12 Number Patterns

12.1 Simple Number Patterns

A list of numbers which form a pattern is called a *sequence*. In this section, straightforward sequences are continued.

Worked Example 1

Write down the next three numbers in each sequence.

(a) 2, 4, 6, 8, 10, . . . (b) 3, 6, 9, 12, 15, . . .

Solution

(a) This sequence is a list of even numbers, so the next three numbers will be

$$12,\ 14,\ 16.$$

(b) This sequence is made up of the multiples of 3, so the next three numbers will be

$$18,\ 21,\ 24.$$

Worked Example 2

Find the next two numbers in each sequence.

(a) 6, 10, 14, 18, 22, . . . (b) 3, 8, 13, 18, 23, . . .

Solution

(a) For this sequence the difference between each term and the next term is 4.

$$\begin{array}{lccccccccccc} \textit{Sequence} & 6, & & 10, & & 14, & & 18, & & 22, & \ldots \\ \textit{Difference} & & 4 & & 4 & & 4 & & 4 & & \end{array}$$

So 4 must be added to obtain the next term in the sequence. The next two terms are

$$22 + 4 = 26$$

and $$26 + 4 = 30,$$

giving $$6,\ 10,\ 14,\ 18,\ 22,\ 26,\ 30,\ \ldots$$

(b) For this sequence, the difference between each term and the next is 5.

$$\begin{array}{lccccccccccc} \textit{Sequence} & 3, & & 8, & & 13, & & 18, & & 23, & \ldots \\ \textit{Difference} & & 5 & & 5 & & 5 & & 5 & & \end{array}$$

Adding 5 gives the next two terms as

$$23 + 5 = 28$$

and $$28 + 5 = 33,$$

giving $$3,\ 8,\ 13,\ 18,\ 23,\ 28,\ 33,\ \ldots$$

Exercises

1. Write down the next four numbers in each list.

(a) 1, 3, 5, 7, 9, ...
(b) 4, 8, 12, 16, 20, ...
(c) 5, 10, 15, 20, 25, ...
(d) 7, 14, 21, 28, 35, ...
(e) 9, 18, 27, 36, 45, ...
(f) 6, 12, 18, 24, 30, ...
(g) 10, 20, 30, 40, 50, ...
(h) 11, 22, 33, 44, 55, ...
(i) 8, 16, 24, 32, 40, ...
(j) 20, 40, 60, 80, 100, ...
(k) 15, 30, 45, 60, 75, ...
(l) 50, 100, 150, 200, 250, ...

2. Find the difference between terms for each sequence and hence write down the next two terms of the sequence.

(a) 5, 8, 11, 14, 17, ...
(b) 2, 10, 18, 26, 34, ...
(c) 7, 12, 17, 22, 27, ...
(d) 6, 17, 28, 39, 50, ...
(e) 8, 15, 22, 29, 36, ...
(f) $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, ...
(g) 4, 13, 22, 31, 40, ...
(h) 26, 23, 20, 17, 14, ...
(i) 20, 16, 12, 8, 4, ...
(j) 18, 14, 10, 6, 2, ...
(k) 11, 8, 5, 2, -1, ...
(l) -5, -8, -11, -14, -17, ...

3. In each part, find the answers to (i) to (iv) *with* a calculator and the answer to (v) *without* a calculator.

(a)
(i) $2 \times 11 = ?$
(ii) $22 \times 11 = ?$
(iii) $222 \times 11 = ?$
(iv) $2222 \times 11 = ?$
(v) $22222 \times 11 = ?$

(b)
(i) $99 \times 11 = ?$
(ii) $999 \times 11 = ?$
(iii) $9999 \times 11 = ?$
(iv) $99999 \times 11 = ?$
(v) $999999 \times 11 = ?$

(c)
(i) $88 \times 11 = ?$
(ii) $888 \times 11 = ?$
(iii) $8888 \times 11 = ?$
(iv) $88888 \times 11 = ?$
(v) $8888888 \times 11 = ?$

(d)
(i) $7 \times 9 = ?$
(ii) $7 \times 99 = ?$
(iii) $7 \times 999 = ?$
(iv) $7 \times 9999 = ?$
(v) $7 \times 999999 = ?$

4. (a) (i) Complete the following number pattern:

$$\begin{array}{ccc} 11 & = & 11 \\ 11 \times 11 & = & 121 \\ 11 \times 11 \times 11 & = & ? \\ ? & = & ? \end{array}$$

(ii) Look at the numbers in the right hand column. Write down what you notice about these numbers.

(b) Use your calculator to work out the next line of the pattern. What do you notice now?

(NEAB)

5.

$$\begin{aligned} 9 \times 2 &= 18 \\ 9 \times 3 &= 27 \\ 9 \times 4 &= 36 \\ 9 \times 5 &= 45 \\ 9 \times 6 &= 54 \\ 9 \times 7 &= 63 \\ 9 \times 8 &= 72 \\ 9 \times 9 &= 81 \end{aligned}$$

(a) (i) Complete the statement,

The units digits in the right hand column, in order, are 8, 7, 6, . . .

(ii) Complete the statement,

The tens digits in the right-hand column, in order, are 1, 2, 3, . . .

(iii) What is the connection between the answers to parts (i) and (ii)?

(b) The numbers in the right-hand column go up by 9 each time. What else do you notice about these numbers?

(MEG)

12.2 Recognising Number Patterns

This section looks at how the terms of a sequence are related. For example the *Fibonacci* sequence:

$$1, \ 1, \ 2, \ 3, \ 5, \ 8, \ 13, \ \ldots$$

is obtained by adding together two consecutive terms to obtain the next term.

$$\begin{aligned} 1 + 1 &= 2 \\ 1 + 2 &= 3 \\ 2 + 3 &= 5 \\ 3 + 5 &= 8 \\ 5 + 8 &= 13 \end{aligned}$$

Worked Example 1

The sequence of square numbers is

$$1, \ 4, \ 9, \ 16, \ 25, \ 36, \ \ldots$$

Explain how to obtain the next number in the sequence.

Solution

The next number can be obtained in one of two ways.

(1) The sequence could be written,

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ 5^2,\ 6^2,\ \ldots$$

and so the next term would be $7^2 = 49$.

(2) Calculating the differences between the terms gives

Sequence	1,		4,		9,		16,		25,		36, ...
Difference		3		5		7		9		11	

The difference increases by 2 each time so the next term would be $36 + 13 = 49$.

Worked Example 2

Describe how to obtain the next term of each sequence below.

(a) 3, 10, 17, 24, 31, ...   (b) 3, 6, 11, 18, 27, ...

(c) 1, 5, 6, 11, 17, 28, ...

Solution

(a) Finding the differences between the terms gives

Sequence	3,		10,		17,		24,		31, ...
Difference		7		7		7		7	

All the differences are the same so each term can be obtained by adding 7 to the previous term. The next term would be

$$31 + 7 = 38.$$

(b) Here again the differences show a pattern.

Sequence	3,		6,		11,		18,		27, ...
Difference		3		5		7		9	

Here the differences increase by 2 each time, so to find further terms add 2 more than the previous difference. The next term would be

$$27 + 11 = 38.$$

(c) The differences again help to see the pattern for this sequence.

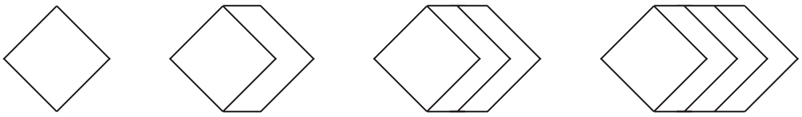
Sequence	1,		5,		6,		11,		17,		28, ...
Difference		4		1		5		6		11	

If the first difference, 4, is ignored, the remaining pattern of the differences is the same as the sequence itself. This shows that the difference between any two terms is equal to the previous term. So a new term is obtained by adding together the two previous terms. The next term of the sequence will be

$$17 + 28 = 45.$$

Worked Example 3

A sequence of shapes is shown below.



Write down a sequence for the number of line segments and explain how to find the next number in the sequence.

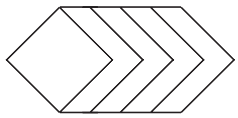
Solution

The sequence for the number of line segments is

$$4, \ 8, \ 12, \ 16, \ \ldots$$

The difference between each pair of terms is 4, so to the previous term add 4. Then the next term is $16 + 4 = 20$.

This corresponds to the shape opposite, which has 20 line segments.



Exercises

1. Find the next two terms of each sequence below, showing the calculations which have to be done to obtain them.

   (a) 5, 11, 17, 23, 29, . . .
   (b) 6, 10, 15, 21, 28, . . .
   (c) 22, 19, 16, 13, 10, . . .
   (d) 30, 22, 15, 9, 4, . . .
   (e) 50, 56, 63, 71, 80, . . .
   (f) 2, 2, 4, 8, 14, 22, . . .
   (g) 6, 11, 16, 21, 26, . . .
   (h) 0, 3, 8, 15, 24, . . .
   (i) 3, 6, 12, 24, 48, . . .
   (j) 1, 4, 10, 22, 46, . . .
   (k) $4, \ -1, \ -11, \ -26, \ -46, \ldots$

2. A sequence of numbers is

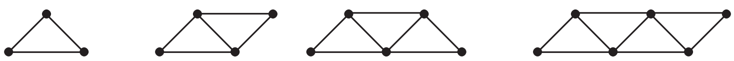
   $$1, \ 8, \ 27, \ 64, \ 125, \ \ldots$$

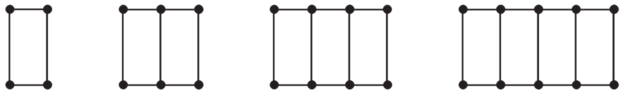
   By considering the differences between the terms, find the next two terms.

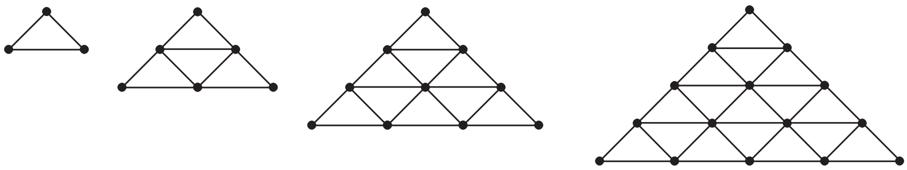
3. Each sequence of shapes below is made up of lines which join two points.
   For each sequence:

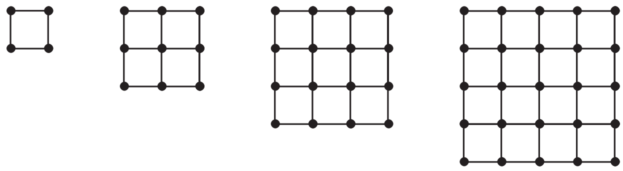
   (i) write down the number of lines, as a sequence;
   (ii) explain how to obtain the next term of the sequence;
   (iii) draw the next shape and check your answer.

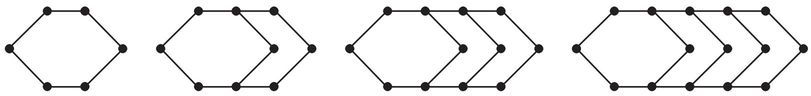
   (a)



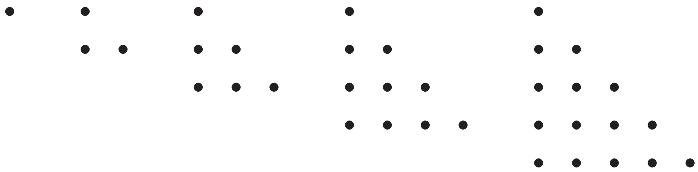
(b) 

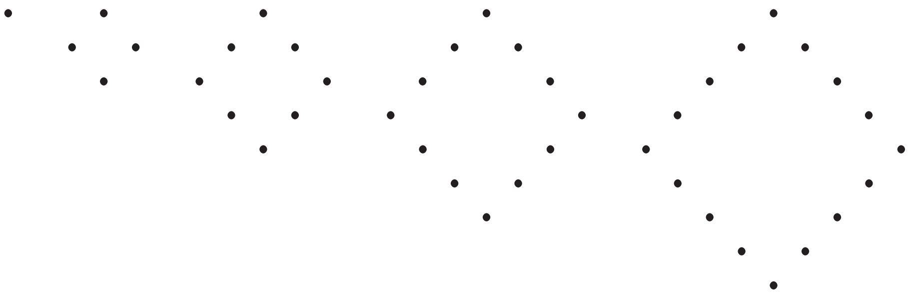
(c) 

(d) 

(e) 

4. Write down a sequence for the number of dots in each pattern. Then explain how to get the next number.

(a) 

(b) 

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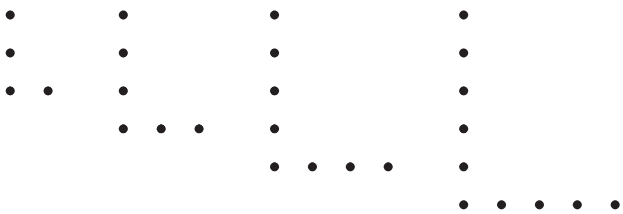
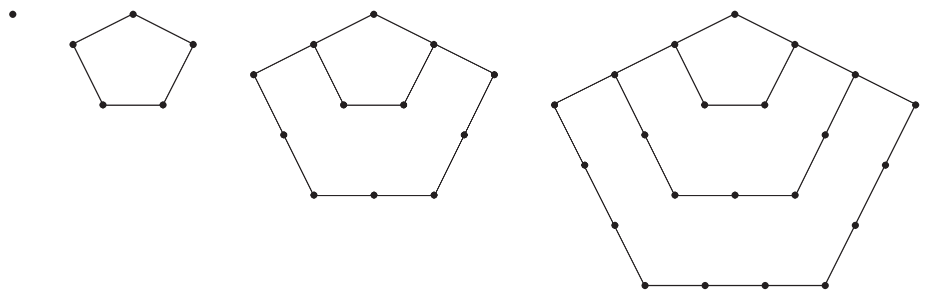
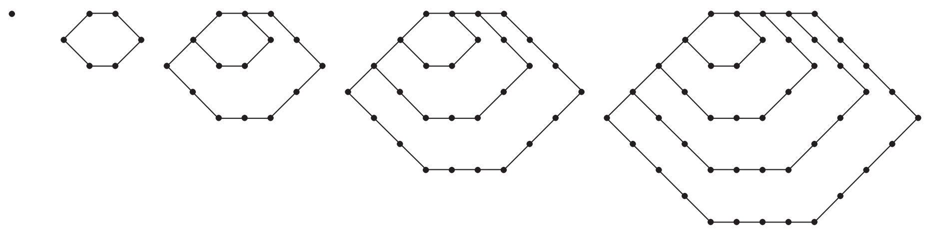
(d) 

(e) 

(f) 

5. (a) Describe how the sequence

$$1,\ 4,\ 9,\ 16,\ 25,\ 36.\ \ldots$$

is formed.

(b) What is the relationship between the sequence in (a) and the sequences below? For each sequence explain how to calculate the terms.

(i) $3,\ 6,\ 11,\ 18,\ 27,\ 38,\ \ldots$ (ii) $-1,\ 2,\ 7,\ 14,\ 23,\ 34,\ \ldots$

(iii) $2,\ 8,\ 18,\ 32,\ 50,\ 72,\ \ldots$ (iv) $4,\ 16,\ 36,\ 64,\ 100,\ 144,\ \ldots$

6. (a) Describe how the sequences below are related.

(i) $1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots$

(ii) $4,\ 4,\ 5,\ 6,\ 8,\ 11,\ 16,\ 24,\ \ldots$

(iii) $-1,\ -1,\ 0,\ 1,\ 3,\ 6,\ 11,\ 19,\ \ldots$

(b) Describe how to find the next term of each sequence.

7. A computer program prints out the following numbers.

$$1 \quad 2 \quad 4 \quad 8 \quad 11 \quad 16 \quad 22$$

When one of these numbers is changed, the numbers will form a pattern.

Circle the number which has to be changed and correct it.

Give a reason why your numbers now form a pattern.

(SEG)

8. Here are the first four numbers of a number pattern.

$$7,\ 14,\ 21,\ 28,\ \ldots$$

(a) Write down the next two numbers in the pattern.

(b) Describe, in words, the rule for finding the next number in the pattern.

(LON)

9. (a) To generate a sequence of numbers, Paul multiplies the previous number in the sequence by 3, then subtracts 1.

Here are the first four numbers of his sequence.

$$1,\ 2,\ 5,\ 14,\ \ldots$$

Find the next two numbers in the sequence.

(b) Here are the first four numbers of the sequence of cube numbers.

$$1,\ 8,\ 27,\ 64,\ \ldots$$

Find the next two numbers in the sequence.

(MEG)

10.

Row 1	12		19		28		39		52		q
Row 2		7		9		11		13		p	

The numbers in *Row 2* of the above pattern are found by using pairs of numbers from *Row 1*. For example,

$$7 = 19 - 12$$
$$9 = 28 - 19$$

(a) By considering the sequence in *Row 2*, write down the value of p.

(b) Find the value of q.

(MEG)

11. (a) A number pattern begins

$$4,\ 8,\ 12,\ 16,\ 20,\ 24,\ \ldots$$

Describe the number pattern.

(b) Another number pattern begins

$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ \ldots$$

(i) Describe this number pattern.

(ii) What is the next number in this pattern?

Each number in this pattern is changed to make a new number pattern.
The new number pattern begins

$$-1,\ 2,\ 7,\ 14,\ 23,\ 34,\ \ldots$$

(iii) What is the next number in the new pattern?
Explain how you found your answer.

(SEG)

12. (a) (i) Write down the multiples of 5, from 5 to 40.

(ii) Describe the pattern of the units digits.

(b) Sequence P is

$$3,\ 6,\ 9,\ 12,\ 15,\ 18,\ 21,\ \ldots$$

Explain how sequence P is produced.

(c) Copy and complete the table below.

Sequence P	$\rightarrow$	Add 1 and then multiply by 2		$\rightarrow$	*Sequence Q*
3	$\rightarrow$	$3 + 1 = 4$,	$4 \times 2 = 8$	$\rightarrow$	8
6	$\rightarrow$	$6 + 1 = 7$,	$7 \times 2 = 14$	$\rightarrow$	14
9	$\rightarrow$	?	?	$\rightarrow$	?
12	$\rightarrow$	?	?	$\rightarrow$	?
15	$\rightarrow$	?	?	$\rightarrow$	?
18	$\rightarrow$	?	?	$\rightarrow$	?

(d) (i) Find the next two terms in the sequence

$$1,\ 4,\ 10,\ 19,\ 31,\ 46,\ 64,\ \dots$$

(ii) Explain how you obtained your answer to part (d) (i).

(MEG)

12.3 Extending Number Patterns

A formula or rule for extending a sequence can be used to work out any term of a sequence without working out all the terms. For example, the 100th term of the sequence,

$$1,\ 4,\ 7,\ 10,\ 13,\ \dots$$

can be calculated as 298 without working out any other terms.

Worked Example 1

Find the 20th term of the sequence

$$8,\ 16,\ 24,\ 32,\ \dots$$

Solution

The terms of the sequence can be obtained as shown below.

$$\begin{aligned} \text{1st term} &= 1 \times 8 = 8 \\ \text{2nd term} &= 2 \times 8 = 16 \\ \text{3rd term} &= 3 \times 8 = 24 \\ \text{4th term} &= 4 \times 8 = 32 \end{aligned}$$

This pattern can be extended to give

$$\text{20th term} = 20 \times 8 = 160$$

Worked Example 2

Find the 10th and 100th terms of the sequence

$$3,\ 5,\ 7,\ 9,\ 11,\ \dots$$

Solution

The terms above are given by

$$\begin{aligned}
\text{1st term} &= 3 \\
\text{2nd term} &= 3+2 = 5 \\
\text{3rd term} &= 3+2\times 2 = 7 \\
\text{4th term} &= 3+3\times 2 = 9 \\
\text{5th term} &= 3+4\times 2 = 11
\end{aligned}$$

This can be extended to give

$$\begin{aligned}
\text{10th term} &= 3+9\times 2 = 21 \\
\text{100th term} &= 3+99\times 2 = 201.
\end{aligned}$$

Worked Example 3

Find the 20th term of the sequence

$$2,\ 5,\ 10,\ 17,\ 26,\ 37,\ \ldots$$

Solution

The terms of this sequence can be expressed as

$$\begin{aligned}
\text{1st term} &= 1^2+1 \\
\text{2nd term} &= 2^2+1 \\
\text{3rd term} &= 3^2+1 \\
\text{4th term} &= 4^2+1 \\
\text{5th term} &= 5^2+1
\end{aligned}$$

Extending the pattern gives

$$\text{20th term} = 20^2+1 = 401.$$

Exercises

1. Find the 10th and 20th terms of each sequence below.

(a) $4,\ 8,\ 12,\ 16,\ 20,\ \ldots$

(b) $5,\ 10,\ 15,\ 20,\ 25,\ \ldots$

(c) $11,\ 21,\ 31,\ 41,\ 51,\ \ldots$

(d) $7,\ 9,\ 11,\ 13,\ 15,\ \ldots$

(e) $5,\ 9,\ 13,\ 17,\ 21,\ \ldots$

(f) $20,\ 19,\ 18,\ 17,\ 16,\ \ldots$

(g) $50,\ 44,\ 38,\ 32,\ 26,\ \ldots$

(h) $22,\ 25,\ 28,\ 31,\ \ldots$

(i) $8,\ 7,\ 6,\ 5,\ 4,\ \ldots$

(j) $-4,\ 0,\ 4,\ 8,\ 12,\ \ldots$

(k) $7,\ 12,\ 17,\ 22,\ 27,\ \ldots$

(l) $3,\ -2,\ -7,\ -12,\ \ldots$

2. (a) Find the 10th term for each of the two sequences below.

   (i) 3, 6, 11, 18, 27, . . .   (ii) 5, 6, 7, 8, 9, . . .

   (b) Hence find the 10th term of the sequences,

   (i) 8, 12, 18, 26, 36, . . .   (ii) 15, 36, 77, 144, 243, . . .

   (iii) -2 0, 4, 10, 18, . . .

3. (a) Find the 20th term of the sequences below.

   (i) 4, 9, 14, 19, 24, . . .   (ii) 3, 5, 7, 9, 11, . . .

   (b) Use your answers to (a) to find the 20th term of the sequence

   12, 45, 98, 171, 264, . . .

4. (a) By considering the two sequences

   1, 4, 9, 16, 25, . . .
   1, 2, 3, 4, 5, . . .

   find the 10th term of the sequence

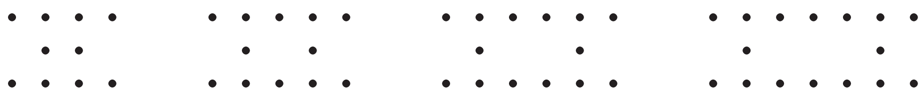
   0, 2, 6, 12, 20, . . .

   (b) Find the 10th term of the sequence

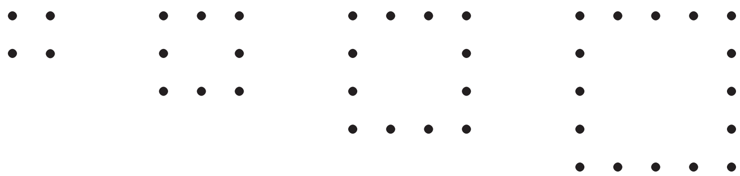
   0, 6, 24, 60, 120, . . .

5. For each sequence of shapes below find the number of dots in the 10th shape.

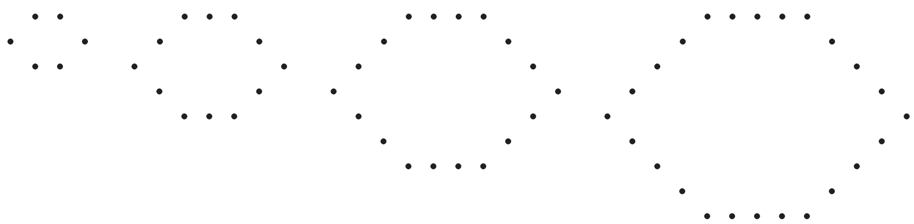
   (a)



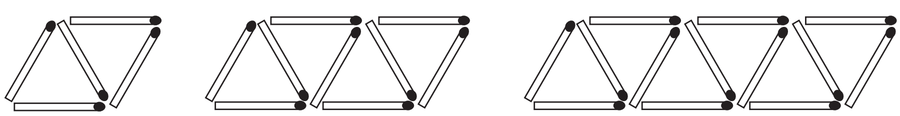
   (b)



   (c)



6. Patterns of triangles are made using sticks. The first three patterns are drawn below.



Pattern 1   *Pattern 2*   *Pattern 3*

Pattern number	1	2	3
Number of sticks	5	9	13

(a) How many sticks has *Pattern 4*?

(b) A pattern needs 233 sticks. What is the number of this pattern?

(c) (i) How many sticks are needed to make *Pattern 100*?

(ii) Explain how you found your answer.

(SEG)

12.4 Formulae and Number Patterns

This section considers how the terms of a sequence can be found using a formula and how a formula can be found for some simple sequences. The terms of a sequence can be described as

$$\boxed{u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ \ldots}$$

where u_1 is the first term, u_2 is the second term and so on. Consider the sequence

$$1,\ 4,\ 9,\ 16,\ 25,\ \ldots$$

$$\begin{aligned} u_1 &= 1 = 1\times 1 \\ u_2 &= 4 = 2\times 2 \\ u_3 &= 9 = 3\times 3 \\ u_4 &= 16 = 4\times 4 \\ u_5 &= 25 = 5\times 5 \end{aligned}$$

This sequence can be described by the general formula

$$u_n = n^2.$$

Worked Example 1

Find the first 5 terms of the sequence defined by the formula $u_n = 3n + 6$.

Solution

$$\begin{aligned} u_1 &= 3\times 1+6 = 9 \\ u_2 &= 3\times 2+6 = 12 \\ u_3 &= 3\times 3+6 = 15 \\ u_4 &= 3\times 4+6 = 18 \\ u_5 &= 3\times 5+6 = 21 \end{aligned}$$

So the sequence is $9,\ 12,\ 15,\ 18,\ 21,\ \ldots$

Note that the terms of the sequence increase by 3 each time and that the formula contains a '$3n$'.

Worked Example 2

Find the first 5 terms of the sequence defined by the formula

$$u_n = 5n - 4.$$

Solution

$$\begin{aligned} u_1 &= 5 \times 1 - 4 = 1 \\ u_2 &= 5 \times 2 - 4 = 6 \\ u_3 &= 5 \times 3 - 4 = 11 \\ u_4 &= 5 \times 4 - 4 = 16 \\ u_5 &= 5 \times 5 - 4 = 21 \end{aligned}$$

So the sequence is

$$1, \ 6, \ 11, \ 16, \ 21, \ \ldots$$

Here the terms increase by 5 each time and the formula contains a '$5n$'.

In general, if the terms of a sequence increase by a constant amount, d, each time, then the sequence will be defined by the formula

$$\boxed{u_n = dn + c}$$

where c is a constant number.

Worked Example 3

Find a formula to describe each of the sequences below.

(a) 13, 20, 27, 34, 41, 48, . . . (b) 1, 12, 23, 34, 45, 56, . . .

Solution

(a) First find the differences between the terms.

Sequence	13,		20,		27,		34,		41,		48. . . .
Difference		7		7		7		7		7	

As the difference between each term is always 7, the formula will contain a '$7n$' and be of the form

$$u_n = 7n + c.$$

To find the value of c consider any term. The first one is usually easiest to use.

Here, using $n = 1$ and $u_1 = 13$ gives

$$\begin{aligned} 13 &= 7 \times 1 + c \\ 13 &= 7 + c \\ c &= 6, \end{aligned}$$

so the formula is

$$u_n = 7n + 6.$$

Note

Always *check* that the formula is correct for other terms, e.g. $n = 2$ and $n = 3$. In this case,

$$u_2 = 7 \times 2 + 6 = 20$$

and

$$u_3 = 7 \times 3 + 6 = 27$$

so the formula holds.

(b) Again start by finding the differences between the terms of the sequence.

Sequence 1, 12, 23, 34, 45, 56, . . .

Difference 11 11 11 11 11

The difference is always 11 so the formula will contain '11*n*' and will be

$$u_n = 11n + c.$$

Using the first term, i.e. $n = 1$ and $u_1 = 1$, gives

$$1 = 11 \times 1 + c$$
$$1 = 11 + c$$
$$c = -10,$$

so the formula is

$$u_n = 11n - 10.$$

[*Check*: $u_2 = 11 \times 2 - 10 = 12$ and $u_3 = 11 \times 3 - 10 = 23$, which are correct.]

Worked Example 4

The 7th, 8th and 9th terms of a sequence are 61, 69 and 77 respectively. Find a formula to describe this sequence.

Solution

Looking at the differences,

Sequence 61, 69, 77,

Difference 8 8

you can conclude that the sequence must be of the form

$$u_n = 8n + c.$$

For $n = 7$, $u_7 = 56 + c$, giving $c = 5$.

Thus the formula to describe the sequence is

$$u_n = 8n + 5.$$

Note

A more general way of tackling this type of problem is to fit the linear sequence

$$u_n = dn + c$$

to the given information.

In this case,

$$u_7 = 7d + c = 61$$

and $$u_8 = 8d + c = 69.$$

Subtracting u_7 from u_8 gives $d = 8$ and substituting for d in either of the two equations gives $c = 5$.

Thus, as before,

$$u_n = 8n + 5.$$

Exercises

1. Use the formulae below to find the first 6 terms of each sequence.

   (a) $u_n = 4n + 1$ (b) $u_n = 5n - 7$ (c) $u_n = 10n + 2$

   (d) $u_n = n^2 - 1$ (e) $u_n = 2n^2 + 1$ (f) $u_n = 2^n$

2. The sequences described by the formulae,

   $$u_n = 8n - 2, \quad u_n = 3n + 5, \quad u_n = n^2 + 1, \quad u_n = n^3 - 1$$

   are given below. Select the formula that describes each sequence.

   (a) 0, 7, 26, 63, 124, . . . (b) 8, 11, 14, 17, 20, . . .

   (c) 6, 14, 22, 30, 38, . . . (d) 2, 5, 10, 17, 26, . . .

3. Find the 20th term for each sequence below.

   (a) $u_n = 4n$ (b) $u_n = 3n - 50$ (c) $u_n = 4n + 7$

   (d) $u_n = 2n - 40$ (e) $u_n = n^2 - 4$ (f) $u_n = \frac{n}{40}$

4. Consider the formula for the sequence below.

   8, 15, 22, 29, 36, 43. . . .

   (a) Explain why the formula contains $7n$.

   (b) Find the formula for the sequence.

5. Find the formula which describes each sequence below.

(a) 4, 9, 14, 19, 24, ... (b) 11, 14, 17, 20, 23, ...

(c) -2, 4, 10, 16, 22, ... (d) 100, 98, 96, 94, 92, ...

(e) 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, ... (f) 5, -2, -9, -16, -23, ...

(g) 10, $9\frac{1}{2}$, 9, $8\frac{1}{2}$, 8, ...

6. Which of the sequences below are described by a formula of the form

$$u_n = dn + c?$$

Where possible, give the formula.

(a) 1, 1, 2, 3, 5, 8, ... (b) 2, 10, 18, 28, 46, 74, ...

(c) 0, 3, 8, 15, 24, 35, 48, ... (d) 1, 2, 4, 7, 11, 16, ...

(e) 1, 1.1, 1.2, 1.3, 1.4, 1.5, ...

(f) 1, 1.1, 1.21, 1.331, 1.4641, 1.61051, ...

(g) 3, 7, 11, 15, 19, 23, ...

7. Write down the first 6 terms of the sequence $u_n = n^2$. Then use your answer to write down formulae for the following sequences.

(a) 3, 6, 11, 18, 27, 38, ... (b) -4, -1, 4, 11, 20, 31, ...

(c) 2, 8, 18, 32, 50, 72, ... (d) 0, 6, 16, 30, 48, 70, ...

(e) 99, 96, 91, 84, 75, 64, ...

8. By considering the sequence described by $u_n = n^3$, find formulae to describe the following sequences.

(a) 0, 7, 26, 63, 124, ... (b) 10, 17, 36, 73, 134, ...

(c) 199, 192, 173, 136, 75, ...

9. The 10th, 11th and 12th terms of a sequence are 50, 54 and 58.

Find a formula to describe this sequence and write down the first 5 terms.

10. The 100th, 101st and 102nd terms of a sequence are 608, 614 and 620.

Find a formula to describe this sequence and find the 10th term.

11. The 10th, 12th, 14th and 16th terms of a sequence are 52, 62, 72 and 82.

Find a formula to describe this sequence and find its first term.

12. Consider the sequence,

$$1, \ 5, \ 9, \ 13, \ 17, \ 21, \ 25, \ \dots$$

(a) Find the next term in the sequence and explain how you obtained your answer.

(b) The n th term in the sequence is $4n - 3$. Solve the equation

$$4n - 3 = 397$$

and explain what the answer tells you.

(MEG)

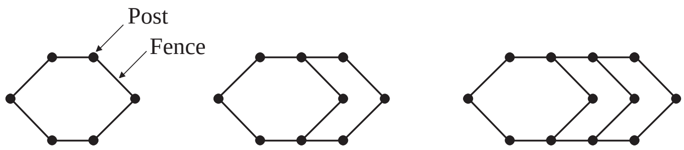
13. Here are the first four terms of a number sequence.

$$7, \ 11, \ 15, \ 19,$$

Write down the n th term of the sequence.

(LON)

14. Sheep enclosures are built using fences and posts. The enclosures are always built in a row.



One enclosure *Two enclosures* *Three enclosures*

(a) Sketch

(i) four enclosures in a row (ii) five enclosures in a row.

(b) Copy and complete the table below.

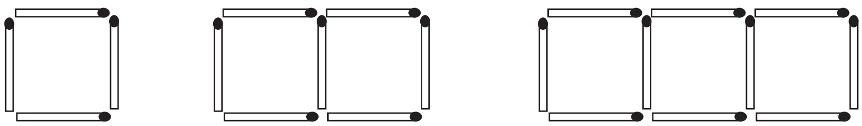
Number of enclosures	1	2	3	4	5	6	7	8
Number of posts	6	9	12					

(c) Work out the number of posts needed for 20 enclosures in a row.

(d) Write down an expression to find the number of posts needed for n enclosures in a row.

(LON)

15. Patterns of squares are formed using sticks. The first three patterns are drawn below.



Pattern 1 *Pattern 2* *Pattern 3*

The table shows the number of sticks needed for each pattern.

Pattern	1	2	3
Number of sticks	4	7	10

(a) (i) Draw *Pattern 4*.

(ii) How many sticks are needed for *Pattern 4*?

(b) How many more sticks are needed to make *Pattern 5* from *Pattern 4*?

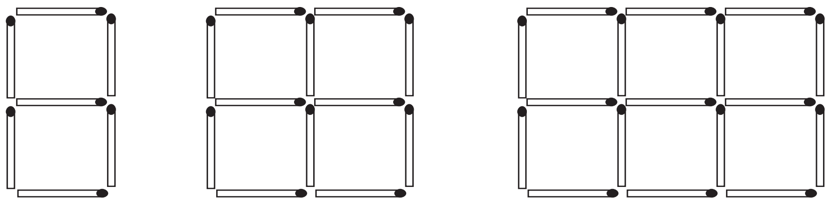
(c) There is a rule for finding the number of sticks needed to make any of these patterns of squares.

If the number of squares in a pattern is s, write down the rule.

(SEG)

16. (a) Sticks are arranged in shapes.

Shape 1 *Shape 2* *Shape 3*



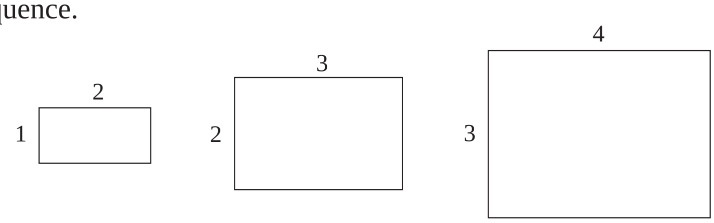
7 sticks 12 sticks 17 sticks

The number of sticks form a sequence.

(i) Write down a rule for finding the next number in the sequence.

(ii) Find a formula in terms of n for the number of sticks in the n th shape.

(b) Find a formula, in terms of n, for the area of the n th rectangle in this sequence.



(SEG)

Just for Fun

John and Julie had a date one Saturday. They agreed to meet outside the cinema at 8 pm.

Julie thought that her watch was 5 minutes fast but in actual fact it was 5 minutes slow.

John thought that his watch was 5 minutes slow but in actual fact it was 5 minutes fast.

Julie deliberately turned up 10 minutes late while John decided to turn up 10 minutes early.

Who turned up first and how long had he/she to wait for the other to arrive?

12.5 General Laws

This section considers sequences which are formed in various ways and uses iterative formulae that describe how one term is obtained from the previous term. The behaviour of the sequences with huge numbers of terms is also considered to see whether they increase indefinitely or approach a fixed value.

Worked Example 1

Find a formula to generate the terms of these sequences:

(a) $\frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \frac{6}{11}, \ldots$

(b) $4, 6, 9, 13.5, 20.25, \ldots$

What happens to these sequences for large values of n?

Solution

(a) This can be approached by looking at the numerators and denominators of the fractions.

The numbers $2, 3, 4, 5, 6, \ldots$ are from the sequence $u_n = n + 1$.

The numbers $3, 5, 7, 9, 11, \ldots$ are from the sequence $u_n = 2n + 1$.

Combining these gives the formula for the sequence

$$\frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \frac{6}{11}, \ldots$$

as

$$u_n = \frac{n+1}{2n+1}$$

As the value of n becomes larger and larger, this sequence produces terms that get closer and closer to $\frac{1}{2}$. Consider the terms below:

$$u_{100} = \frac{101}{201} = 0.502\,487\,6$$

$$u_{1000} = \frac{1001}{2\,001} = 0.500\,249\,9$$

$$u_{10000} = \frac{10\,001}{20\,001} = 0.500\,025\,0$$

$$u_{100000} = \frac{100\,001}{200\,001} = 0.500\,002\,5$$

We say that the sequence *converges* to $\frac{1}{2}$.

Note

A more rigorous approach is to divide both the numerator and denominator of the expression for u_n by n, giving

$$u_n = \frac{n+1}{2n+1} = \frac{1+\frac{1}{n}}{2+\frac{1}{n}}.$$

Now as n becomes larger, the term $\frac{1}{n} \to 0$, giving

$$u_n \to \frac{1+0}{2+0} = \frac{1}{2}$$

We write $u_n \to \frac{1}{2}$ as $n \to \infty$ (infinity) and say that

' u_n tends to $\frac{1}{2}$ as n tends to infinity.'

(b) The terms of this sequence are multiplied by a factor of 1.5 to obtain the next term.

$$4 \xrightarrow{\times 1.5} 6 \xrightarrow{\times 1.5} 9 \xrightarrow{\times 1.5} 13.5 \xrightarrow{\times 1.5} 20.25$$

Considering each term helps to see the general formula.

$$\begin{aligned}
u_1 &= 4 = 4 \times 1.5^0 \\
u_2 &= 4 \times 1.5 = 4 \times 1.5^1 \\
u_3 &= 4 \times 1.5 \times 1.5 = 4 \times 1.5^2 \\
u_4 &= 4 \times 1.5 \times 1.5 \times 1.5 = 4 \times 1.5^3 \\
u_5 &= 4 \times 1.5 \times 1.5 \times 1.5 \times 1.5 = 4 \times 1.5^4
\end{aligned}$$

So the general term is $u_n = 4 \times 1.5^{n-1}$.

The terms of this sequence become larger and larger, never approaching a fixed value as in the last example. We say the sequence *diverges*. In fact, any sequence which does *not* converge is said to diverge.

Worked Example 2

Find iterative formulae for each of the following sequences.

(a) 7, 11, 15, 19, 23, 27, . . .

(b) 6, 12, 24, 48, 96, . . .

(c) 1, 1, 2, 3, 5, 8, 13, 21, . . .

Solution

This group of sequences show how one term is related to the previous term, so the formulae give u_{n+1} in terms of the previous term, u_n.

(a)
$$\begin{aligned} u_1 &= 7 \\ u_2 &= 7+4 = u_1+4 \\ u_3 &= 11+4 = u_2+4 \\ u_4 &= 15+4 = u_3+4 \end{aligned}$$

So $u_{n+1} = u_n + 4$ with $u_1 = 7$.

(b)
$$\begin{aligned} u_1 &= 6 \\ u_2 &= 2\times 6 = 2\times u_1 \\ u_3 &= 2\times 12 = 2\times u_2 \\ u_4 &= 2\times 24 = 2\times u_3 \end{aligned}$$

So $u_{n+1} = 2u_n$ with $u_1 = 6$.

(c)
$$\begin{aligned} u_1 &= 1 \\ u_2 &= 1 \\ u_3 &= 1+1 = u_1+u_2 \\ u_4 &= 1+2 = u_2+u_3 \\ u_5 &= 2+3 = u_3+u_4 \\ u_6 &= 3+5 = u_4+u_5 \end{aligned}$$

So $u_{n+1} = u_n + u_{n-1}$, with $u_1 = 1$ and $u_2 = 1$.

Worked Example 3

Find the first 4 terms of the sequence defined iteratively by

$$u_{n+1} = \frac{1}{2}\left(u_n + \frac{4}{u_n}\right)$$

starting with $u_1 = 1$. Show that the sequence converges to 2.

Solution

$$\begin{aligned} u_1 &= 1 \\ u_2 &= \frac{1}{2}\left(1+\frac{4}{1}\right) = 2.5 \\ u_3 &= \frac{1}{2}\left(2.5+\frac{4}{2.5}\right) = 2.05 \\ u_4 &= \frac{1}{2}\left(2.05+\frac{4}{2.05}\right) = 2.000\,609\,75 \end{aligned}$$

These terms appear to be getting closer and closer to 2. If the sequence does converge to a particular value then as n becomes larger and larger, u_{n+1} becomes approximately the same as u_n.

So if $u_{n+1} = u_n = d$, say, then

$$u_{n+1} = \frac{1}{2}\left(u_n + \frac{4}{u_n}\right)$$

becomes
$$\begin{aligned} d &= \frac{1}{2}\left(d + \frac{4}{d}\right) \\ 2d &= d + \frac{4}{d} \\ d &= \frac{4}{d} \\ d^2 &= 4 \\ d &= 2 \text{ or } -2. \end{aligned}$$

So the sequence does converge to 2. If the sequence had started with $u_1 = -1$, then it would have converged to -2, the other possible value of d.

Exercises

1. Find formulae to generate the terms of each sequence below.

(a) $\frac{1}{10}, \frac{4}{11}, \frac{9}{12}, \frac{16}{13}, \frac{25}{14}, \ldots$

(b) $\frac{5}{2}, \frac{7}{3}, \frac{9}{4}, \frac{11}{5}, \frac{13}{6}, \ldots$

(c) 2, 6, 18, 54, 162, . . .

(d) 1, 0.9, 0.81, 0.729, 0.6561, . . .

(e) 1, 1.2, 1.44, 1.728, 2.0736, . . .

(f) $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \frac{2}{81}, \ldots$

(g) 2, 4, 8, 16, 32, . . .

(h) 3, 5, 9, 17, 33, . . .

(i) $\frac{3}{4}, \frac{6}{5}, \frac{9}{6}, \frac{12}{7}, \frac{15}{8}, \ldots$

Which of the above sequences converge and which diverge? For those which converge, find the value to which they converge.

2. Find iterative formulae for each of the following sequences.

(a) 8, 5, 2, -1, -4, . . .

(b) 5, 20, 80, 320, 1280, . . .

(c) 2, 3, 5, 9, 17, . . .

(d) 4000, 2000, 1000, 500, 250, . . .

(e) 3, 3, 6, 9, 15, 24, 39, . . .

(f) 1, 1, 1, 3, 5, 9, 17, 31, . . .

3. The iterative formula
$$u_{n+1} = \frac{1}{2}\left(u_n + \frac{6}{u_n}\right)$$
can be used with $u_1 = 1$ to define the sequence.

(a) Find the first 5 terms of the sequence.

(b) Show that the sequence converges to $\sqrt{6}$.

4. Does the sequence defined by

$$u_{n+1} = \frac{1}{3}\left(2u_n + \frac{27}{u_n^2}\right)$$

with $u_1 = 2$ converge? If it does, find the value to which it converges. What happens if u_1 is a different value?

5. Find the first 5 terms of the sequence

$$u_{n+1} = u_n(2 - 7u_n)$$

starting with $u_1 = 0.1$. To what value does this sequence converge? (Give your answer as a fraction.)

6. To what value does the sequence

$$u_{n+1} = u_n(2 - 3u_n)$$

with $u_1 = \frac{1}{2}$, converge? Calculate the first four terms of the sequence to check your answer.

7. Javid and Anita try to find different ways of exploring the sequence

$$4, \ 10, \ 18, \ 28, \ 40, \ \ldots$$

(a) Javid writes

1st number	$4 = 1 \times 4 = 1 \times (1 + 3)$
2nd number	$10 = 2 \times 5 = 2 \times (2 + 3)$
3rd number	$18 = 3 \times 6 = 3 \times (3 + 3)$
4th number	$28 = 4 \times 7 = 4 \times (4 + 3)$

How would Javid write down (i) the 5th number (ii) the *n*th number?

(b) Anita writes

1st number	$4 = 2 \times 3 - 2$
2nd number	$10 = 3 \times 4 - 2$
3rd number	$18 = 4 \times 5 - 2$
4th number	$28 = 5 \times 6 - 2$

How would Anita write down (i) the 5th number (ii) the *n*th number?

(c) Show how you would prove that Javid's expression and Anita's expression for the *n*th number are the same.

(MEG)

8. (a) Write down the next number in this sequence.

$$1, \ 2, \ 4, \ 8, \ 16, \ 32, \ \ldots$$

(b) Describe how the sequence is formed.

(c) One number in the sequence is 1024. Describe how you can use the number 1024 to find the number in the sequence which comes just before it.

(SEG)

9.

Row 1	1	Sum $= 1$
Row 2	3 5	Sum $= 8 = 2^3$
Row 3	7 9 11	Sum $= 27 = 3^3$

(a) Write down the numbers and sum which continue the pattern in *Row 4*.

(b) Which row will have a sum equal to 1000?

(c) What is the sum of *Row 20*?

(d) The first number in a row is x. What is the second number in this row? Give your answer in terms of x.

(MEG)

12.6 Quadratic Formulae

Consider the sequence generated by the formula $u_n = n^2 + n$.

$$2,\ 6,\ 12,\ 20,\ 30,\ 42,\ \ldots$$

The differences between terms can be considered as below.

	2,		6,		12,		20,		30,		42,	. . .
First differences		4		6		8		10		12		
Second differences			2		2		2		2			

The first differences increase, but the second differences are all the same. Whenever the second differences are constant a sequence can be described by a quadratic formula of the form

$$\boxed{u_n = an^2 + bn + c}$$

where a, b and c are constants.

Worked Example 1

Find the formula which describes the sequence

$$1,\ 8,\ 21,\ 40,\ 65,\ 96.$$

Solution

First examine the differences.

	1,		8,		21,		40,		65,		96,	. . .
First differences		7		13		19		25		31		
Second differences			6		6		6		6			

As the second differences are constant, the sequence can be described by a quadratic formula of the form

$$u_n = an^2 + bn + c.$$

To find the values of a, b and c consider the first 3 terms.

$$\begin{array}{llllll} \text{Using} & u_1 = 1 & \text{gives} & 1 & = a + b + c & (1) \\ \text{Using} & u_2 = 8 & \text{gives} & 8 & = 4a + 2b + c & (2) \\ \text{Using} & u_3 = 21 & \text{gives} & 21 & = 9a + 3b + c & (3) \end{array}$$

These are three simultaneous equations. To solve them, subtract equation (1) from equation (2) to give equation (4) and from equation (3) to give equation (5) as below.

$$\begin{array}{lrcll} & 8 & = & 4a + 2b + c & (2) \\ \text{subtract} & 1 & = & a + b + c & (1) \\ \hline & 7 & = & 3a + b & (4) \end{array} \qquad \begin{array}{rcll} 21 & = & 9a + 3b + c & (3) \\ 1 & = & a + b + c & (1) \\ \hline 20 & = & 8a + 2b & (5) \end{array}$$

Then subtracting $2 \times$ equation (4) from equation (5) gives

$$\begin{array}{lrcll} & 20 & = & 8a + 2b & (5) \\ & 14 & = & 6a + 2b & (4) \times 2 \\ \hline & 6 & = & 2a & \\ \text{so} & a & = & 3. & \end{array}$$

Substituting for a in equation (4) gives

$$\begin{array}{lrcl} & 7 & = & 3 \times 3 + b \\ \text{so} & b & = & -2. \end{array}$$

Finally, substituting for a and b in equation (1) gives

$$\begin{array}{lrcl} & 1 & = & 3 - 2 + c \\ \text{so} & c & = & 0. \end{array}$$

The sequence is then generated by the formula

$$u_n = 3n^2 - 2n.$$

Note

You should, of course, check the first few terms:

$$\begin{array}{l} n = 1 \rightarrow u_1 = 3 \times 1 - 2 \times 1 = 1 \\ n = 2 \rightarrow u_2 = 3 \times 2^2 - 2 \times 2 = 8 \\ n = 3 \rightarrow u_3 = 3 \times 3^2 - 2 \times 3 = 21. \end{array}$$

Investigation

Find the next term in the sequence

$$\frac{1}{2},\ 1,\ \frac{9}{4},\ \frac{27}{5},\ \frac{27}{2}$$

Alternative Approach

Solving sets of simultaneous equations like these can be quite hard work. Examining carefully the differences leads to an easier method. The first four terms of the sequence

$$u_n = an^2 + bn + c$$

are

$$\begin{aligned} u_1 &= a + b + c \\ u_2 &= 4a + 2b + c \\ u_3 &= 9a + 3b + c \\ u_4 &= 16a + 4b + c \end{aligned}$$

Consider the differences for these terms.

$$\begin{array}{lccccccc} & a+b+c, & & 4a+2b+c, & & 9a+3b+c, & & 16a+4b+c \\ \textit{First difference} & & 3a+b & & 5a+b & & 7a+b & \\ \textit{Second difference} & & & 2a & & 2a & & \end{array}$$

Note that the second difference is equal to $2a$, the first of the first differences is $3a + b$ and the first term is $a + b + c$. This can be used to create a much easier approach to finding a, b and c, as shown in the next example.

Worked Example 2

Find the formula which generates the sequence

$$6,\ 11,\ 18,\ 27,\ 38,\ 51,\ \ldots$$

Solution

First find the differences.

$$\begin{array}{lccccccccccc} & 6, & & 11, & & 18, & & 27, & & 38, & & 51, \ \ldots \\ \textit{First differences} & & 5 & & 7 & & 9 & & 11 & & 13 & \\ \textit{Second differences} & & & 2 & & 2 & & 2 & & 2 & & \end{array}$$

As the second differences are constant, the sequence is generated by the quadratic formula

$$u_n = an^2 + bn + c.$$

Using the results from the differences considered in the *alternative approach* in Worked Example 1 gives

$$\begin{aligned} 2a &= 2 \\ 3a + b &= 5 \\ a + b + c &= 6. \end{aligned}$$

This gives $a = 1$, $b = 2$, and $c = 3$, so the formula is

$$u_n = n^2 + 2n + 3.$$

Exercises

1. Show that each sequence below has a constant second difference and use this to find the next 2 terms.

   (a) 5, 10, 17, 26, 37, . . . (b) 0, 10, 28, 54, 88, . . .

   (c) 2, 14, 34, 62, 98, . . . (d) $2\frac{1}{2}$, 4, $6\frac{1}{2}$, 10, $14\frac{1}{2}$, . . .

   (e) -1, -2, -1, 2, 7, . . .

2. The third, fourth and fifth terms of a quadratic sequence are 6, 10 and 16.

   Find the first, second and sixth terms of the sequence.

3. A sequence begins 12, 24, 42, 66, . . . Find the 10th term of this sequence.

4. The first and third terms of a sequence are 6 and 48. If the second difference is constant and equal to 10, find the second term of the sequence.

5. For each of the following sequences, state whether or not they are generated by a quadratic formula and if they are, give the formula.

   (a) 3, 2, 3, 6, 11, 18, . . . (b) 0, -4, 6, 66, 236, 600, . . .

   (c) 1, 3, 7, 13, 21, 31, . . . (d) 1, 3, 4, 7, 11, 18, . . .

   (e) -5, 0, 7, 16, 27, 40, . . . (f) -1, 3, 7, 11, 15, 19, . . .

   (g) 1, 1, 7, 19, 37, 61, . . .

6. The 2nd, 3rd and 4th terms of a quadratic sequence are 0, 3 and 8.
   Find the 1st and 5th terms of the sequence.

7. In a sequence,

   the 2nd term is 10 more than the 1st term,

   the 3rd term is 15 more than the 2nd term,

   the 4th term is 20 more than the 3rd term.

   Show that the sequence is quadratic and find a formula for the sequence if the first term is 1.

8. (a) The terms of a particular cubic sequence are given by $u_n = 2n^3 - 1$.

   Find the first 6 terms of this sequence and then the first, second and third differences. What do you notice?

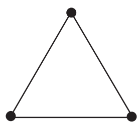
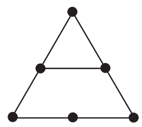
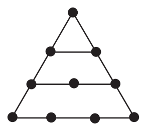
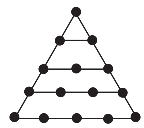
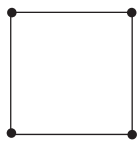
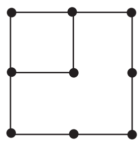
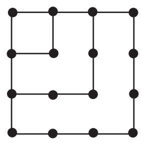
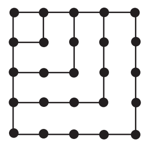
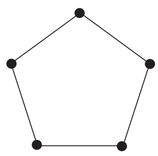
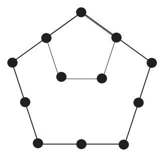
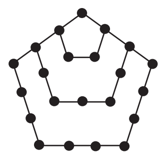
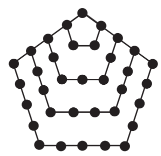
   (b) Check that the result you noted in Part (a) is true for a cubic sequence of your own choice.

   (c) By considering $u_n = an^3 + bn^2 + cn + d$, show that the third difference of any cubic sequence is a constant and give its value in terms of *a*.

(d) Then find the formula which describes the cubic sequence

$$4,\ 15,\ 40,\ 85,\ 156,\ 259,\ \ldots$$

9. The table shows how the first 5 triangle, square and pentagon numbers are formed.

	1	**2**	**3**	**4**	**5**
Triangle	• 1	 3	 6	 10	 15
Square	• 1	 4	 9	 16	 25
Pentagon	• 1	 5	 12	 22	 35

(a) Show that all the sequences formed are quadratic and find expressions for them.

(b) The hexagon numbers give the sequence

$$1,\ 6,\ 15,\ 28,\ 45,\ \ldots$$

Show that the terms of this sequence are given by

$$u_n = 2n^2 - n.$$

(c) By looking at the expressions you have obtained so far, predict formulae for the heptagonal and octagonal numbers. Use the facts that the 8th heptagonal number is 148 and the 8th octagonal number is 176 to check your formulae.

(d) Show that the 8th decagonal number is 232.

10. Look at the three sequences below.

Sequence p	4, 6, 8, 10, 12, . . .
Sequence q	3, 8, 15, 24, 35, . . .
Sequence r	5, 10, 17, . . .

(a) The sequence r is obtained from sequences p and q as follows.

$$\sqrt{4^2+3^2} = 5, \qquad \sqrt{6^2+8^2} = 10, \qquad \sqrt{8^2+15^2} = 17$$

and so on.

(i) Use the numbers 10 and 24 to calculate the fourth term of sequence r.

(ii) Calculate the fifth term of sequence r.

(b) (i) Find the tenth term of sequence p.

(ii) Find the sixth term of sequence q.

(c) (i) Write down the nth term of sequence p.

(ii) The nth term of sequence q is

$$n^2 + kn$$

where k represents a number. Find the value of k.

(NEAB)

11. The first four diagrams of a sequence are shown below.

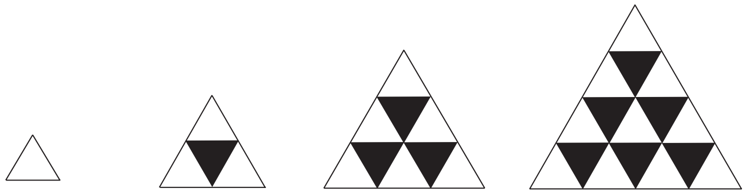


Diagram 1 *Diagram 2* *Diagram 3* *Diagram 4*

The table below shows the number of black and white triangles for the first three diagrams.

Diagram number	1	2	3	4	5
Number of white triangles	1	3	6		
Number of black triangles	0	1	3		
Total number of triangles	1	4	9		

(a) Complete the table, including the column for a fifth diagram.

(b) What will be the total number of triangles in diagram 10?

(c) (i) On a grid, plot the number of white triangles against the diagram numbers.

(ii) On the same grid, plot the number of black triangles for each diagram number in your table.

(iii) What do you notice about the two sets of points?

(d) Two pupils are trying to find a general rule to work out the number of white triangles. One rule they suggest is

$$\text{Number of white triangles } = d(d+1)$$

where d is the diagram number. Is this rule correct? Show any calculations that you make.

(NEAB)