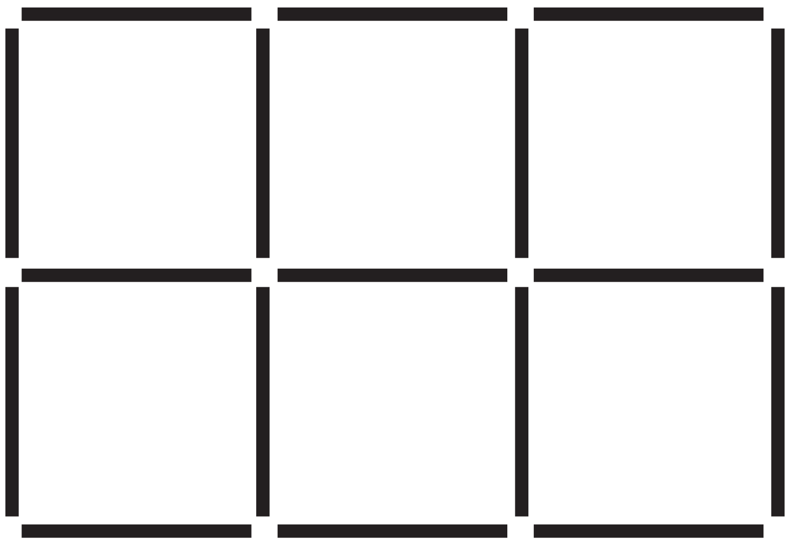
UNIT 2 *Formulae* Activities

Activities

ACTIVITY 2.1 *Rectangular Grids*

To make a rectangular grid of 2 rows of 3 squares, you would need 17 matches:



1. Investigate the number of matches needed to make rectangular grids of different sizes.

2. Find a rule/formula connecting the size of the grid (rows × columns) and the number of matches needed (*M*) for

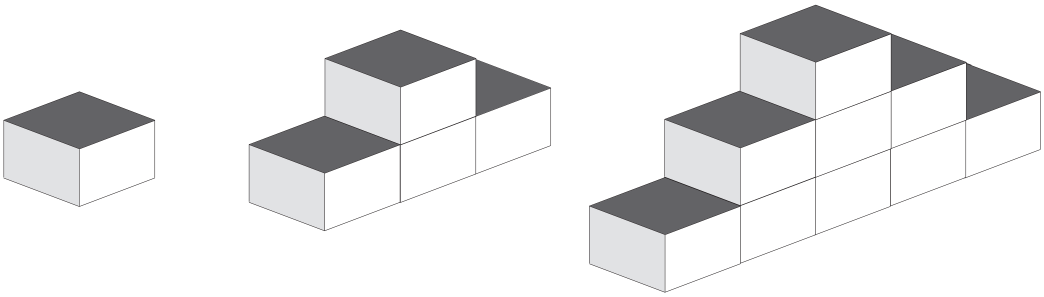
   (a) square $(n \times n)$ grids (b) rectangle $(n \times m)$ grids

   (The grid shown is 2 rows × 3 columns.)

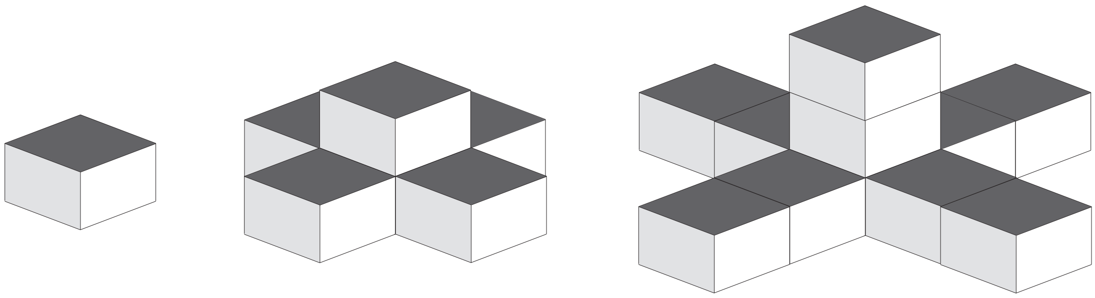
Extension
Extend your rule to other shapes, e.g. triangles.

ACTIVITY 2.2 *Monumental Towers*

Ancient civilizations used to make towers for memorials to remember their rulers.
Some might have looked like this ...



Others could have looked like this ...



The height of the tower depended on the importance of the ruler! The more important the ruler, the taller the tower.
Imagine you are the architect for an ancient civilization.

1. Design your own growing pattern of monuments. What sort if tower would you make? Make or draw a few of them.

The rulers were always changing their minds about how high their tower should be.

2. As the architect, you need to know how many bricks you need to make similar towers of any height, How can you work this out for the n^{th} tower for the two designs above?

ACTIVITY 2.3 *Physical Fitness*

P.E. experts have spent much time trying to find a single test which best measures physical fitness. Research has shown the connection between inactivity and coronary heart disease, and so emphasis has been placed on taking regular exercise. A key concept in testing physical fitness is that of a person's *pulse rate*, and in particular, how quickly it returns to its normal rate after excessive exercise.

Gallaher and Braihe Test

In this test, the exerciser steps up on to a bench (or stair) of height 18 inches for boys, 16 inches for girls. The tester shouts out " up - 2 - 3 - 4 " continuously to tell the exerciser when to step up and down; the "up" command comes every 2 seconds for 4 minutes. The exerciser continues for as long as possible up to four minutes.

The *pulse rate* is taken at the following times after the person stops exercising:

$1 - 1\frac{1}{2}$ minutes: $2 - 2\frac{1}{2}$ minutes; $3 - 3\frac{1}{2}$ minutes

In each case the number of beats per half minute is multiplied by 2 to give the pulse rate.

The *fitness index* is evaluated from

$$\text{Index} = \frac{50 \times T}{\left(p_1 + p_2 + p_3\right)}$$

where T is the duration of the exercise in seconds, and p_1, p_2, p_3 are the measured pulse rates. The grading is given in the table below.

Index	**Grade**
< 50	Very poor
50 - 60	Poor
60 - 70	Fair
70 - 80	Good
89 - 90	Excellent
> 90	Superb

Use this index to test your physical fitness, and keep testing over a period of time to monitor your progress.

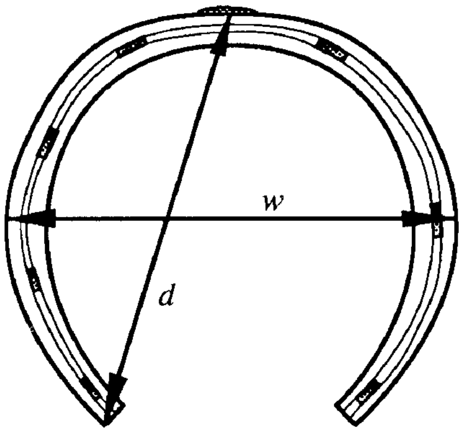
Extension

Construct a graph of T against total pulse rate to show the different grades.

ACTIVITY 2.4 *Horseshoes*

Farriers can buy ready-made horseshoes, but many make their own so that the fit is better. The horseshoe is made up from a straight strip of iron which is forged into the familiar horseshoe shape.

The problem that the farrier has to solve is to determine the length of iron (l) needed to make the appropriate sized shoe.



Two formulae are commonly used:

A $l = 2w + 2$

B $l = w + d + 1.5$

where w is the width of the shoe, in inches, and d is the diagonal measured in inches from the toe to the heel, as shown.

Data for Horseshoes

Horse	*Width* (w)	*Diagonal* (d)
Crystal	5.15	5.50
Honey	5.25	5.75
Frosty	5.50	5.80
William	5.75	6.00
Smudger	6.00	6.40

1. Determine the length of iron required to shoe each horse using
   (a) formula *A*, (b) formula *B*.

   Comparing the lengths required by each formula, what do you notice?

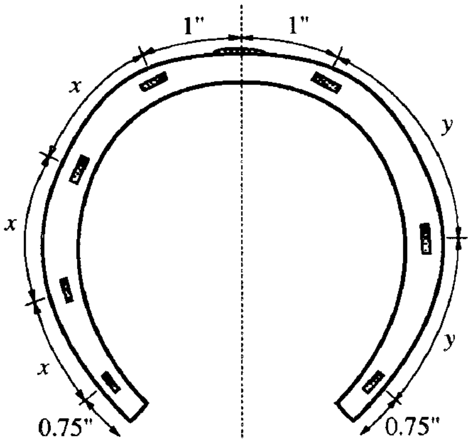
2. What is the condition on d and w which ensures that the two formula are the same?

3. The cost of the shoe is directly related to the length of iron, l, used. If *Crystal* has a new set of horseshoes 8 times a year, what saving is made in iron using formula *B* rather than formula *A*? What is the percentage saving?

Ready-made shoes often use 8 nail holes symmetrically placed so that they can be used on either foot, but a handmade shoe is made specifically for the left or right foot, with 4 nails on the outside half and 3 on the inside half.

4. The nails for *Honey's* front left shoe are shown opposite, with distances between them as illustrated.

   Determine the appropriate values to take for x and y.



ACTIVITY 2.5 *Hill Walking*

You all know that going up a hill slows down your pace, but by how much?
We will try to provide a mathematical model to describe this situation.

1. What factors will affect your average walking time up a hill?

   In this analysis, we will assume that the walking time depends on four factors:

   - horizontal distance travelled (map distance) d miles
   - vertical distance travelled (height of the hill) h feet
   - speed for horizontal walking x mph
   - speed for vertical climbing y feet per hour

The model will be assumed to be of the form

$$T = \frac{d}{x} + \frac{h}{y}$$

where T is the time taken.

Map distance d (miles)	Height climbed h (feet	Walking time T (hours)
12	1500	$5\frac{1}{2}$
15	2000	7

2. Use the experimental data given opposite to find appropriate values for x and y.

3. Use the model $T = \frac{d}{3} + \frac{h}{1000}$

   to determine how long it might take to climb *Mount Snowdon*, 3560 feet high, if the horizontal distance from your starting point is 3 miles.

4. If it takes you 8 hours to climb a hill and your map distance is 4 miles, estimate the height of the hill.

Extensions

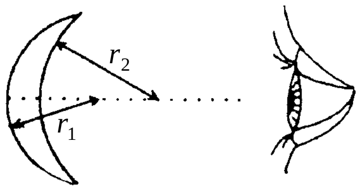
1. Apply this test to a local situation, using your own experimental data, to see if this model works in practice.

2. Can you produce a model for the time it takes to come *down* a hill?

ACTIVITY 2.6 *Spectacles Power*

It is now easy to choose spectacles from displays by trying them on and deciding which pair help your sight the most. One of the important parameters is the *power* of the lens which is measured in units called *diopters*.
This is in fact the *inverse* of the *focal length* of the lens and the formula is given by

$$P = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$



Here $P =$ power (in diopters)
$r_1 =$ radius of *outside* lens surface (in metres)
$r_2 =$ radius of *inside* lens surface (in metres)
$n =$ refraction index of the glass (about 1.5 to 1.6)

So, for example, if $r_1 = 10$ cm, $r_2 = 12$ cm and $n = 1.5$,

then $$P = (1.5-1)\left(\frac{1}{0.10} - \frac{1}{0.12}\right) = \frac{5}{6} \approx 1.5$$

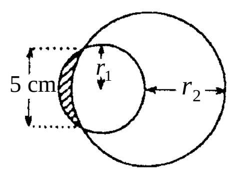
1. Find the powers (in diopters) for the following lenses:
   (a) $r_1 = 10$ cm, $r_2 = 11$ cm $n = 1.5$
   (b) $r_1 = 10$ cm, $r_2 = 13$ cm $n = 1.5$
   (c) $r_1 = 10$ cm, $r_2 = 15$ cm $n = 1.5$
   (d) $r_1 = 10$ cm, $r_2 = 20$ cm $n = 1.5$

2. For $r_1 = 10$ cm, $n = 1.5$, draw a graph of P against r_2.
   Use your graph to estimate the value of r_2 needed to give a power value of
   (a) 1 diopter (b) 2 diopters.

Using a graph to estimate values is a quick, but not necessarily accurate, way of solving the problem. In fact, we can rearrange the formula to give r_2 in terms of the other parameters.

3. (a) Find a formula for $\frac{1}{r_2}$ in terms of n, r_1 and P.
   (b) Hence deduce the formula for r_2 in terms of n, r_1 and P.

4. Use this formula to find r_2 when $r_1 = 10$ cm, $n = 1.5$ and P is
   (a) 2 diopters (b) 3 diopters.



Extensions

1. You can get some idea of the actual shape of a lens by drawing two circles of radius r_1 and r_2, cutting them out, overlapping them and moving them until an appropriate lens height (say 5 cm) is found.

2. With $n = 1.5$, plot graphs of r_1 against r_2 for power values 1, 1.5, 2 and 2.5.

ACTIVITY 2.7 *Bode's Law*

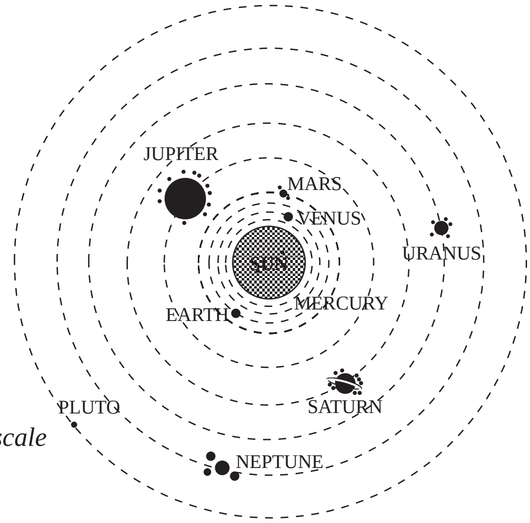
Although we regard our present understanding of the universe as advanced, there are still many mysteries. One of these is known as *Bode's Law*. It is an empirical law, which means it is based on observed data and not on a theoretical understanding.

The law, published in 1772, relates the ratio

$$\frac{R}{R_e} = \frac{\text{Distance of planet from Sun}}{\text{Distance of earth from Sun}}$$

to the number n, where

$n = 0$ - *Venus*
$n = 1$ - *Earth*
$n = 2$ - *Mars*
$n = 3$
$n = 4$ - *Jupiter*
$n = 5$ - *Saturn*
$n = 6$ - *Uranus*
$n = 7$ - *Neptune*
$n = 8$ - *Pluto*



Not to scale

1. For each planet, find the ratio R/R_e to one place decimal and plot a graph of

   R/R_e against n.

2. Bode's Law is given by the formula

   $$R/R_e = 0.4 + 0.3 \times 2^n$$

   Check the accuracy of the values from this formula with the actual values.

   Do they all agree?

3. What value should be taken for n for Mercury?

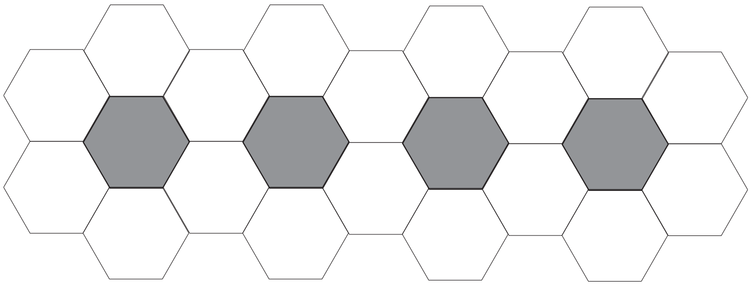
4. A large number of asteroids are found at distance 433.8×10^6 km from the Sun.

   What can you conjecture from this?

5. Does the data support the view that Neptune and Pluto were once a single planet?

Planet	**Distance from Sun** (in millions of km)
Mercury	57.9
Venus	108.2
Earth	149.6
Mars	227.9
Jupiter	778.3
Saturn	1427.0
Uranus	2870.0
Neptune	4497.0
Pluto	5907.0

ACTIVITY 2.8 *Flower Beds*

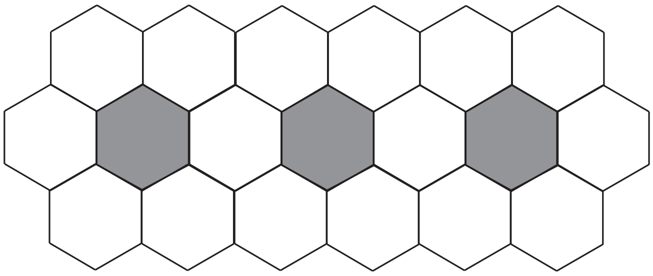


The council wish to create 10 flower beds and surround them with hexagonal paving slabs according to the pattern shown above. (In this pattern 18 slabs surround 4 flower beds.)

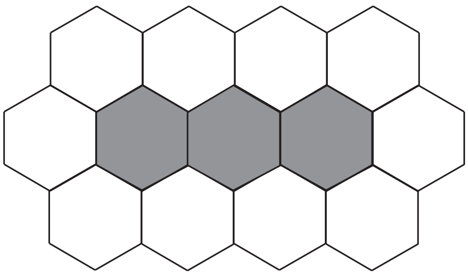
1. How many slabs will the council need?

2. Find a formula that the council can use to decide the number of slabs needed for any number of flower beds.

There are many other ways of surrounding flower beds with hexagonal paving slabs.



3. Invent your own examples and find general formulae for the number of slabs needed for any number of them.



ACTIVITY 2.9 *Pendulums*

You can easily observe that the time of swing of a pendulum depends on the length of the pendulum – the longer the length, the longer the swing. But how exactly does the pendulum time, T, (the time for a complete cycle) and its length, L, correspond?

Perform the following experiments, and then analyse the data to answer this question.

1. Time the swing, (a complete cycle) of pendulums of lengths 1 m, 2 m, 3 m, 4 m, etc. (You need to do this as accurately as possible, so it is probably best to time 5 swings and then divide by 5.)

2. Plot a graph of T (on the y-axis) against lengths, L, on the x-axis. What sort of relationship does this suggest?

3. Plot a graph of T against $\sqrt{L}$. What shape is this?

Your graph in question 3 should be approximately a straight line, so the formula is of the form

$$T = k\sqrt{L}$$

The constant k is the shape of the straight line.

4. Estimate your value of k from your data.

   Use your formula to predict then length of pendulum that has a period of 2 seconds.

   Check your result experimentally.

ACTIVITY 2.10 *Chill Factor*

When the temperature drops to near zero ($0\ °C$), UK weather forecasters give both the expected air temperature and the wind chill factor. This is because a person will feel colder when there is a significant wind blowing, particularly at low temperatures.

The wind chill temperature is particularly important for mountain walkers, and for farmers, who might need to protect their livestock, e.g. new born lambs.

The formula gives the WIND CHILL temperature, $T\ °C$, in terms of

AIR temperature, $t\ °C$ and WIND speed, v mph

and takes the form $T = 33 + f(v)\,(t - 33)$, where the wind effect function, $f(v)$, is given by

$$f(v) = 0.45 + 0.29\sqrt{v} - 0.02\,v$$

Although this looks complicated, it is straightforward to evaluate.

For example, if $t = -5\ °C$ and $v = 10$ mph, then

$$f(10) = 0.45 + 0.29\sqrt{10} - (0.02)10 = 1.167$$

and $$T = 33 + (1.167)\,(-5 - 33) = -11.3\ °C$$

1. Find the wind chill temperature when
   (a) $t = 0\ °C,\ v = 10$ mph (b) $t = -10\ °C,\ v = 5$ mph (c) $t = 2\ °C,\ v = 20$ mph

2. Calculate $f(5)$. What does this imply about the wind chill temperature?

The sensations felt by people can be characterised by the following wind chill temperature ranges.

***T* °C**	**Sensation**	
less than 25	Freezing cold	(exposed flesh freezes)
$-25 < T < -15$	Bitterly cold	(full protective clothing required)
$-15 < T < -10$	Very cold	(some protective clothing needed)
$-10 < T < -0$	Cold	(plenty of warm clothes needed)
$0 < T < -10$	Very cool	(some warm clothes needed)

We will find out what combinations of air temperature, t and wind speed, v, produce wind chill temperatures on the *Freezing cold* and *Bitterly cold* ranges.

3. Show that when $T = -25\ °C$, then $t = 33 - \dfrac{58}{f(v)}$. For $v = 10, 20, ..., 50$, find the value of t, and sketch the graph of v against t.

4. Show that when $T = -15\ °C$, then $t = 33 - \dfrac{58}{f(v)}$.
   Repeat the procedure in question 3.

5. From your graph, find which sensations are produced by the following combinations of air temperature and wind speed.
   (a) $t = -15\ °C,\ v = 15$ mph (b) $t = -5\ °C,\ v = 50$ mph (c) $t = 0\ °C,\ v = 40$ mph

ACTIVITY 2.11 *Heptathlon*

The heptathlon is a competition for women athletes who take part in *seven* separate events (usually over a two-day period).

100 m Hurdles	High Jump	Shot Put
200 m	Long Jump	Javelin
800 m		

For each activity, there is a points system of scoring, based on the idea that a very good competitor will score about 1000 points in each event.

For *track* events, the points are given by the formula

$$P = a(b - M)^c$$

where *a*, *b* and *c* are given by the table opposite and *M* is the time in seconds taken by the competitor.

Event	*a*	*b*	*c*
100 m Hurdles	9.23076	26.70	1.835
200 m	4.99087	42.50	1.81
800 m	0.11193	254.00	1.88

In the 1988 Seoul Olympic Games, the following performances were achieved.

Name	100 m Hurdles s	High Jump cm	Shot Put m	200 m s	Long Jump cm	Javelin m	800 m s
A. Behmer	13.20	183	14.20	23.105	668	44.54	124.20
N. Chaubenkova	13.51	174	14.76	23.93	632	47.46	127.90
S. John	12.85	180	16.23	23.65	671	42.56	126.14
J. Joyner-Kersee	12.69	186	15.80	22.56	727	45.66	128.51
R. Sablovskaite	13.61	180	15.23	23.92	625	42.78	132.24

For the 100 m Hurdles, the points score of A. Behmer is

$$\begin{aligned} 9.23076(26.70 - 13.20)^{1.835} &= 9.23076 \times (13.50)^{1.835} \\ &= 9.23076 \times 118.621 \\ &= 1094 \text{ points} \end{aligned}$$

Note:
Points score is always rounded *down*.

1. Find the points scored by A. Behmer for the 200 m and 800 m.

For *field* events, the model is modified to $P = a(M - b)^c$ Why does it take this form?

2. Given the parameters opposite for the field events, find the points scored by A. Behmer for each event and hence her *total* for the heptathlon.

Event	*a*	*b*	*c*
High Jump	1.84523	75.00	1.348
Long Jump	0.188807	210.00	1.41
Shop Put	56.0211	1.50	1.05
Javelin	15.9803	3.80	1.04

3. Who won the Gold Medal?

Extension
For each event, find the time or distance needed to score 1000 points.

ACTIVITY 2.12 *Algebraic Manipulation*

Find the path from START to FINISH that passes only through boxes with correct factorisations, expansions or simplifications.

Finish

$(r+s)^2 = r^2 + rs + 2rs$	$(3x+y)(z-q) = 3xz - 3xq + yz - yq$	$\frac{9}{2p} + \frac{8}{2p} = \frac{17}{4p}$	$(a-2b)(a-3b) = a^2 - ab + 6b^2$
$4p^2 + 4pr + r^2 = (2p+2r)^2$	$ps - 2p^2 - 3rs + 6rp = (s-2p)(p-3r)$	$\frac{3+y}{y+2} - \frac{3}{y+2} = \frac{y}{y+2}$	$\left(a + \frac{3}{4}b\right)\left(a - \frac{3}{4}b\right) = a^2 - \frac{9}{16}b^2$
$\frac{7m}{5n} - \frac{4m}{3n} = \frac{3m}{2n}$	$(h+2k)(h-2k) = h^2 - 2k^2$	$\frac{12k}{5l} \times \frac{10l}{3k} = 8k$	$\frac{15x}{22y} \div \frac{5x}{11y} = \frac{3}{2}$
$6x^2 - 17x - 10 = (2x+1)(3x-10)$	$\frac{2}{p} + \frac{3}{5p} = \frac{13}{5p}$	$(4e-g)(2k+l) = 8ek + 4el - 2gk - gl$	$bz - az + by - ay = (b-a)(z+y)$
$\frac{18ab}{3bc} = \frac{6a}{c}$	$12k^2 + 3km + 4kl + lm = (2k+l)(6k+m)$	$(2p+r)(p-m) = 2p^2 + 2pm + rp - rm$	$\frac{12r^2}{5p} \div \frac{5p^2}{6r} = 2pr$

Start