°*

First draw an angle of 60° using the previous construction, then bisect that angle using the method described in Worked Example 4 earlier in this section.



Construction 4: *To draw an angle of 45°*

Start by constructing a 90° angle using one of the methods described in Worked Examples 1, 2 and 3. Then bisect that angle using the method described earlier in Worked Example 4.

**Worked Example 5**

Construct a triangle ABC with $AB = 8$ cm, the angle $BAC = 60^\circ$ and $ABC = 45^\circ$.

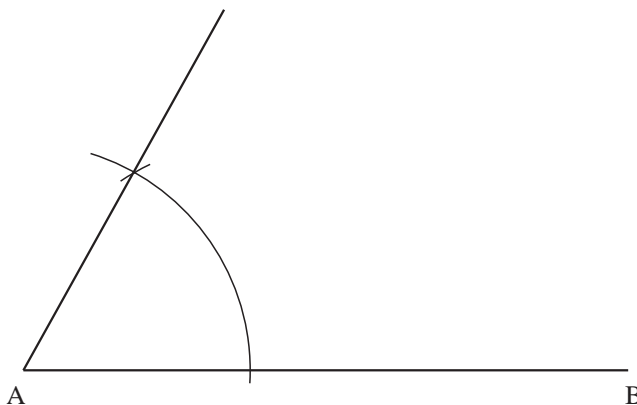
Then measure

- BC
- AC
- the angle ACB.

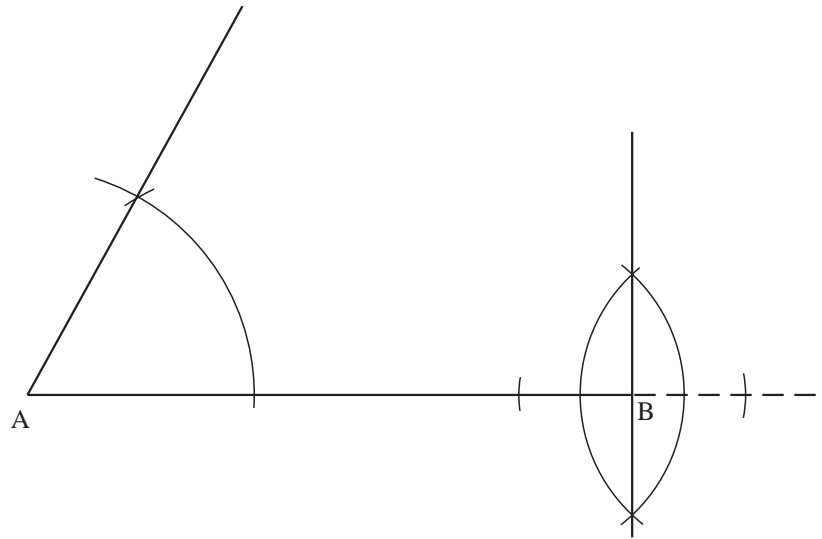
**Solution**

First draw a line of length 8 cm and label it AB.

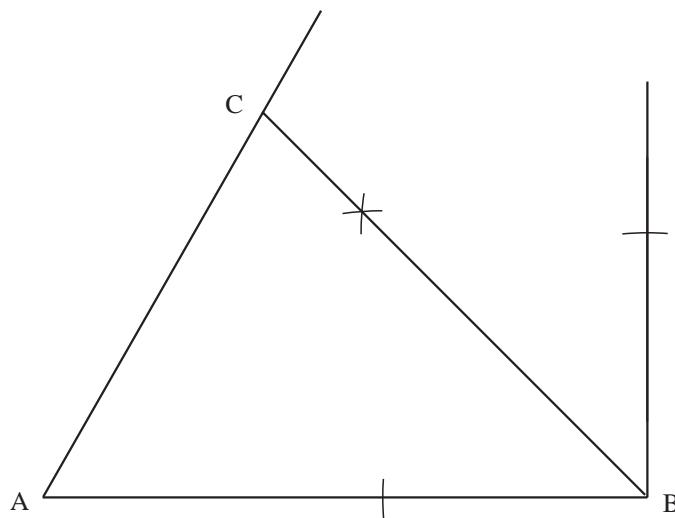
Then construct an angle of 60° at the end A as shown below (see construction 2 above).



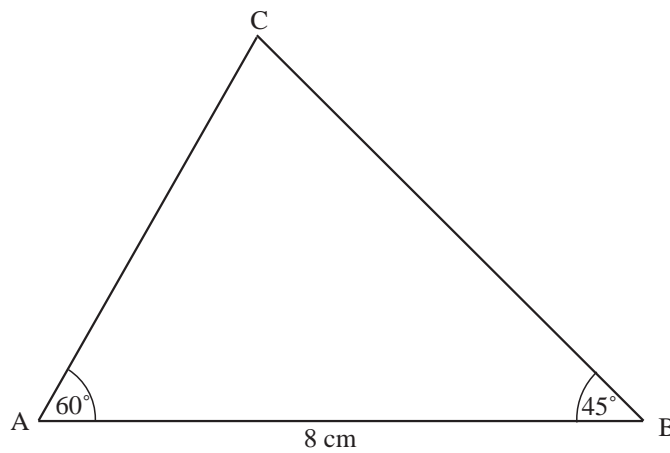
Extend the line AB and draw the line through B perpendicular to AB, using the method described in Worked Example 1.



Using the method described in Worked Example 4, this right angle can now be bisected to give the 45° angle that is required. This is shown in the following diagram and completes the triangle.



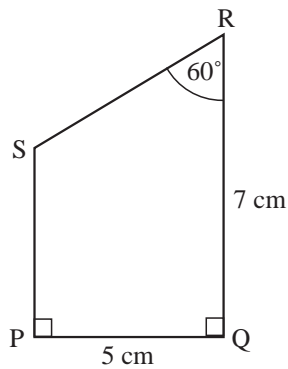
The triangle is shown below:



- (a) $BC = 7.2 \text{ cm}$ (b) $AC = 5.9 \text{ cm}$ (c) $\angle ACB = 75^\circ$



Worked Example 6



Use a ruler and compasses to construct the shape PQRS as shown opposite.

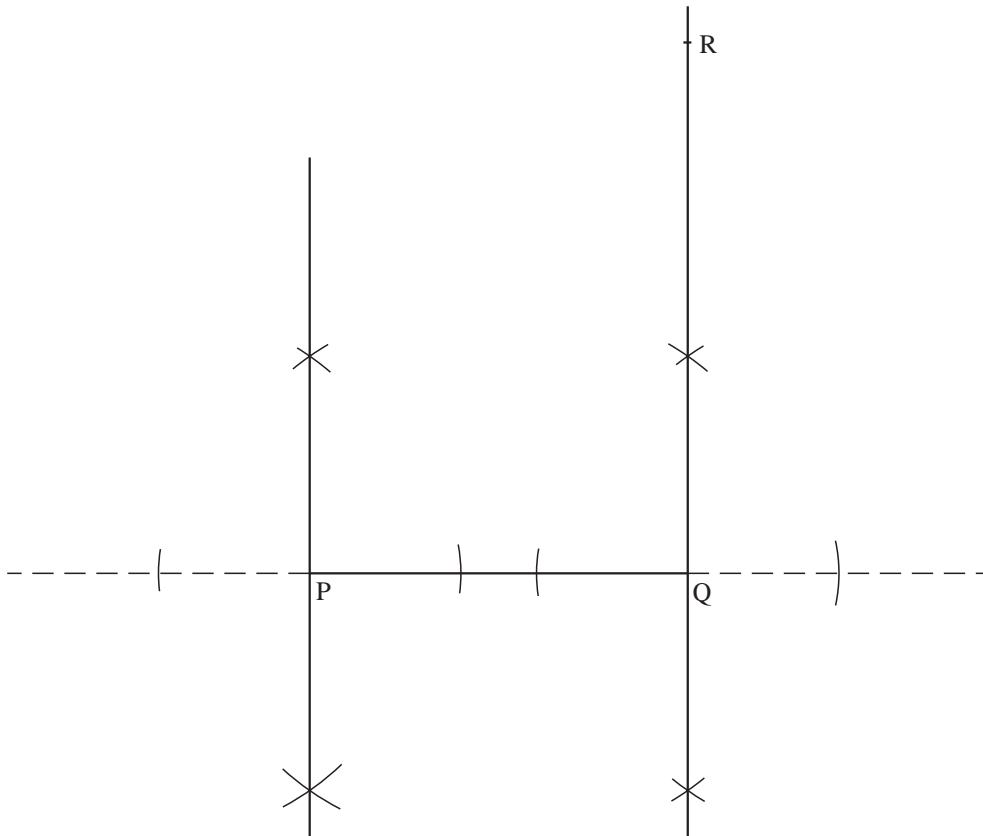
Then measure

- (a) RS
- (b) PS
- (c) the angle PSR.

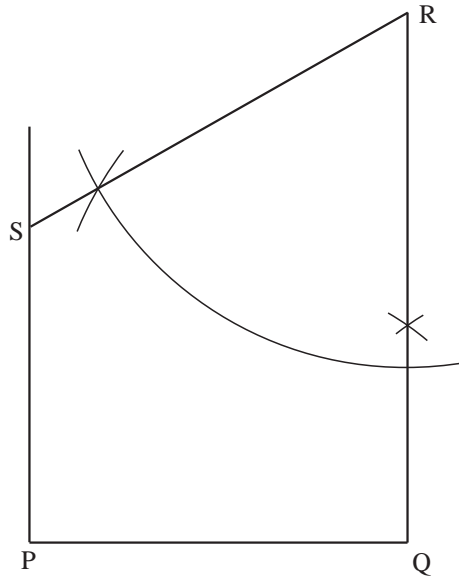


Solution

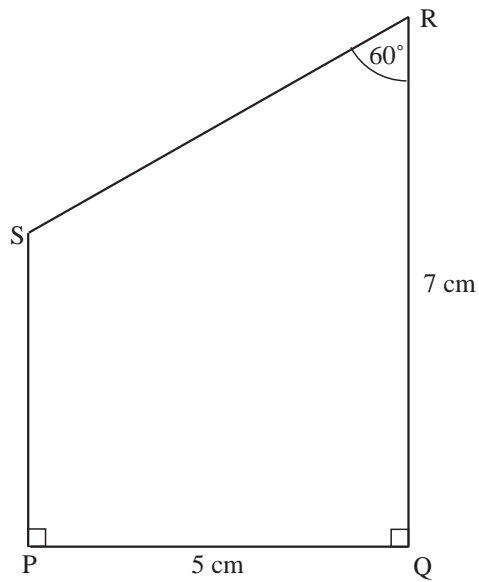
First draw the line PQ, with length 5 cm, and then construct perpendicular lines through P and Q using the method described in Worked Example 1. Measure 7 cm along the perpendicular through Q and mark in the point R.



Now, using the method described in Construction 2, construct a 60° angle at R to complete the shape.



The shape PQRS is shown below:

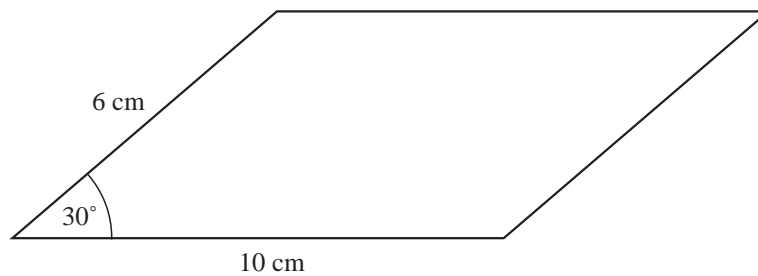


- (a) $RS = 5.8 \text{ cm}$
- (b) $PS = 4.1 \text{ cm}$
- (c) $\angle PSR = 120^\circ$



Exercises

- Construct an angle of 135° . (*Hint:* $135^\circ = 90^\circ + 45^\circ$)
- Construct an angle of 15° .
- Construct an angle of 75° .
- A triangle ABC is such that $AB = 10$ cm, $\angle ABC = 30^\circ$ and $\angle BAC = 60^\circ$.
 - Draw the triangle.
 - Measure AC and BC.
 - Measure $\angle ACB$.
- Draw triangle LMN with $LM = 9$ cm, $\angle LMN = 30^\circ$ and $LN = 7$ cm. Show that there are two possible solutions.
- Use a ruler and compasses to construct this parallelogram.



- Take any triangle ABC. Set your compasses to length BC and draw an arc centred on A. Now set your compasses to length AB and draw an arc centred on C. Label the point of intersection, D. Join D to A and C. Explain why this construction produces the line through A parallel to BC and the line through C parallel to BA.
- Take any triangle XYZ. Construct the perpendicular bisectors of XY, YZ and ZX. Show that these meet at a single point, W. Draw the circle, centre W, radius WX. Show that this circle passes through all three vertices of the triangle. (It is called the *circumscribed circle* of $\triangle XYZ$.)
- Take any triangle JKL. Construct the angle bisectors of $\angle JKL$, $\angle KLJ$ and $\angle LJK$. Show that these meet at a single point, I. Draw the circle, centre I, such that this circle just touches all three sides of the triangle. (It is called the *inscribed circle* of $\triangle JKL$.)
- Using straight edge and compasses:
 - construct a regular hexagon,
 - construct a regular octagon.
- Take any triangle PQR. Construct the perpendicular from P to QR meeting it at S, the perpendicular from Q to PR meeting it at T, and the perpendicular from R to PQ meeting it at U. Show that the lines PS, QT and RU all meet at a single point. (It is called the *orthocentre* of $\triangle PQR$.)

12. Take any triangle ABC . Construct the perpendicular bisector of BC to obtain the midpoint F of BC . Construct the midpoint G of CA , and the midpoint H of AB , in the same way. Join AF , BG and CH . Show that these meet at a single point, M . Show, by measuring, that

$$AM = \frac{2}{3} AF, \quad BM = \frac{2}{3} BG, \quad CM = \frac{2}{3} CH$$

(M is called the *centroid* of $\triangle ABC$. If you cut $\triangle ABC$ out of card then M would be the centre of mass of $\triangle ABC$.)

Answers

14.5A Straight Edge and Compasses Constructions

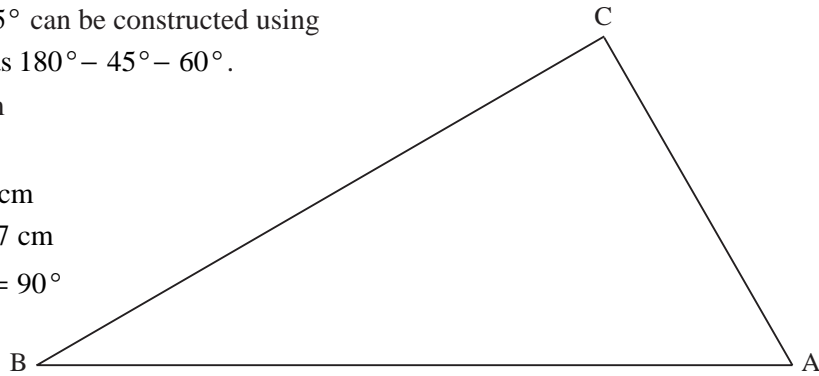
Teachers will need to set appropriate accuracy levels for their pupils when they attempt these constructions. Depending on the number of steps required to complete a given construction, accuracy levels of ± 1 mm or ± 2 mm, and $\pm 1^\circ$ or $\pm 2^\circ$ should be demanded.

Teachers will need to check pupils' constructions on an individual basis.

- The hint leads pupils to construct and combine two separate angles of 90° and 45° . Having done it this way, point out that the angle of 135° could also have been constructed using $180^\circ - 45^\circ$, which would require the construction of only one angle.
- An angle of 15° can be constructed using $45^\circ - 30^\circ$ or as $60^\circ - 45^\circ$.
- An angle of 75° can be constructed using $45^\circ + 30^\circ$ or as $180^\circ - 45^\circ - 60^\circ$.

- (a) Diagram

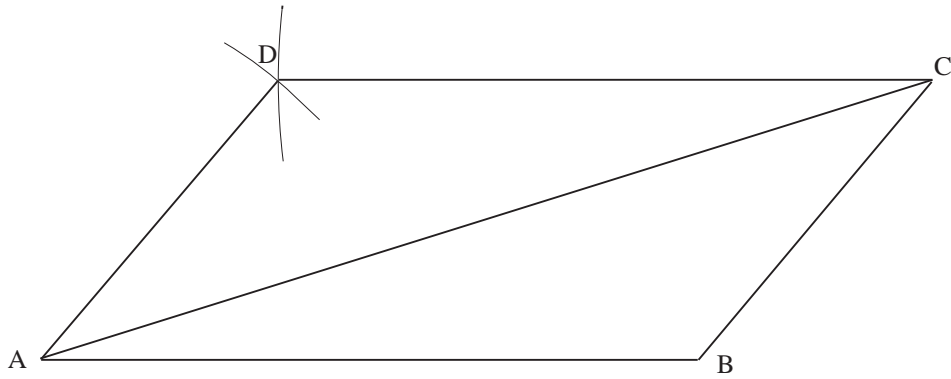
- $AC = 5$ cm
 $BC = 8.7$ cm
- $\angle ACB = 90^\circ$



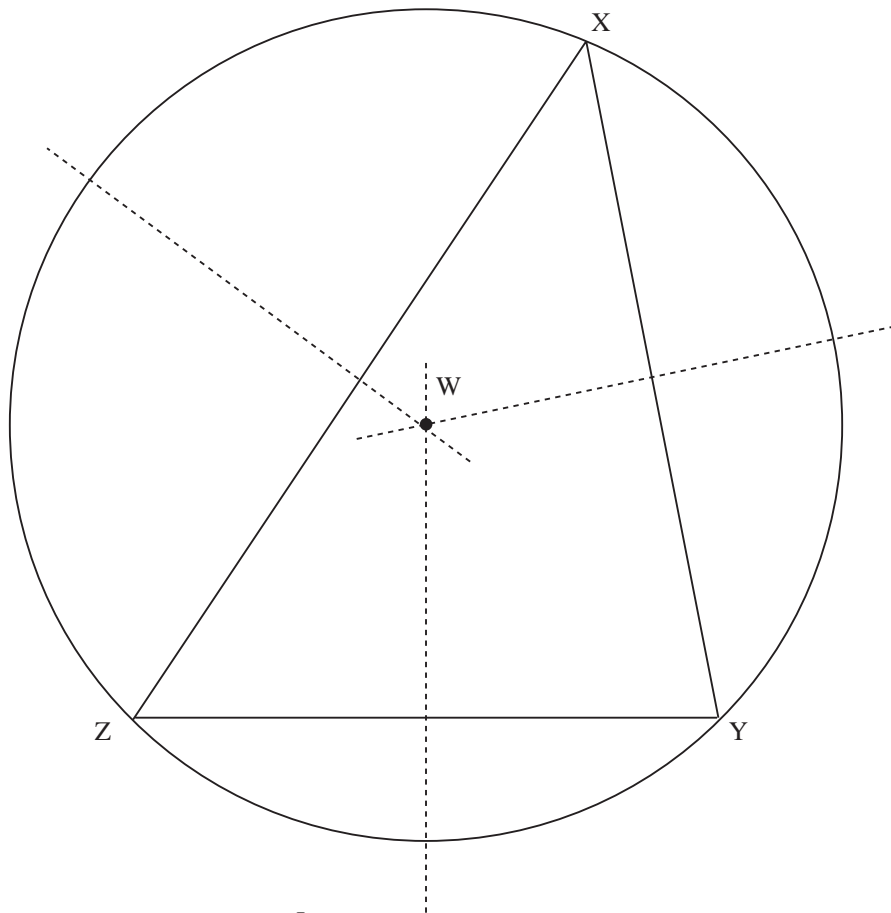
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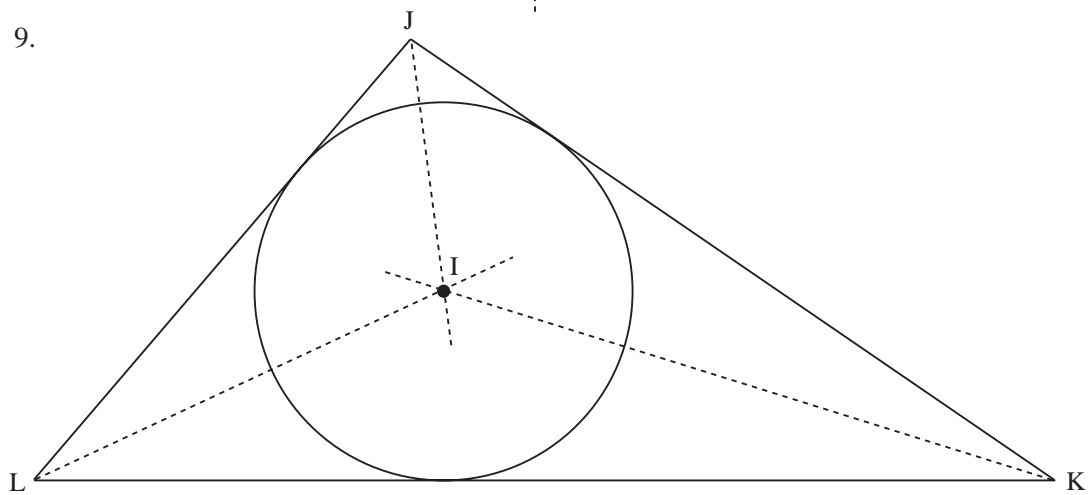
7. The figure drawn is a parallelogram.



- 8.

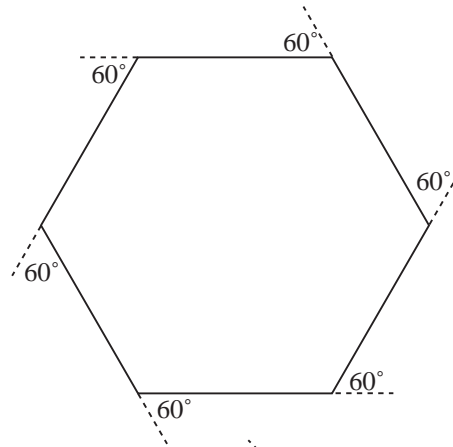


- 9.



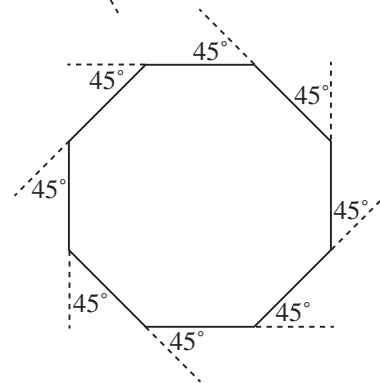
10. (a) To construct a regular hexagon requires repetition 5 times of the following two steps:

- measure the length of the edge
- at one end of the edge, measure an angle of 60° .



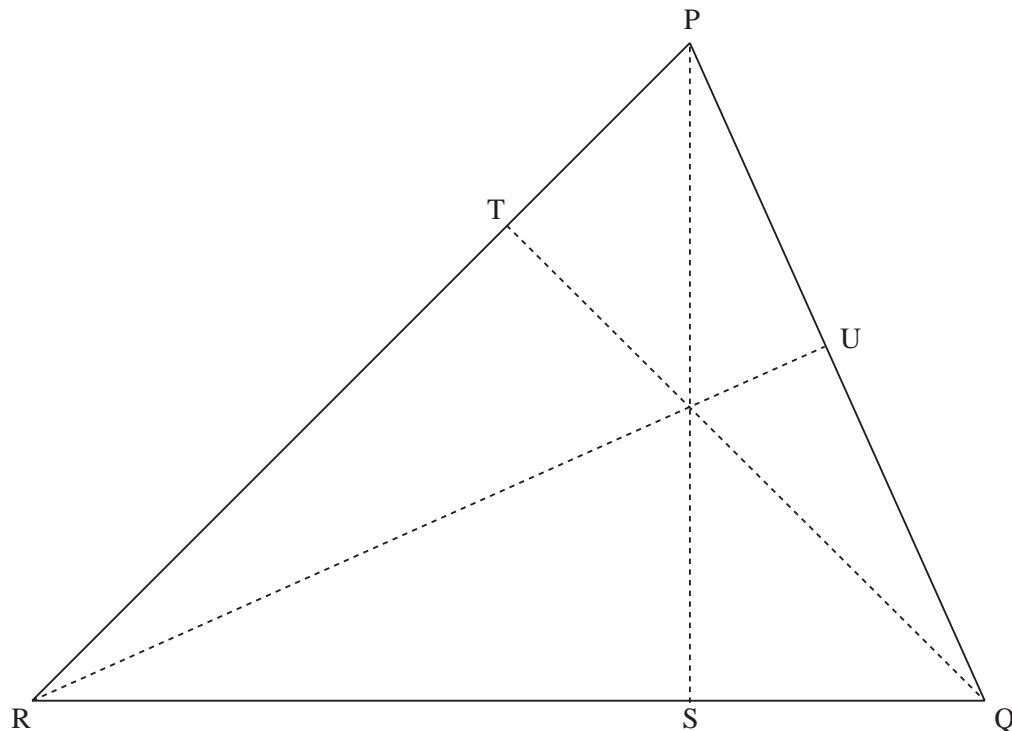
(b) To construct a regular octagon requires repetition 7 times of the following two steps:

- measure the length of the edge
- at one end of the edge, measure an angle of 45° .



N.B. There are equally acceptable variations on these two constructions.

11.



12.

