7 Mensuration

This Unit is concerned with measuring, calculating and estimating lengths, areas and volumes, as well as the construction of three-dimensional (3D) objects.

7.1 Units and Measuring

Different units can be used to measure the same quantities. It is important to use sensible units. Some important units are listed below.

1 km = 1000 m
1 m = 100 cm
1 m = 1000 mm
1 cm = 10 mm

1 tonne = 1000 kg
1 kg = 1000 g

1 litre = 1000 ml
1 m^3 = 1000 litres
1 cm^3 = 1 ml

Worked Example 1

What would be the best units to use when measuring,

(a) the distance between Birmingham and Manchester,
(b) the length of a matchbox,
(c) the mass of a person,
(d) the mass of a letter,
(e) the mass of a lorry,
(f) the volume of medicine in a spoon,
(g) the volume of water in a swimming pool?

Solution

(a) Use km (or miles).
(b) Use mm or cm.
(c) Use kg.
(d) Use grams.
(e) Use tonnes
(f) Use ml.
(g) Use m^3.

Worked Example 2

(a) How many mm are there in 3.72 m?

(b) How many cm are there in 4.23 m?

(c) How many m are there in 102.5 km?

(d) How many kg are there in 4.32 tonnes?

Solution

(a) 1 m = 1000 mm

So

$$\begin{aligned} 3.72 \text{ m} &= 3.72 \times 1000 \\ &= 3720 \text{ mm} \end{aligned}$$

(b) 1 m = 100 cm

So

$$\begin{aligned} 4.23 \text{ m} &= 4.23 \times 100 \\ &= 423 \text{ cm} \end{aligned}$$

(c) 1 km = 1000 m

So

$$\begin{aligned} 102.5 \text{ km} &= 102.5 \times 1000 \\ &= 102\,500 \text{ m} \end{aligned}$$

(d) 1 tonne = 1000 kg

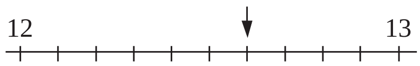
So

$$\begin{aligned} 4.32 \text{ km} &= 4.32 \times 1000 \\ &= 4320 \text{ kg} \end{aligned}$$

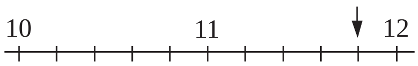
Worked Example 3

What value does each arrow point to?

(a)



(b)



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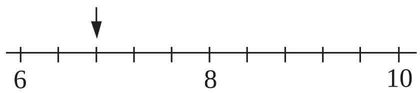


Solution

(a) Here the marks are 0.1 units apart.
So the arrow points to 12.6.

(b) Here the marks are 0.2 units apart.
So the arrow points to 11.8.

(c) Here the marks are 0.4 units apart.
So the arrow points to 6.8.

Exercises

1. Measure each line below. Give the length to the nearest mm.

   (a)

   (b)

   (c)

   (d)

   (e)

2. Which units do you think would be the most suitable to use when measuring:

   (a) the distance between two towns,

   (b) the length of a sheet of paper,

   (c) the mass of a sheet of paper,

   (d) the mass of a sack of cement,

   (e) the volume of a water in a cup,

   (f) the volume of water in a large tank?

3. (a) How many grams are there in 12.3 kg?

   (b) How many mm are there in 4.7 m?

   (c) How many mm are there in 16.4 cm?

   (d) How many m are there in 3.4 km?

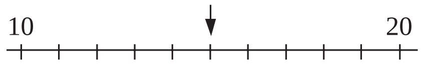
   (e) How many cm are there in 3.7 m?

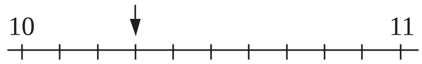
   (f) How many ml are there in 6 litres?

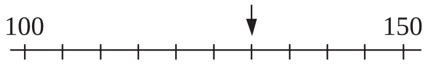
4. Copy and complete the table below.

Length in m	*Length in cm*	*Length in mm*
4		
	311	
		1500
	374	
8.62		

5. Read off the value shown by the arrow on each scale

(a) 

(b) 

(c) 

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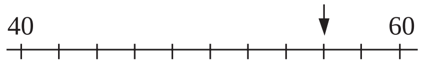
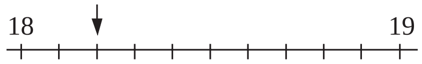
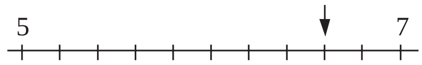
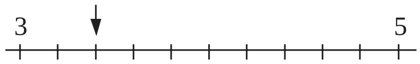
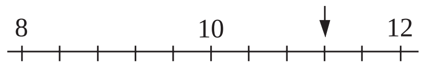
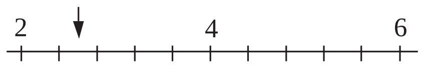
(e) 

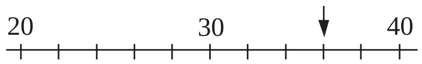
(f) 

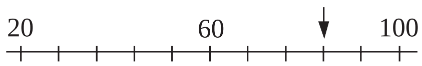
(g) 

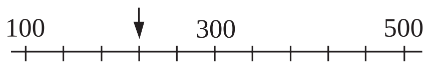
(h) 

(i) 

(j) 

(k) 

(l) 

(m) 

6. A jug contains 1 litre of water.

(a) If 150 ml is poured out, how much water is left?

(b) A glass holds 200 ml of water. How many glasses could be filled from a full jug?

7. State whether the following lengths would be best measured to the nearest m, cm or mm.

(a) Your height.

(b) The length of a ship.

(c) The height of a hill.

(d) The thickness of a book.

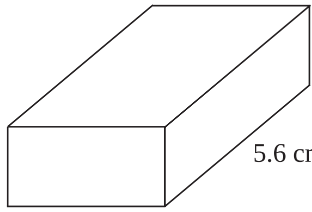
(e) The height of a building.

(f) The length of a matchstick.

(g) The width of a matchstick.

8. A cuboid has sides as shown in the diagram.

Convert the lengths of these sides to mm.



9. Each length below is given in mm. Give each length to the nearest cm.

(a) 42 mm (b) 66 mm (c) 108 mm

(d) 3 mm (e) 7 mm (f) 9.4 mm

10. (a) What metric unit of length would you use to measure the length of a large coach?

(b) Using the unit you gave in part (a) estimate the length of a large coach.

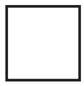
(LON)

Just for Fun

Which is heavier, 1 kg of iron or 1 kg of feathers?

7.2 Estimating Areas

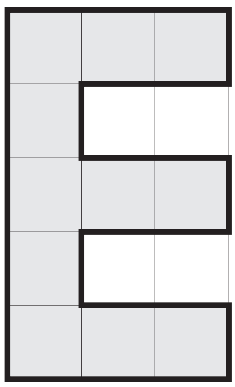
A square with sides of 1 cm has an area of 1 cm^2.



Area = 1 cm^2

Worked Example 1

Find the area of the shaded shape.

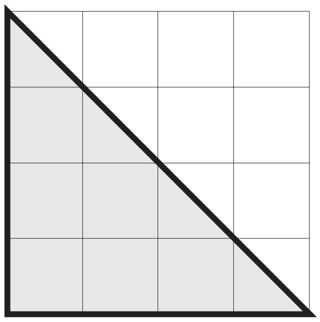


Solution

The shape covers 11 squares, so its area is 11 cm^2.

Worked Example 2

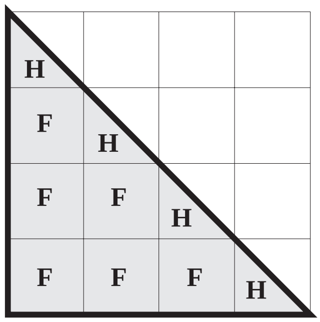
Find the area of the shaded triangle.



Solution

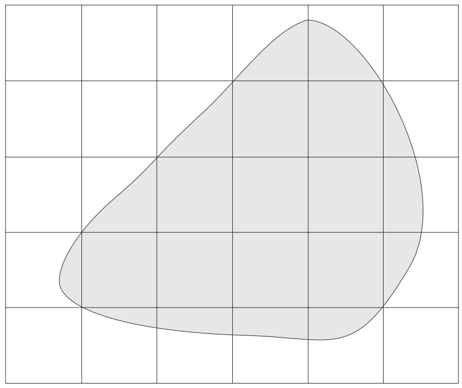
The triangle covers 6 full squares marked **F**, and 4 half squares marked **H**.

$$\text{Area} = 6 + 2$$
$$= 8\,\text{cm}^2$$



Worked Example 3

Estimate the area of the shape shaded in the diagram.



Solution

This is a much more complicated problem as there are only 9 full squares marked **F**, but many other part squares. You need to combine part squares that approximately make a whole square. For example,

the squares marked * make about 1 full square;

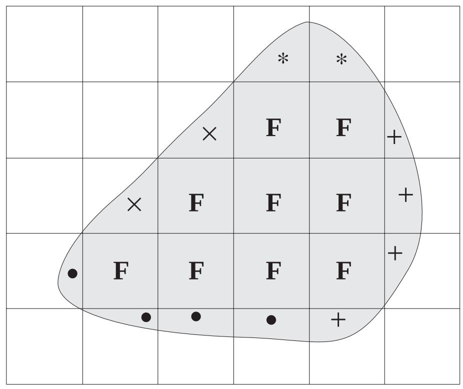
the squares marked × make about 1 full square;

the squares marked + make about 1 full square;

the squares marked • make about 1 full square.

Thus the total area is approximately

$$9 + 4 = 13\text{ cm}^2$$



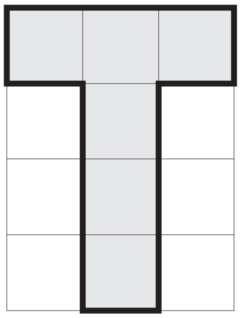
Just for Fun

Use 12 cocktail sticks to form 6 equilateral triangles, all of the same area. Move only 4 cocktail sticks from your figure so as to get 3 equilateral triangles, 2 of which are of the same area.

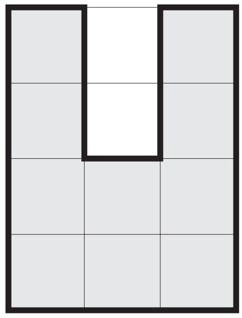
Exercises

1. Find the area of each of the following shapes.

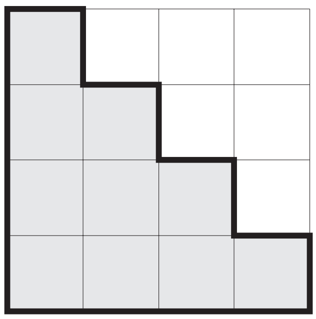
(a)



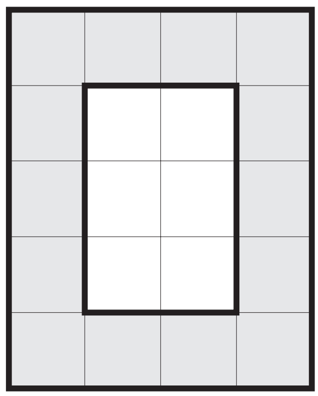
(b)



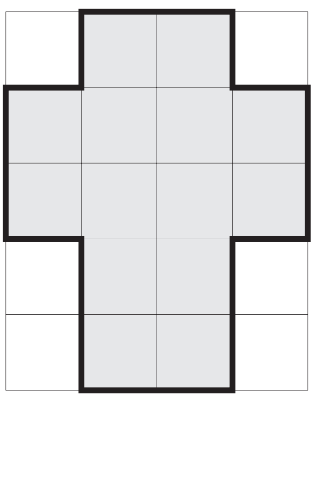
(c)



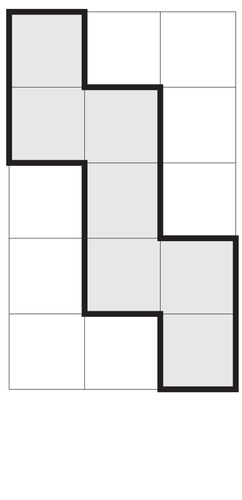
(d)



(e)

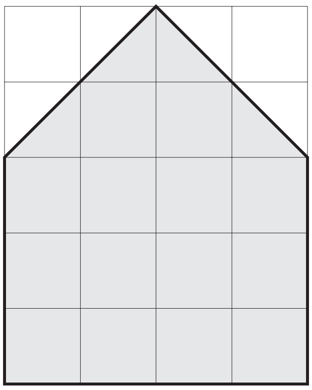


(f)

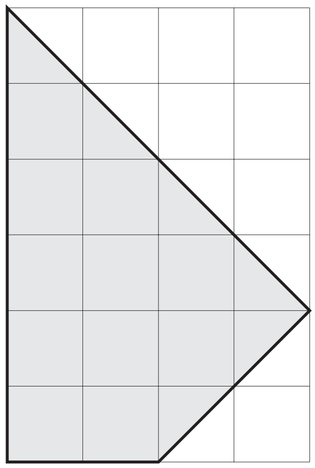


2. By counting the number of whole squares and half squares, find the area of each of the following shapes.

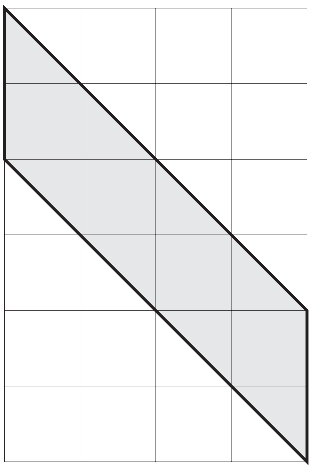
(a)



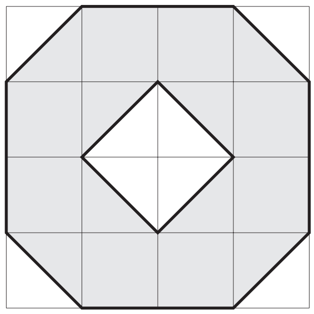
(b)



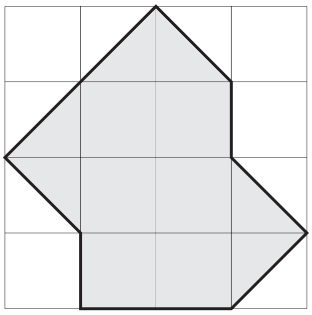
(c)



(d)



(e)



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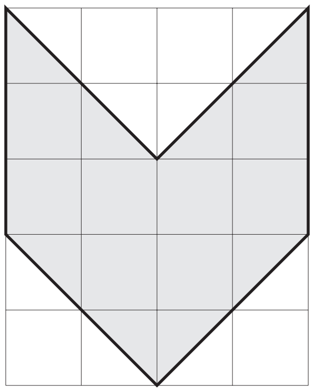
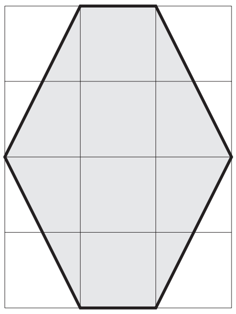
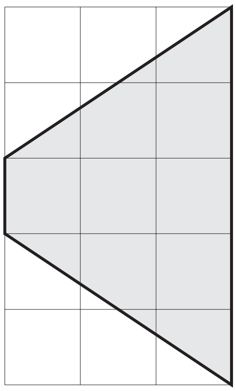


3. Estimate the area of each of the following shapes.

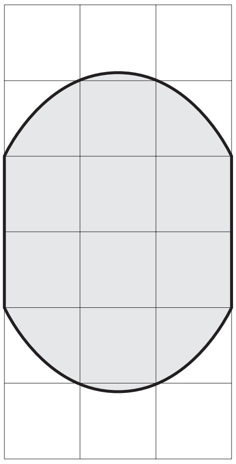
(a)



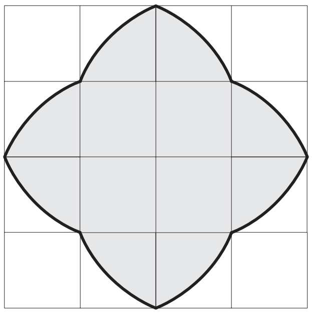
(b)



(c)



(d)



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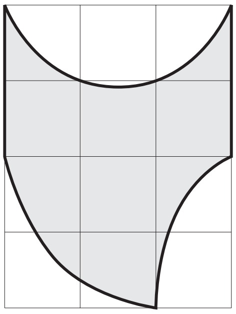

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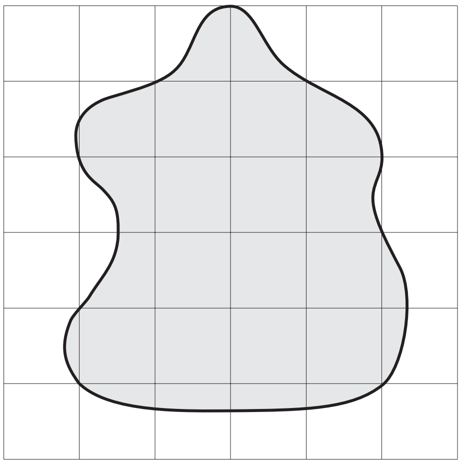
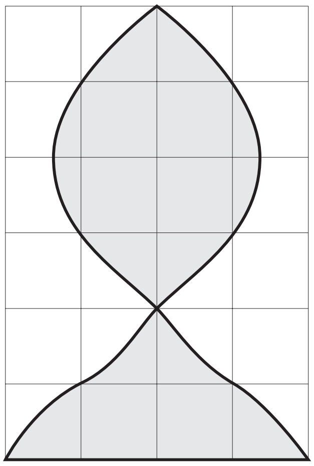


(g)



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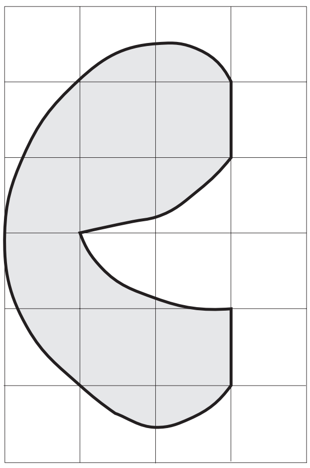
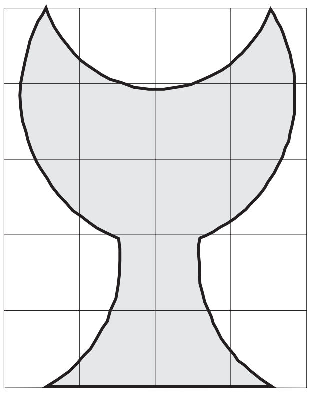
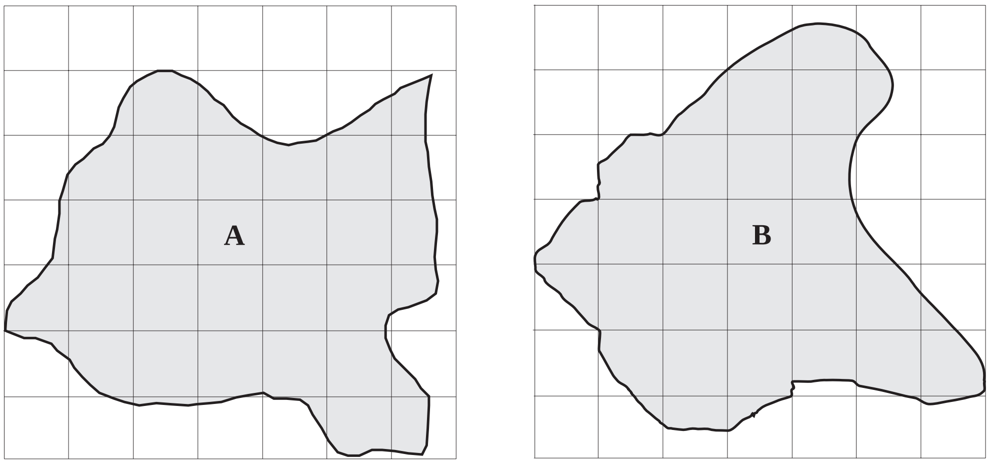


(i)

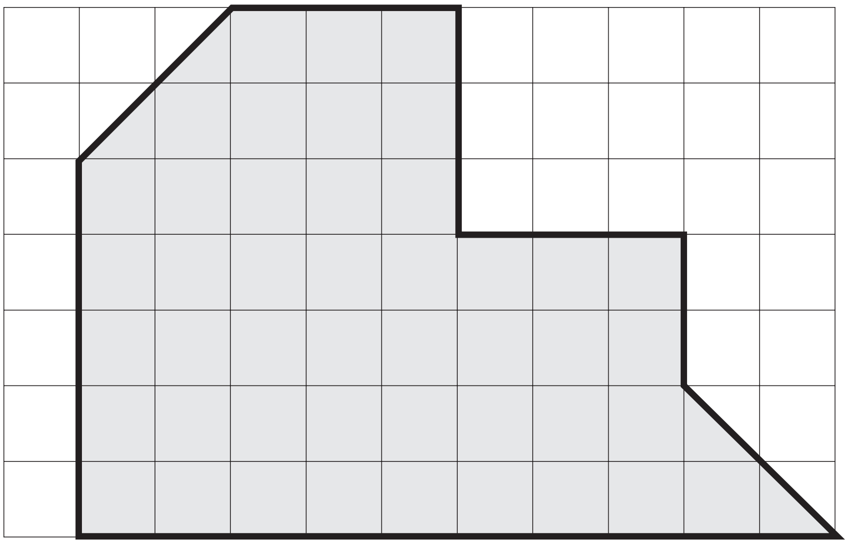


4. The diagrams below shows the outlines of two islands, A and B. The grid squares have sides of length 1 km. Find the approximate area of each island.



5. Each of the squares in this grid has an area of 1 square centimetre.

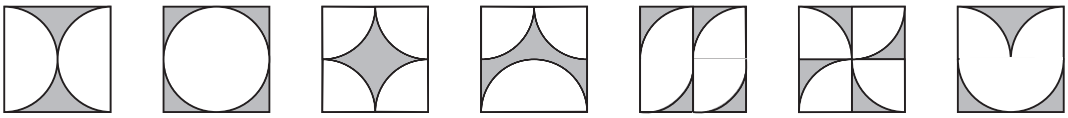
Work out the area of the shaded shape.



(LON)

Investigation

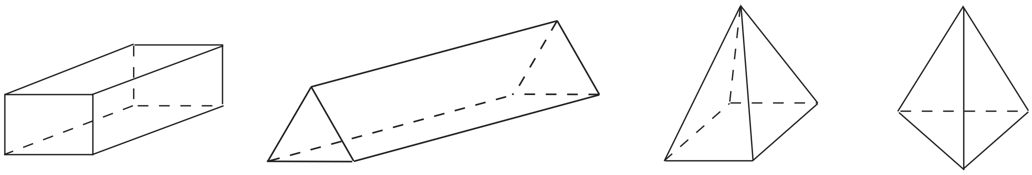
Which of the following shaded figures has the greatest area? The squares are of the same length and the curved lines are all arcs of circles.



7.3 Making Solids Using Nets

A net can be cut out, folded and glued to make a hollow shape.

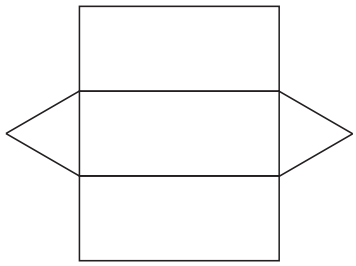
In this Unit, you will be dealing with 3-dimensional shapes such as



cuboid *prism* *pyramid* *tetrahedron*

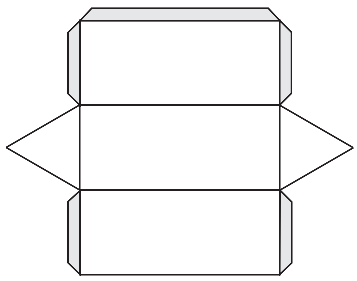
Worked Example 1

What solid is made when the net shown is folded and glued?



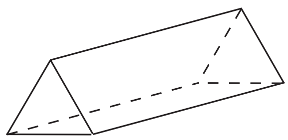
Solution

It is important to add tabs to the net so that it can be glued. You could put tabs on every edge, but this would mean gluing tabs to tabs. The diagram opposite shows one possible position of the tabs.



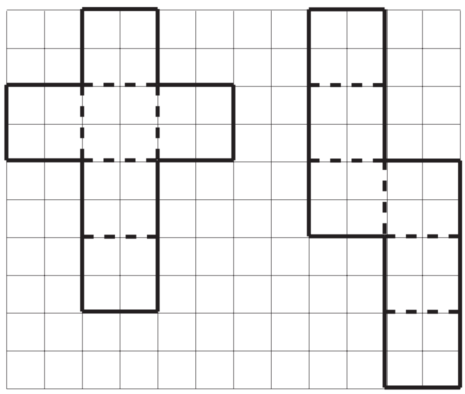
Before gluing, crease all the folds.

The final solid is a triangular prism.



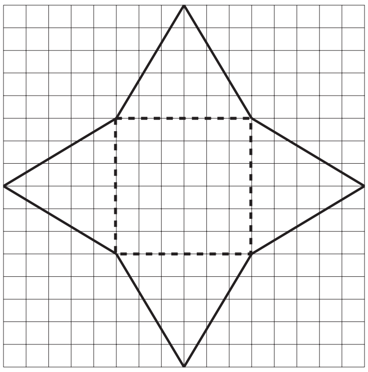
Exercises

1. Copy and cut out larger versions of the following nets. Fold and glue them to obtain cubes. Do not forget to add tabs to the nets.

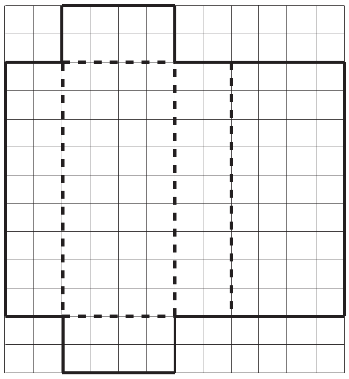


2. Copy each net shown below make it into a solid. State the name of the solid that you make, if it has one.

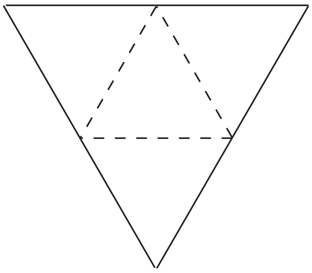
(a)



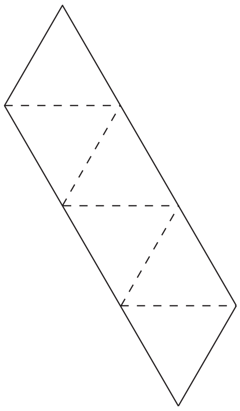
(b)



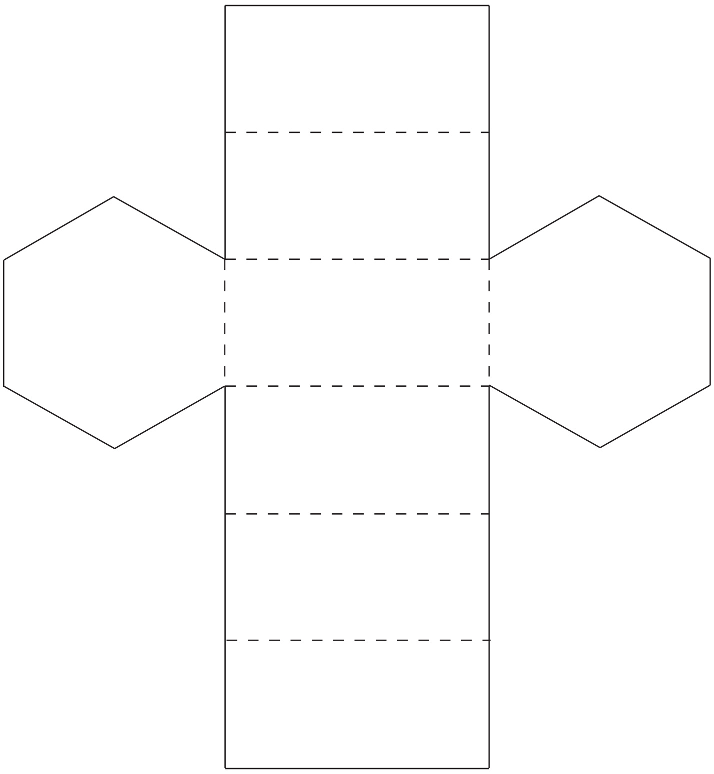
(c)



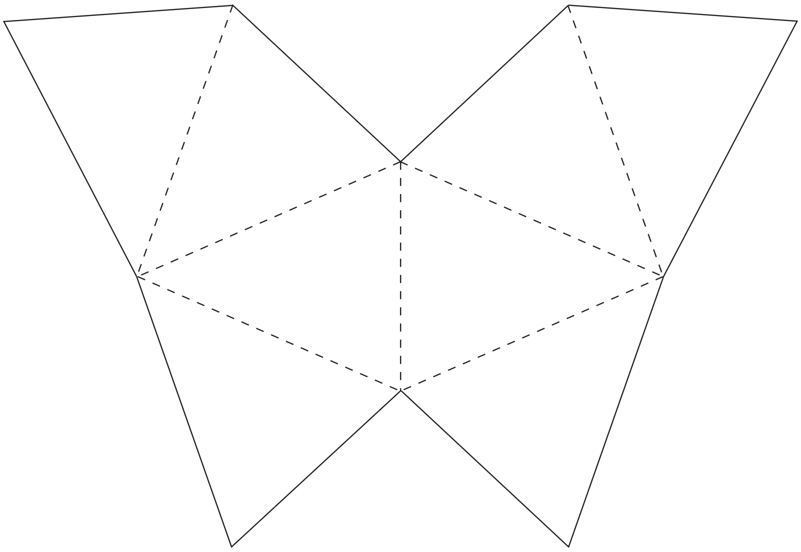
(d)



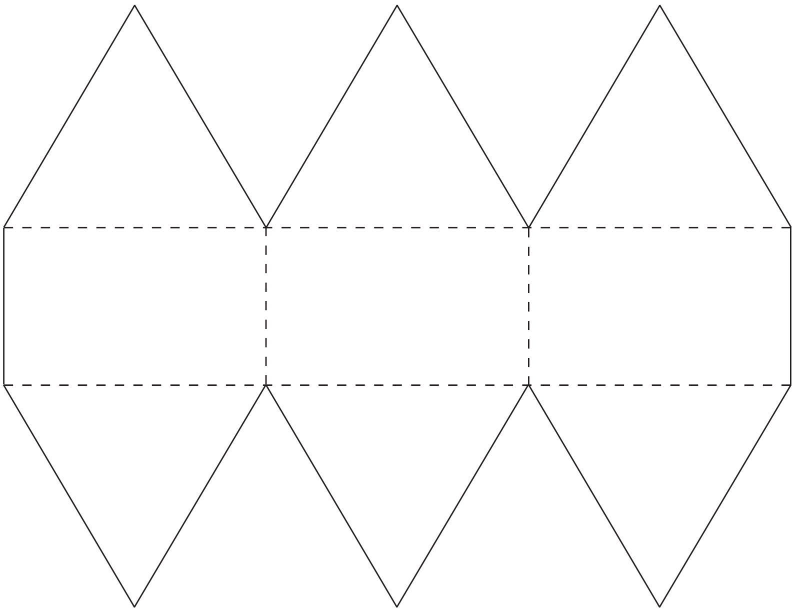
(e)



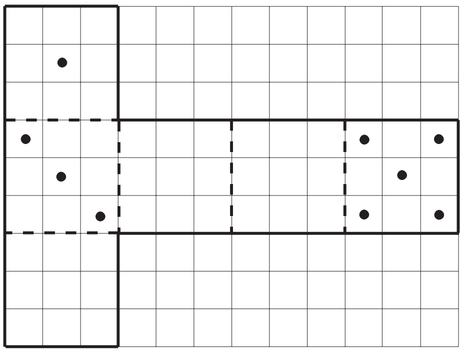
(f)



(g)



3. The diagram shows the net for a dice with some of the spots in place. Fill in the missing spots so that the opposite faces add up to 7. Then make the dice.

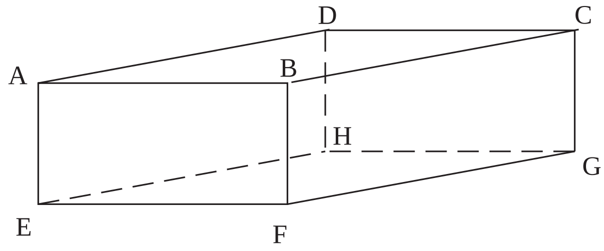


7.4 Constructing Nets

A net for a solid can be visualised by imagining that the shape is cut along its edges until it can be laid flat.

Worked Example 1

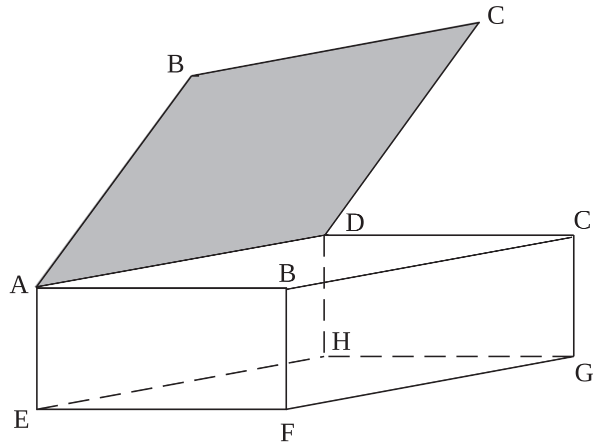
Draw the net for the cuboid shown in the diagram.

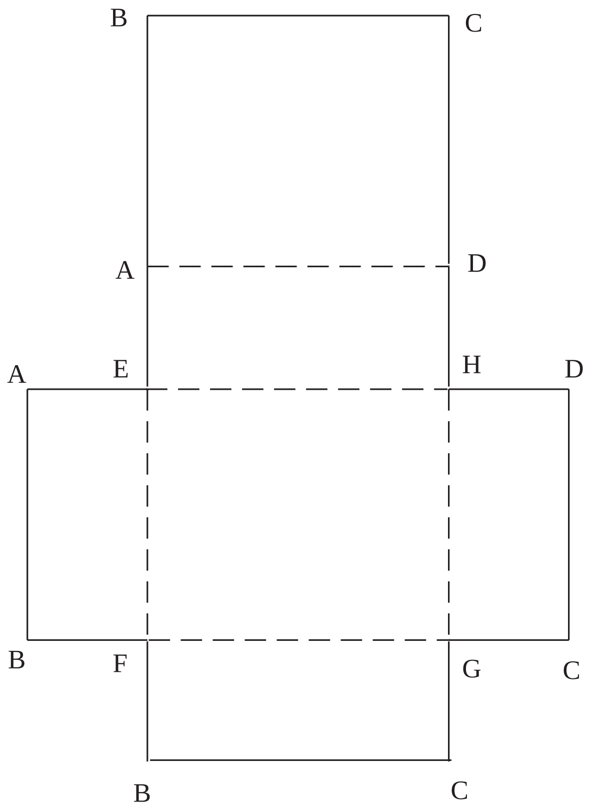


Solution

Imagine making cuts as below:

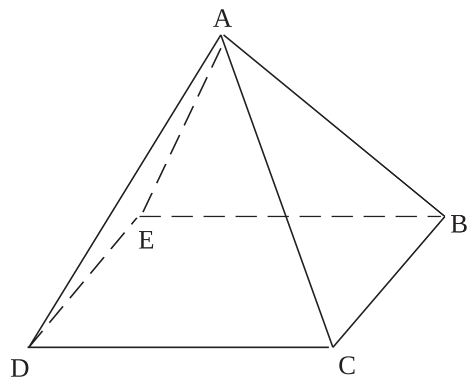
- cut along the edges AB, BC and CD to open the top like a flap.
- then cut down AE, BF, CG and DH, and press flat to give the net below.





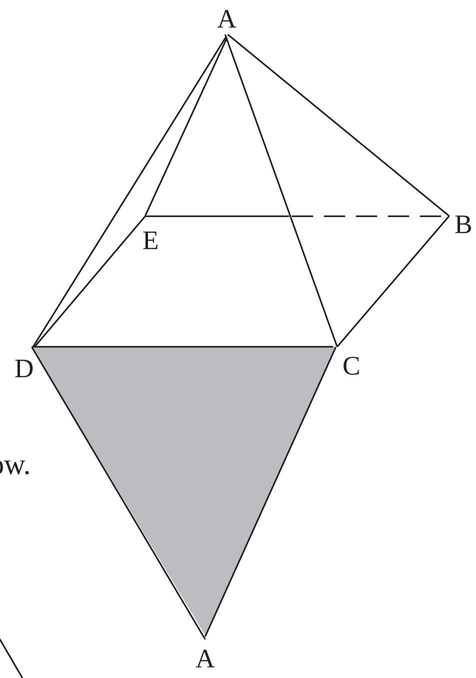
Worked Example 2

Draw the net for this square based pyramid.

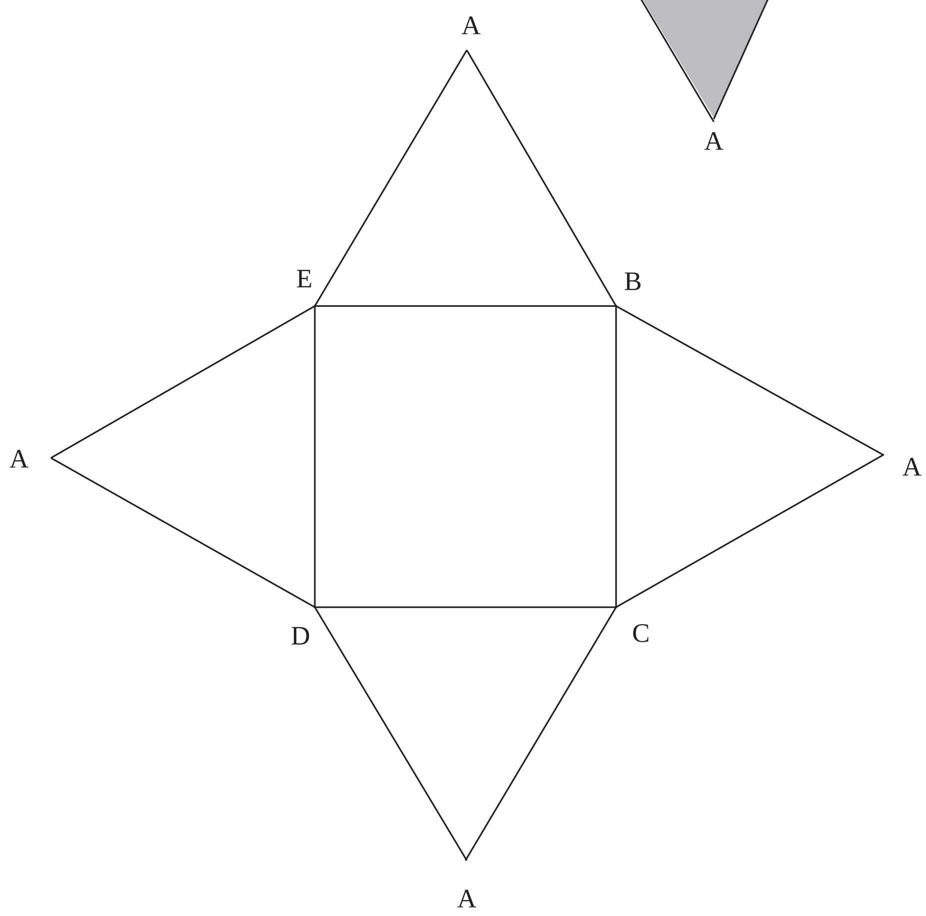


Solution

First imagine cutting down the edges AD and AC and opening out a triangle.



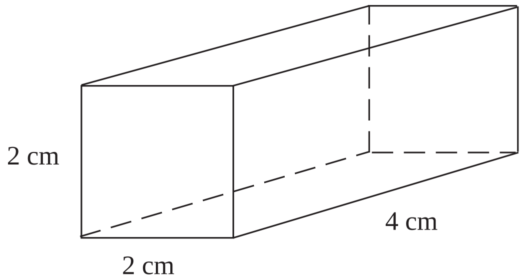
Then cutting down AB and AE gives the net below.



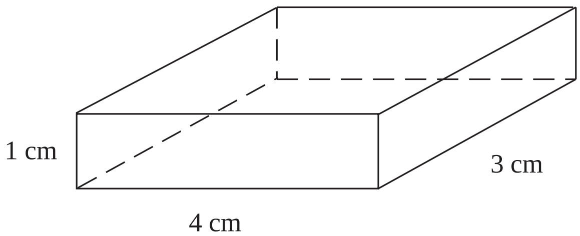
Exercises

1. Draw an accurate net for each cuboid below.

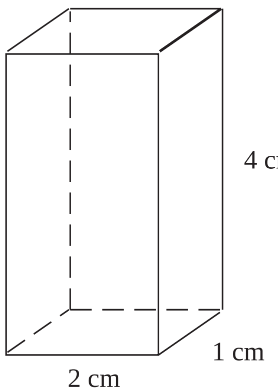
(a)



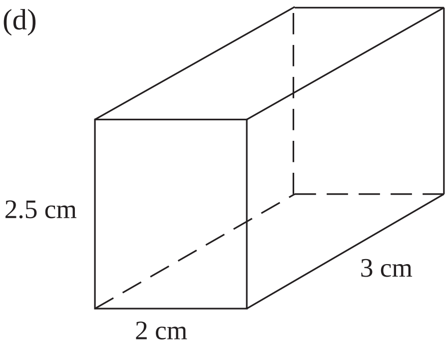
(b)



(c)

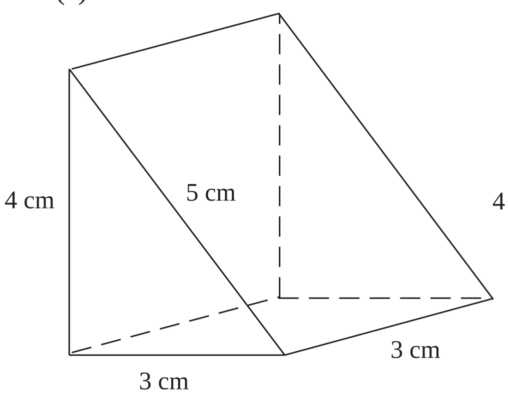


(d)

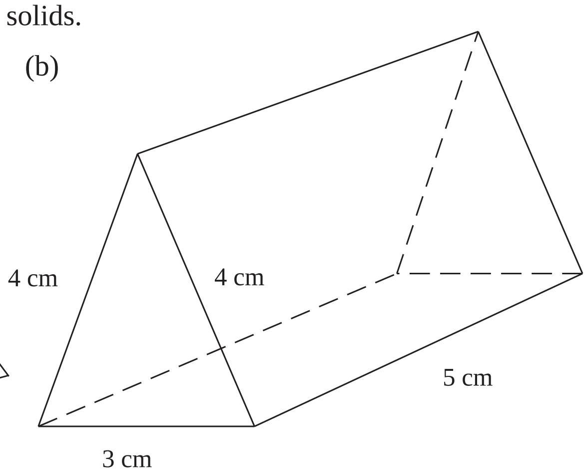


2. Draw a net for each of the following solids.

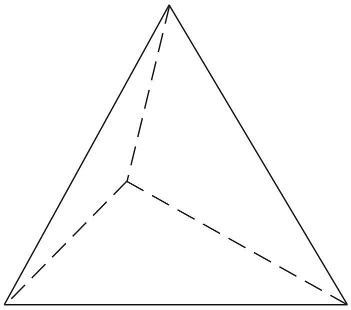
(a)



(b)

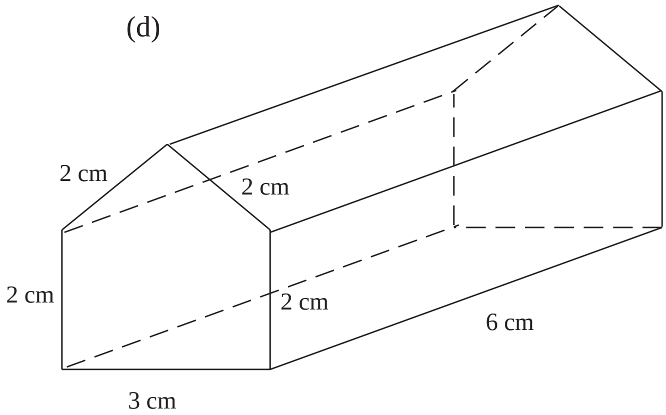


(c)

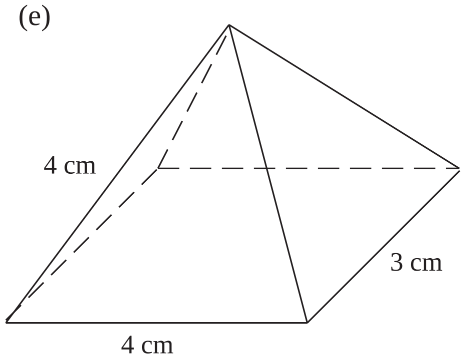


All edges 5 cm

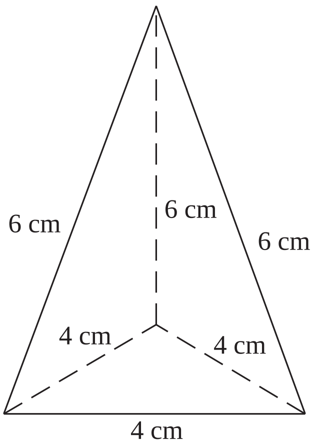
(d)



(e)



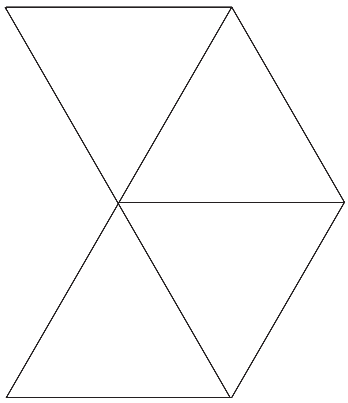
(f)



3. (a) Draw and cut out four equally sized equilateral triangles.

   (b) How many different ways can they be arranged with sides joined together?

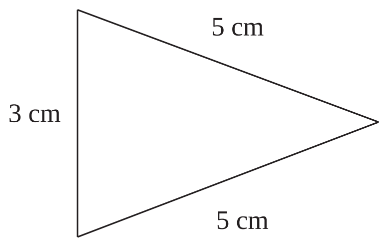
   One example is shown.



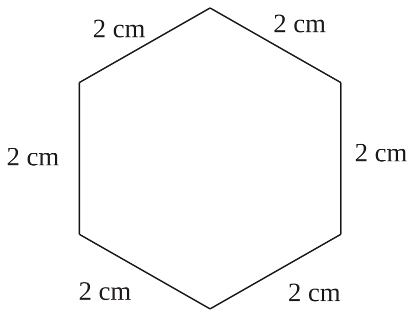
   (c) Which of your arrangements of triangles form a net for a tetrahedron?

4. The diagrams below show the ends of two of prisms that each have length of 8 cm. Draw a net for each prism.

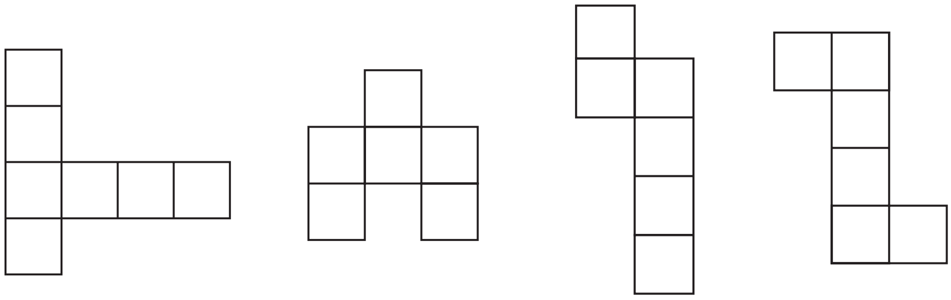
(a)



(b)



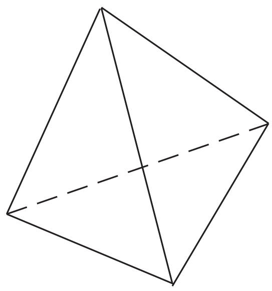
5. Which one of these nets can be folded to make a cube?



P Q R S

(SEG)

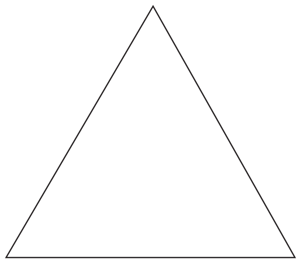
6.



The diagram above shows a pyramid with four equal triangular faces.

Each edge is 4 cm long.

Below is one of the faces.

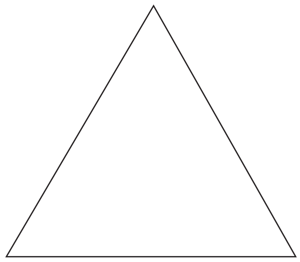


(a) What is the special name given to this kind of triangle?

(b) What is the size of each angle of this triangle?

(c) Construct an accurate net for the pyramid.

One face has been drawn for you.



(NEAB)

7.5 Conversion of Units

It is useful to be aware of both metric and imperial units and to be able to convert between them.

Imperial Units

1 foot	=	12 inches
1 yard	=	3 feet
1 pound (lb)	=	16 ounces
1 stone	=	14 pounds
1 gallon	=	8 pints

Conversion Facts

1 kg is about 2.2 lbs.

1 gallon is about 4.5 litres.
1 litre is about 1.75 pints.

5 miles is about 8 km.
1 inch is about 2.5 cm.
1 foot is about 30 cm.

Worked Example 1

John is measured. His height is 5 feet and 8 inches.

Find his height in:

(a) inches, (b) centimetres (c) metres.

Solution

(a) There are 12 inches in one foot, so

$$\begin{aligned} \text{John's height} &= 5 \times 12 + 8 \\ &= 60 + 8 \\ &= 68 \text{ inches} \end{aligned}$$

(b) 1 inch is about 2.5 cm, so

$$\text{John's height} = 68 \times 2.5$$
$$= 170 \text{ cm}$$

(c) 1 metre = 100 cm, so

$$\text{John's height} = 1.7 \text{ m}$$

Worked Example 2

A family travels 365 miles on holiday. Convert this distance to km.

Solution

As 5 miles is approximately equal to 8 km, first divide by 5 and then multiply by 8.

$$365 \div 5 = 73$$
$$73 \times 8 = 584$$

So 365 miles is approximately the same as 584 km.

Worked Example 3

Jared weighs 8 stone and 5 pounds. Find Jared's weight in:

(a) pounds,

(b) kg.

Solution

(a) There are 14 pounds in 1 stone, so

$$\text{Jared's weight} = 8 \times 14 + 5$$
$$= 112 + 5$$
$$= 117 \text{ lbs}$$

(b) As 1 pound is about 0.45 kg,

$$\text{Jared's weight} = 117 \times 0.45$$
$$= 53 \text{ kg} \quad \text{(to the nearest kg)}$$

Worked Example 4

A line is 80 cm long. Convert this length to inches.

Solution

$$1 \text{ inch} = 2.5 \text{ cm}$$

$\frac{80}{2.5} = 32$, so the line is about 32 inches long.

Exercises

1. Convert each quantity to the units given.

(a)	3 inches to cm	(b)	18 stone to pounds
(c)	6 lbs to ounces	(d)	6 feet 3 inches to inches
(e)	15 kg to lbs	(f)	3 yards to inches
(g)	3 feet to cm	(h)	5 gallons to litres
(i)	120 inches to cm	(j)	45 kg to lbs
(k)	9 litres to pints	(l)	45 gallons to litres
(m)	8 litres to pints	(n)	6 gallons to pints

2. Convert each quantity to the units given. Give answers to 1 decimal place.

(a)	8 lbs to kg	(b)	3 lbs to kg
(c)	16 pints to litres	(d)	10 cm to inches
(e)	400 cm to feet	(f)	80 ounces to pounds
(g)	182 lbs to stones	(h)	50 litres to gallons
(i)	84 inches to feet	(j)	52 cm to inches
(k)	16 litres to gallons	(l)	3 pints to litres
(m)	6 lbs to kg	(n)	212 cm to feet

3. The table gives the distances between some towns in miles. Convert the distances to km, giving your answer to the nearest km.

	Norwich	Great Yarmouth	Lowestoft
Great Yarmouth	19		
Lowestoft	27	11	
Beccles	18	20	9

4. A car travels on average 10 km for every litre of petrol. The car is driven from Leicester to Peterborough, a distance of 41 miles.

   (a) How far does the car travel in km?

   (b) How many litres of petrol are used?

   (c) How many gallons of petrol are used?

5. A recipe for a large cake includes the following ingredients.

$\frac{3}{4}$ pint	orange juice
3 lbs	flour
$\frac{1}{2}$ lb	butter
2 lbs	mixed fruit

Convert these units to litres or kg, giving your answers to 2 decimal places.

6. The Krishnan family is going on holiday with their caravan. The length of their car is 12 feet 10 inches and the length of their caravan is 16 feet 8 inches.

Find the total length of the car and caravan in

(a) inches, (b) cm, (c) metres.

7. James is 6 feet 2 inches tall and weighs 11 stone 5 pounds.

Michael is 180 cm tall and weighs 68 kg.

Who is the taller and who is the heavier?

8. Jane and Christopher go strawberry picking. Jane picks 8 kg and Christopher picks 15 lbs. Who has picked the greater weight of strawberries?

9. A customer asks for a sheet of glass 15 inches by 24 inches, What would be the area of the glass in cm^2?

10. Rohan is going to buy a new car. He tries out two different ones.

The first car he tries out travels 50 miles on 2 gallons of petrol.

The second car travels 100 km on 12 litres of petrol.

Find the petrol consumption in litres per km for both cars.

Which is the more economical?

11. Here is a rule to change miles into kilometres.

Multiply the number of miles by 8 Divide by 5

(a) Use this rule to change 30 miles into kilometres.

(b) Write down an equation connecting kilometres (K) and miles (M).

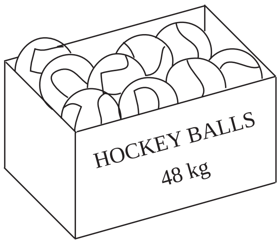
(c) Use your equation to find the value of M when $K = 100$.

(NEAB)

12. (a) Convert 48 kg to grams.

A box contains 280 hockey balls.

The hockey balls weigh 48 kg.



(b) Calculate the weight of one hockey ball to the nearest gram.

One kilogram is approximately 2.2 pounds.

(c) Estimate the weight of the box of hockey balls in pounds.

(SEG)

13. The same quantity can sometimes be measured in different units.

(a) Write out the statement below, filling in the missing unit.

Choose the unit from this list:

millimetres, centimetres, metres, kilometres

1 inch = 2.54 . . . .

(b) Write out the statement below, filling in the missing unit.

Choose the unit from this list:

millimetres, litres, gallons, cubic metres

4 pints = 2.27 . . . .

(MEG)

14. (a) Megan is 5 feet 3 inches tall.

1 cm = 0.394 inches

12 inches = 1 foot

Calculate Megan's height in centimetres. Give your answer to an appropriate degree of accuracy.

(b) An electronic weighing scale gives Megan's weight as 63.4792 kg.

Give her weight correct to an appropriate degree of accuracy.

(SEG)

15. A ball bearing has mass 0.44 pounds.

1 kg = 2.2 pounds.

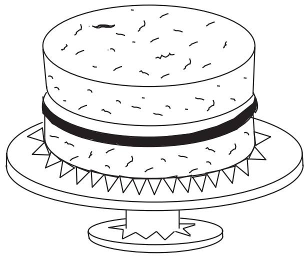
(a) Calculate the mass of the ball bearing in kilograms.

(b) $\text{Density} = \dfrac{\text{Mass}}{\text{Volume}}$

When the mass is measured in kg and the volume is measured in cm^3, what are the units of the density?

(SEG)

16. A recipe for a cake for four people uses



4 eggs

8 ounces sugar

4 ounces butter

14 ounces flour

$\frac{1}{4}$ pint milk

16 ounces = 1 pound

James finds a 500 g bag of flour in the cupboard.

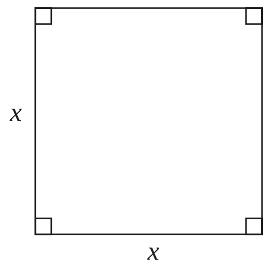
Will he have enough flour for this recipe?

Clearly explain your reasoning.

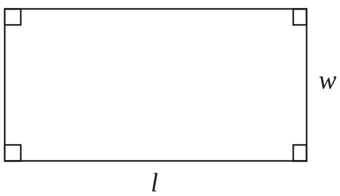
(NEAB)

7.6 Squares, Rectangles and Triangles

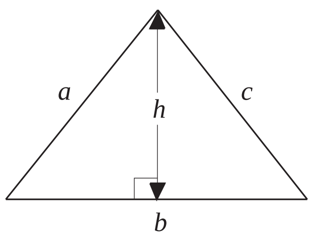
For a *square*, the area is given by $x \times x = x^2$ and the perimeter by $4x$, where x is the length of a side.



For a *rectangle*, the area is given by lw and the perimeter by $2(l + w)$, where l is the length and w the width.



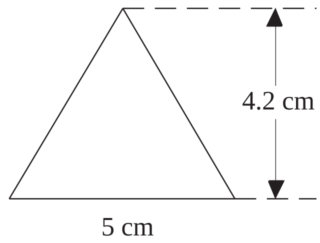
For a *triangle*, the area is given by $\frac{1}{2}bh$ and the perimeter by $a + b + c$, where b is the length of the base, h the height and a and c are the lengths of the other two sides.



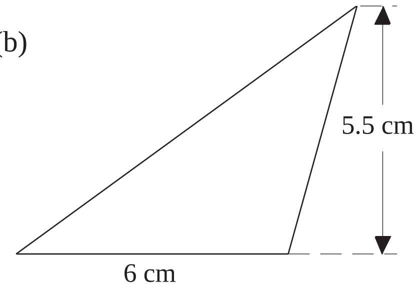
Worked Example 1

Find the area of each triangle below.

(a)



(b)



Solution

Use Area $= \frac{1}{2}bh$ or $\frac{1}{2} \times \text{base} \times \text{height}$.

(a) Area $= \frac{1}{2} \times 5 \times 4.2$

$= 10.5 \text{ cm}^2$

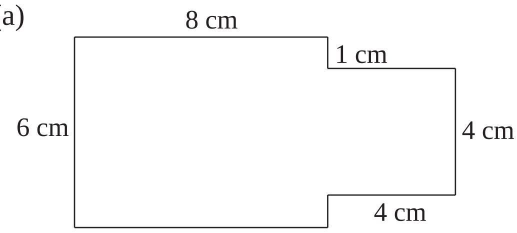
(b) Area $= \frac{1}{2} \times 6 \times 5.5$

$= 16.5 \text{ cm}^2$

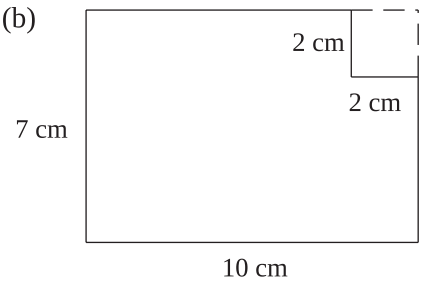
Worked Example 2

Find the perimeter and area of each shape below.

(a)



(b)



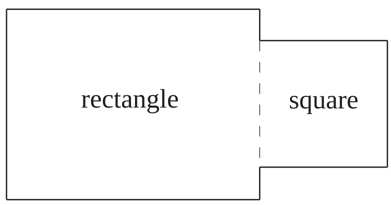
Solution

(a) The perimeter is found by adding the lengths of all the sides.

$P = 6 + 8 + 1 + 4 + 4 + 4 + 1 + 8$

$= 36 \text{ cm}$

To find the area, consider the shape split into a rectangle and a square.



Area $=$ Area of rectangle $+$ Area of square

$= 6 \times 8 + 4^2$

$= 48 + 16$

$= 64 \text{ cm}^2$

(b) Adding the lengths of the sides gives

$$P = 10 + 7 + 8 + 2 + 2 + 5$$
$$= 34 \text{ cm}$$

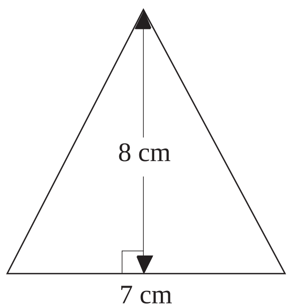
The area can be found by considering the shape to be a rectangle with a square removed from it.

$$\begin{aligned}\text{Area of shape} &= \text{Area of rectangle} - \text{Area of square}\\ &= 7 \times 10 - 2^2\\ &= 70 - 4\\ &= 66 \text{ cm}^2\end{aligned}$$

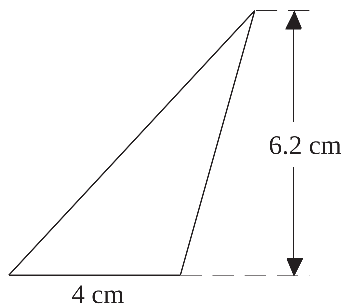
Exercises

1. Find the area of each triangle.

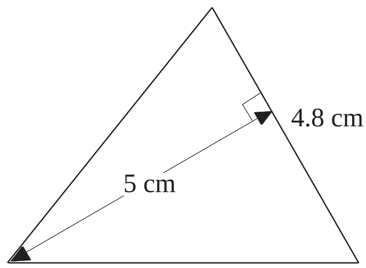
(a)



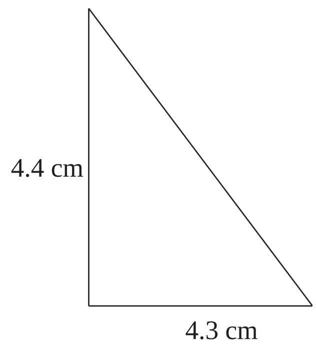
(b)



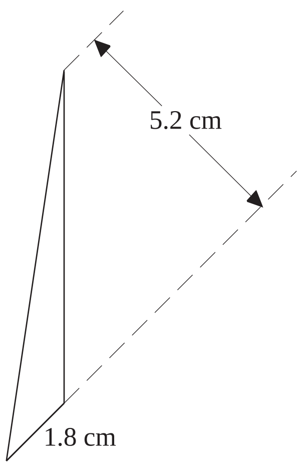
(c)



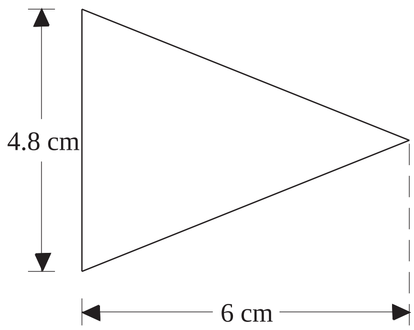
(d)



(e)

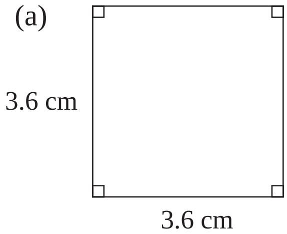


(f)

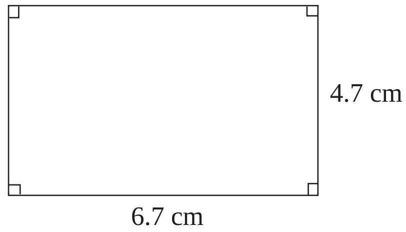


2. Find the perimeter and area of each shape below.

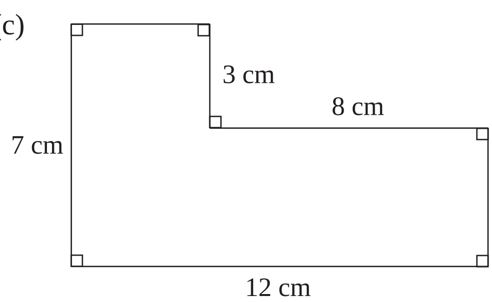
(a)



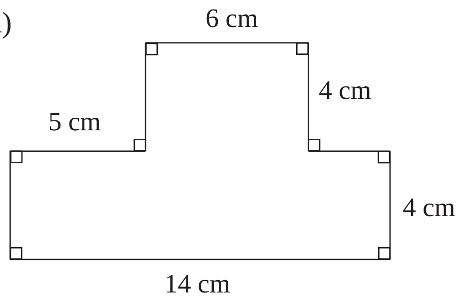
(b)



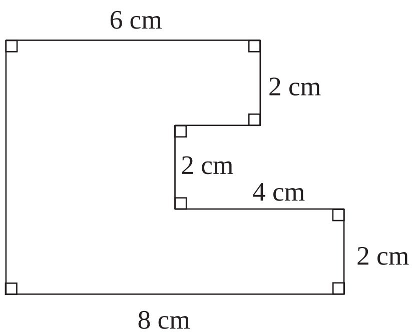
(c)



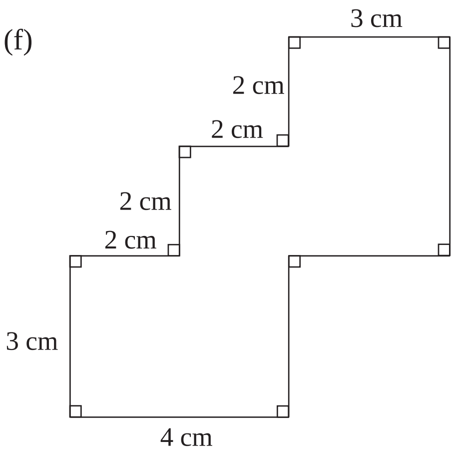
(d)



(e)

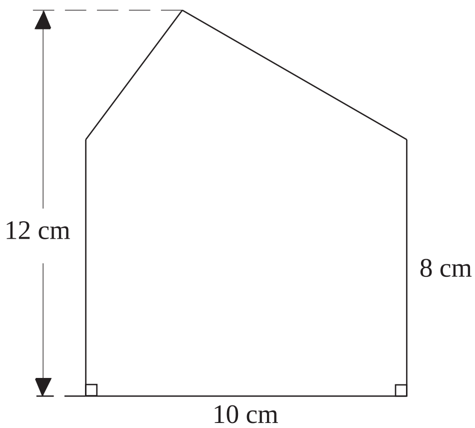


(f)

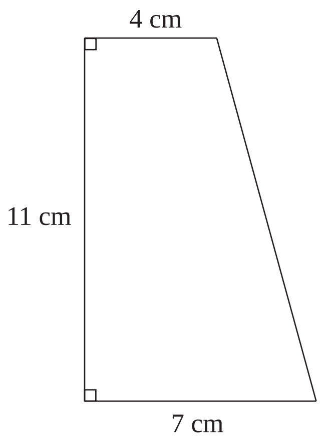


3. Find the area of each shape.

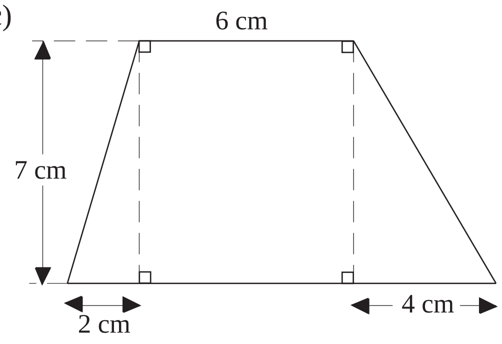
(a)



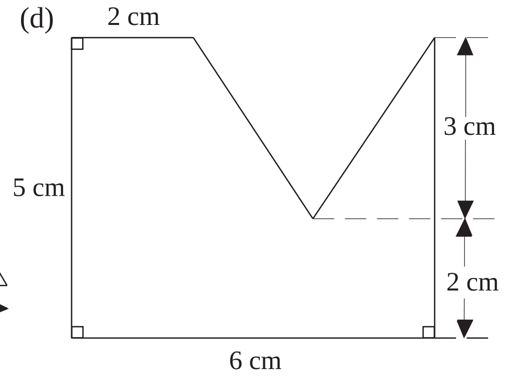
(b)



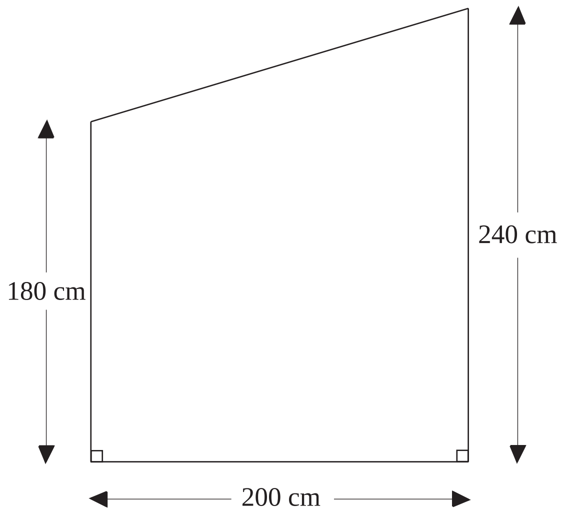
(c)



(d)



4. The diagram shows the end wall of a shed built out of concrete bricks.

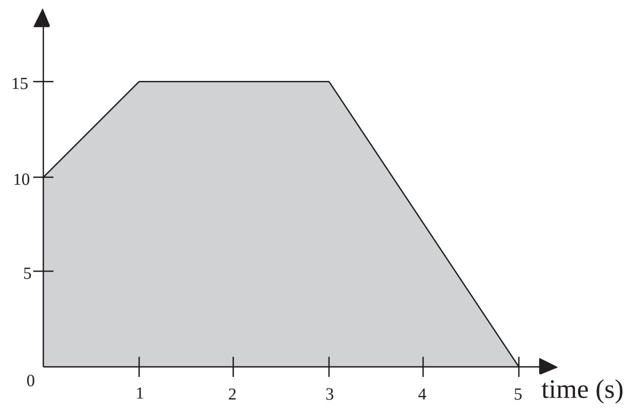


(a) Find the area of the wall.

(b) The blocks are 45 cm by 23 cm in size.

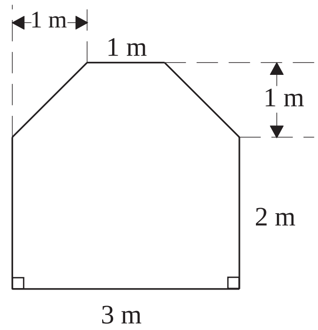
How many blocks would be needed to build the wall? (The blocks can be cut.)

5. The shaded area on the speed time graph represents the distance travelled by a car. Find the distance.



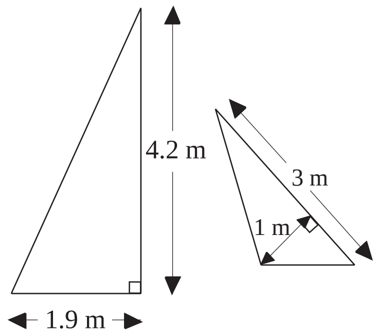
6. The plan shows the base of a conservatory.

   Find the area of the base.



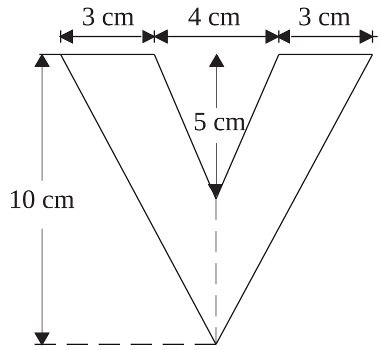
7. The diagram shows the two sails from a dinghy.

   Find their combined area.

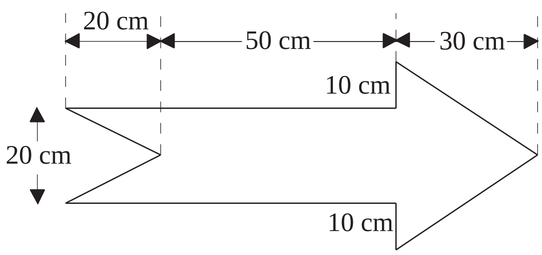


8. The diagram shows the letter V.

   Find the area of this letter.



9. Find the area of the arrow shown in the diagram.

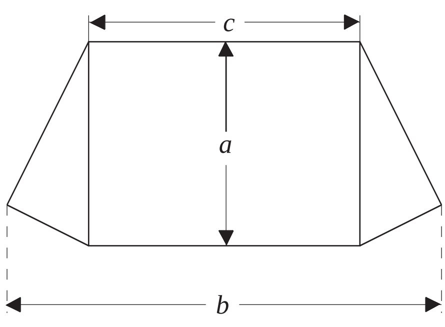


10. The diagram shows how the material required for one side of a tent is cut out.

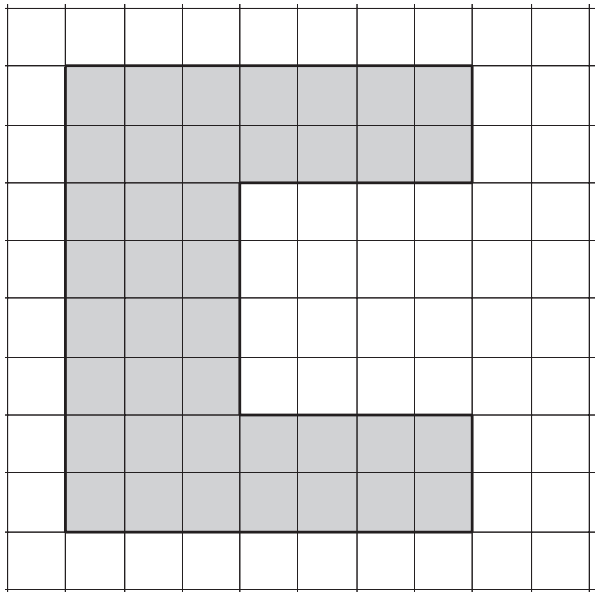
    (a) Find the area of the material shown if: $b = 3.2$ m, $c = 2$ m and

    (i) $a = 1.5$ m (ii) $a = 2$ m

    (b) Find the area if $a = 1.6$ m, $b = 3.4$ m, and $c = 2$ m.



11.



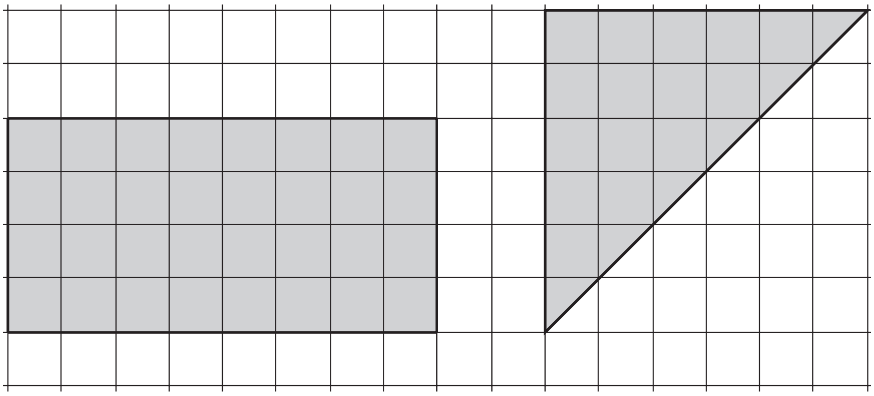
The  shape above is shaded on centimetre squared paper.

(a) Find the perimeter of this shape.

(b) Find the area of this shape.

(MEG)

12.



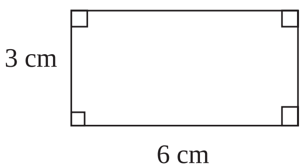
(a) What is the perimeter of the rectangle?

(b) What is the area of the triangle?

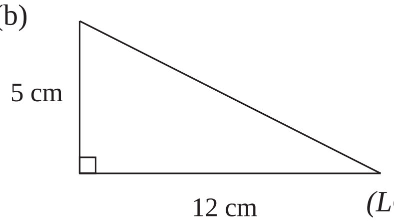
(SEG)

13. Work out the areas of these shapes.

(a)

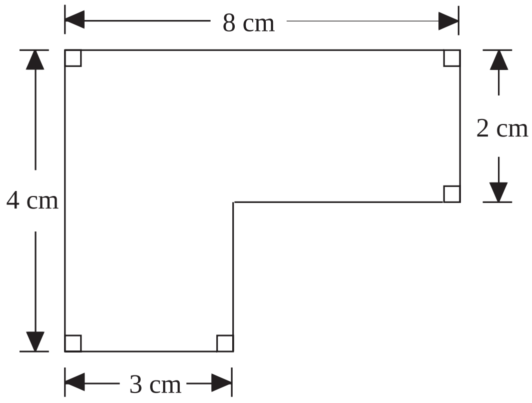


(b)



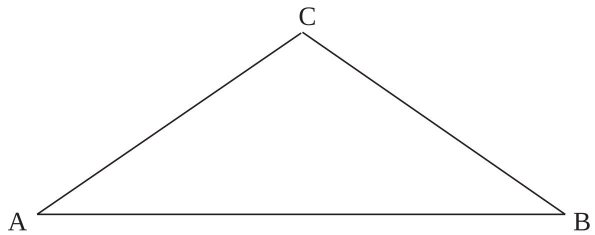
(LON)

14. Calculate the area of this shape.



(LON)

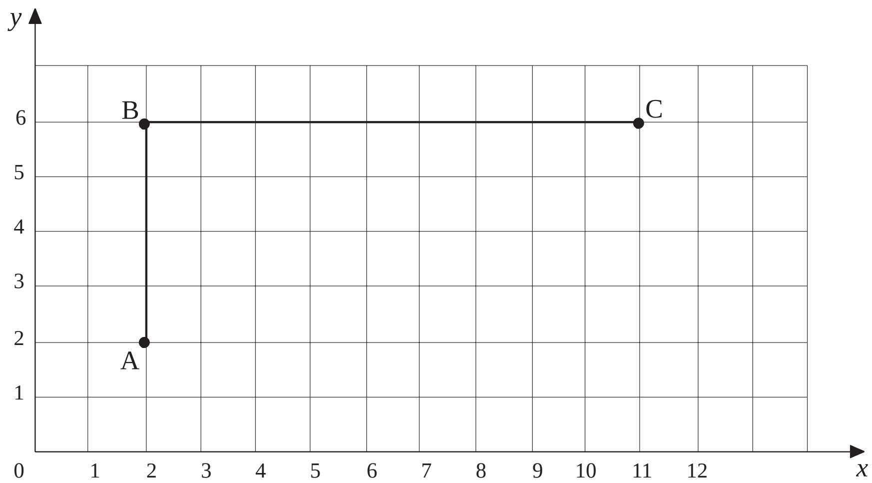
15.



By making and using appropriate measurements, calculate the area of triangle ABC in square centimetres. State the measurements that you have made and show your working clearly.

(MEG)

16.



(a) Write down the coordinates of the mid-point of AC.

(b) Copy the diagram and mark and label a point D so that ABCD is a rectangle.

(c) (i) Find the perimeter of the rectangle ABCD.

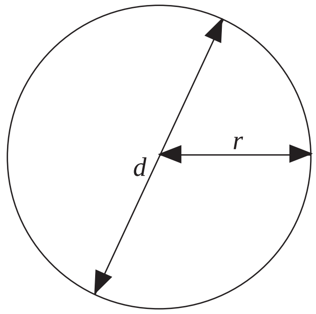
(ii) Find the area of the rectangle ABCD.

(d) The rectangle has reflective (line) symmetry.

Describe another type of symmetry that it has.

(MEG)

7.7 Area and Circumference of Circles



The *circumference* of a circle can be calculated using

$$\boxed{C = 2\pi r \quad \text{or} \quad C = \pi d}$$

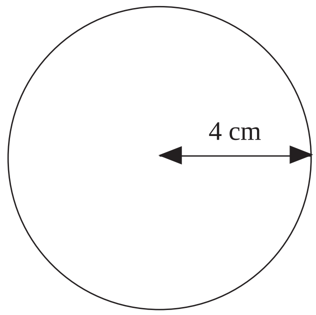
where r is the radius and d the diameter of the circle.

The *area* of a circle is found using

$$\boxed{A = \pi r^2 \quad \text{or} \quad A = \frac{\pi d^2}{4}}$$

Worked Example 1

Find the circumference and area of this circle.



Solution

The circumference is found using $C = 2\pi r$, which in this case gives

$$\begin{aligned} C &= 2\pi \times 4 \\ &= 25.1 \text{ cm} \qquad \text{(to one decimal place)} \end{aligned}$$

The area is found using $A = \pi r^2$, which gives

$$\begin{aligned} A &= \pi \times 4^2 \\ &= 50.3 \text{ cm}^2 \qquad \text{(to one decimal place)} \end{aligned}$$

Worked Example 2

Find the radius of a circle if:

(a) its circumference is 32 cm, (b) its area is 14.3 cm^2.

Solution

(a) Using $C = 2\pi r$ gives

$$32 = 2\pi r$$

and dividing by 2π gives

$$\frac{32}{2\pi} = r$$

so that

$$r = 5.10 \text{ cm} \quad \text{(to 2 decimal places)}$$

(b) Using $A = \pi r^2$ gives

$$14.3 = \pi r^2$$

Dividing by π gives

$$\frac{14.3}{\pi} = r^2$$

Then taking the square root of both sides gives

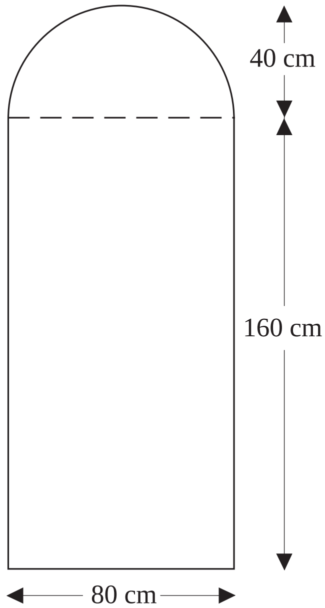
$$\sqrt{\frac{14.3}{\pi}} = r$$

so that

$$r = 2.13 \text{ cm} \quad \text{(to 2 decimal places)}$$

Worked Example 3

Find the area of the door shown in the diagram.
The top part of the door is a semicircle.



Solution

First find the area of the rectangle.

$$\begin{aligned} \text{Area } &= 80 \times 160 \\ &= 12\,800 \text{ cm}^2 \end{aligned}$$

Then find the area of the semicircle.

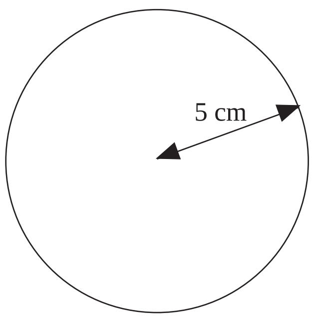
$$\begin{aligned} \text{Area } &= \frac{1}{2} \times \pi \times 40^2 \\ &= 2513 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{Total area } &= 12\,800 + 2513 \\ &= 15\,313 \text{ cm}^2 \quad \text{(to the nearest cm}^2\text{)} \end{aligned}$$

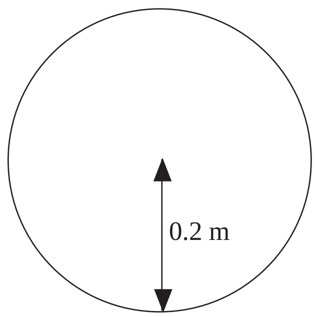
Exercises

1. Find the circumference and area of each circle shown below.

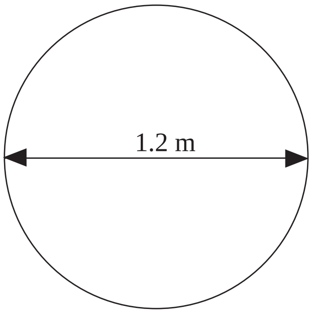
(a)



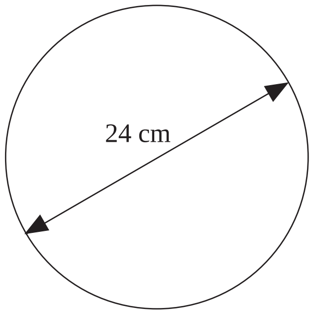
(b)



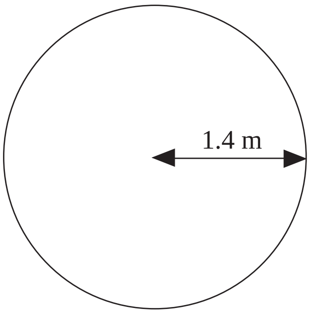
(c)



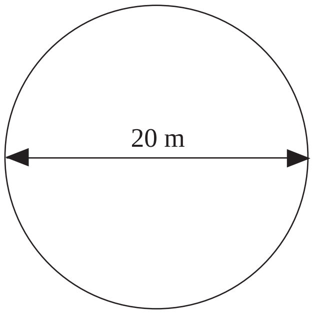
(d)



(e)



(f)



2. Find the radius of the circle which has:

   (a) a circumference of 42 cm,

   (b) a circumference of 18 cm,

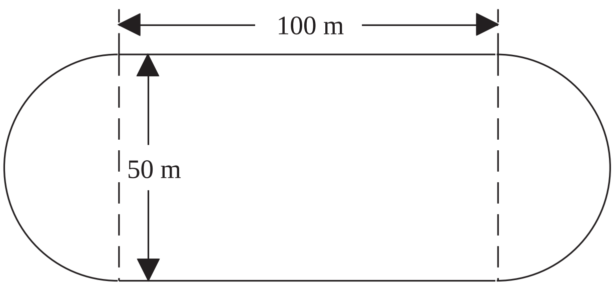
   (c) an area of 69.4 cm^2,

   (d) an area of 91.6 cm^2.

3. The diagram shows a running track.

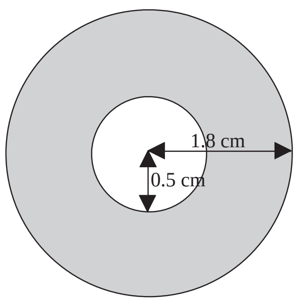
   (a) Find the length of one complete circuit of the track.

   (b) Find the area enclosed by the track.

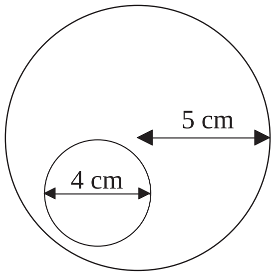


4. A washer has an outer radius of 1.8 cm and an inner radius of 0.5 cm.

   Find the area that has been shaded in the diagram, to the nearest cm^2.



5. An egg, fried perfectly, can be thought of as a circle (the yolk) within a larger circle (the white).



(a) Find the area of the smaller circle that represents the surface of the yolk.

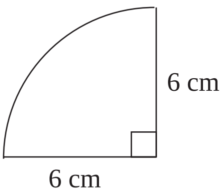
(b) Find the area of the surface of the whole egg.

(c) Find the area of the surface of the white of the egg, to the nearest cm^2 .

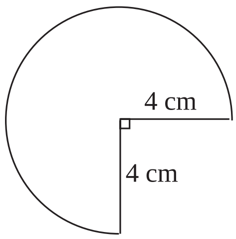
6. The shapes shown below were cut out of card, ready to make cones.

Find the area of each shape.

(a)



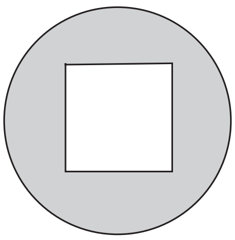
(b)



7. A circular hole with diameter 5 cm is cut out of a rectangular metal plate of length 10 cm and width 7 cm. Find the area of the plate when the hole has been cut out.

8. Find the area of the wasted material if two circles of radius 4 cm are cut out of a rectangular sheet of material that is 16 cm long and 8 cm wide.

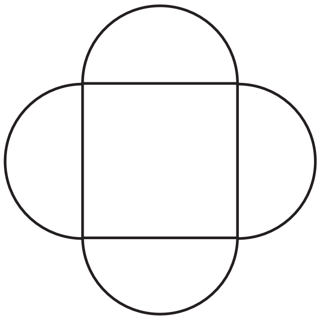
9. A square hole is cut in a circular piece of card to create the shape shown.



(a) Find the shaded area of the card if the radius of the circle is 5.2 cm and the sides of the square are 4.8 cm.

(b) Find the radius of the circle if the shaded area is 50 cm^2 and the square has sides of length 4.2 cm.

10. Four semicircles are fixed to the sides of a square as shown in the diagram, to form a design for a table top.



(a) Find the area of the table top if the square has sides of length 1.5 m.

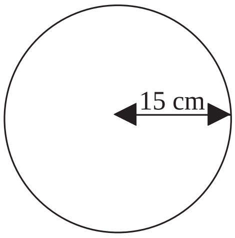
(b) Find the length of the sides of the square and the total area of the table top if the area of each semicircle is 1 m^2.

11. The radius of a circle is 8 cm.

Work out the area of the circle.
(Use $\pi = 3.14$ or the π button on your calculator.)

(LON)

12. A circle has a radius of 15 cm.



Calculate the area of the circle.

Take π to be 3.14 or use the π key on your calculator.

(SEG)

13. Louise does a sponsored bicycle ride.

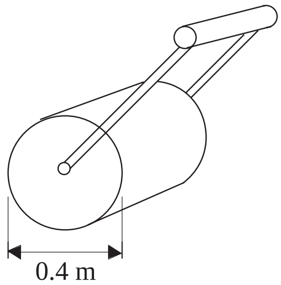
Each wheel of her bicycle is of radius 25 cm.

(a) Calculate the circumference of one of the wheels

(b) She cycles 50 km. How many revolutions does a wheel make during the sponsored ride?

(NEAB)

14. The diameter of a garden roller is 0.4 m.



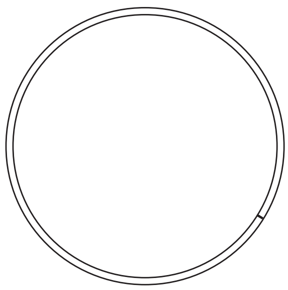
The roller is used on a path of length 20 m.

Calculate how many times the roller rotates when rolling the length of the path once.

Take π to be 3.14 or use the π key on your calculator.

(SEG)

15.



A piece of rope is 12 metres long. It is laid on the ground in a circle, as shown in the diagram.

(a) Using 3.14 as the value of π, calculate the diameter of the circle.

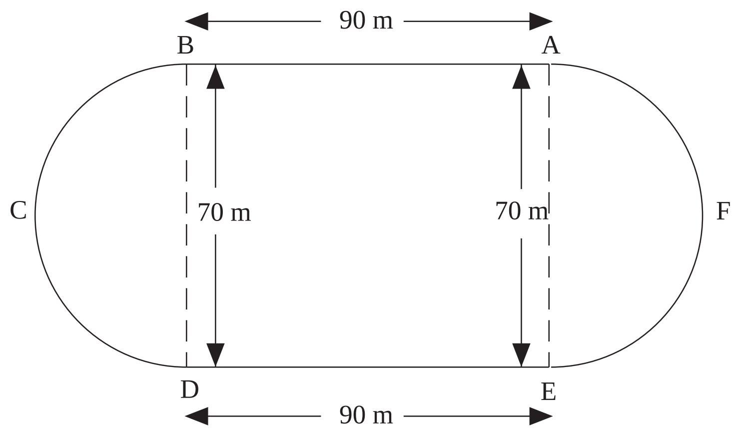
(b) Explain briefly how you would check the answer to part (a) mentally.

The cross-section of the rope is a circle of radius 1.2 cm.

(c) Calculate the area of the cross-section.

(MEG)

16.



The diagram shows a running track.

BA and DE are parallel and straight. They are each of length 90 metres.

BCD and EFA are semicircular. They each have a diameter of length 70 metres.

(a) Calculate the perimeter of the track.

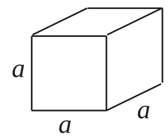
(b) Calculate the total area inside the track.

7.8 Volumes of Cubes, Cuboids, Cylinders and Prisms

The volume of a *cube* is given by

$$\boxed{V = a^3}$$

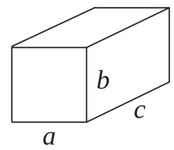
where a is the length of each side of the cube.



For a *cuboid* the volume is given by

$$\boxed{V = abc}$$

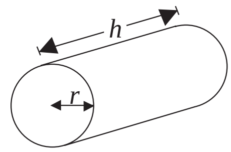
where a, b and c are the lengths shown in the diagram.



The volume of a *cylinder* is given by

$$\boxed{V = \pi r^2 h}$$

where r is the radius of the cylinder and h is its height.



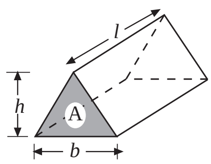
The volume of a *triangular prism* can be expressed in two ways, as

$$\boxed{V = Al}$$

where A is the area of the end and l the length of the prism, or as

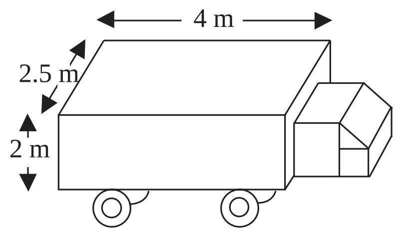
$$\boxed{V = \frac{1}{2}bhl}$$

where b is the base of the triangle and h is the height of the triangle.



Worked Example 1

The diagram shows a lorry.



Find the volume of the load-carrying part of the lorry.

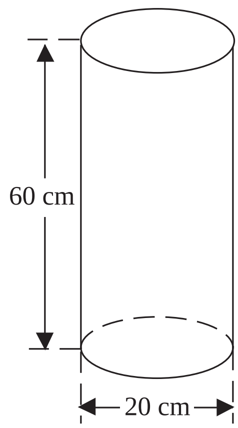
Solution

The load-carrying part of the lorry is represented by a cuboid, so its volume is given by

$$V = 2 \times 2.5 \times 4$$
$$= 20 \text{ m}^3$$

Worked Example 2

The cylindrical body of a fire extinguisher has the dimensions shown in the diagram. Find the maximum volume of water the extinguisher could hold.



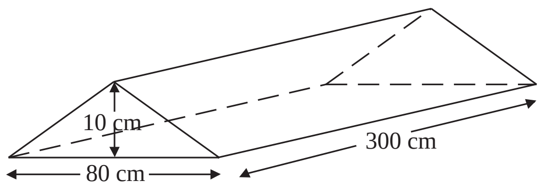
Solution

The body of the extinguisher is a cylinder with radius 10 cm and height 60 cm, so its volume is given by

$$V = \pi \times 10^2 \times 60$$
$$= 18850 \text{ cm}^3 \quad \text{(to the nearest cm}^3\text{)}$$

Worked Example 3

A 'sleeping policeman' (traffic calming road hump) is made of concrete and has the dimensions shown in the diagram. Find the volume of concrete needed to make one 'sleeping policeman'.



Solution

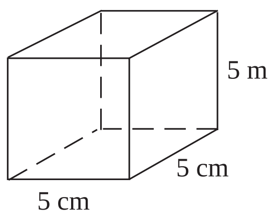
The shape is a triangular prism with $b = 80$, $h = 10$ and $l = 300$ cm. So its volume is given by

$$V = \frac{1}{2} \times 80 \times 10 \times 300$$
$$= 120\,000 \text{ cm}^3.$$

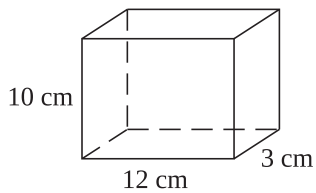
Exercises

1. Find the volume of each solid shown below.

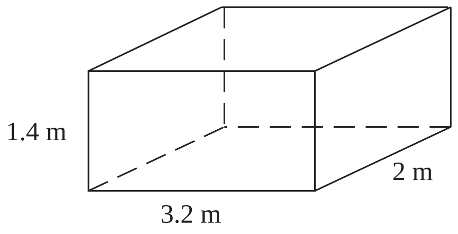
(a)



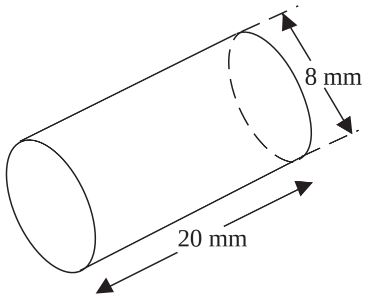
(b)



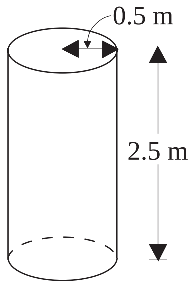
(c)



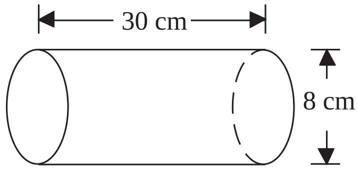
(d)



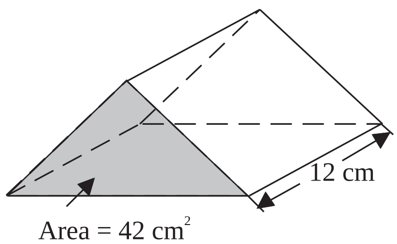
(e)



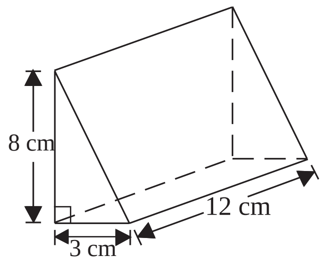
(f)



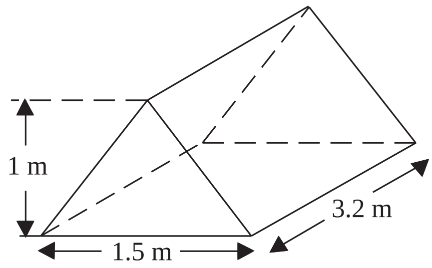
(g)



(h)

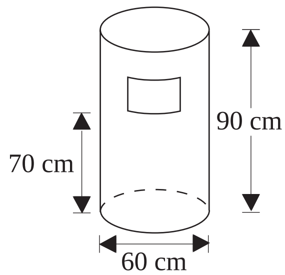


(i)



2. (a) Find the volume of the litter bin shown in the diagram, in m^3 to 2 decimal places.

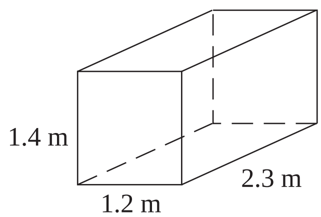
   (b) Find the volume of rubbish that can be put in the bin, if it must all be below the level of the hole in the side, in m^3 to 2 decimal places.



3. A water tank has the dimensions shown in the diagram.

   (a) Find the volume of the tank.

   (b) If the depth of water is 1.2 m, find the volume of the water.



4. A concrete pillar is a cylinder with a radius of 20 cm and a height of 2 m.

   (a) Find the volume of the pillar.

   The pillar is made of concrete, but contains 10 steel rods of length 1.8 m and diameter 1.2 cm.

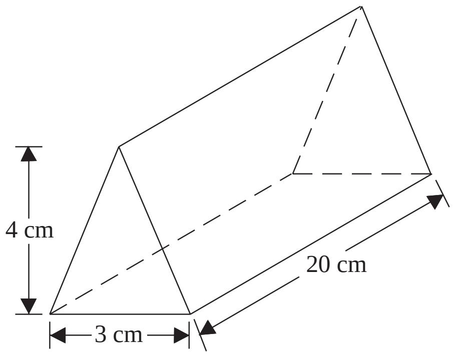
   (b) Find the volume of one of the rods and the volume of steel in the pillar.

   (c) Find the volume of concrete contained in the pillar.

5. The box shown in the diagram contains chocolate.

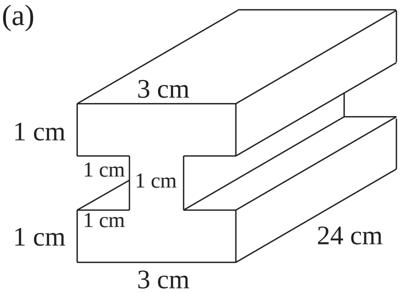
   (a) Find the volume of the box.

   (b) If the box contains 15 cm^3 of air, find the volume of the chocolate.

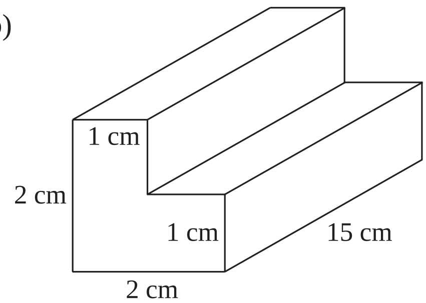


6. Find the volume of each prism below.

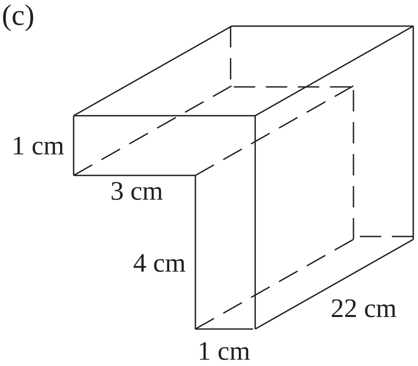
(a)



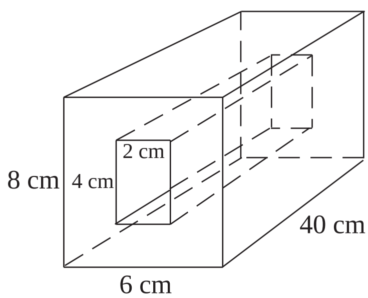
(b)



(c)

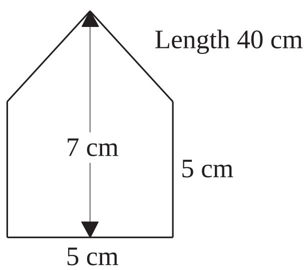


(d)

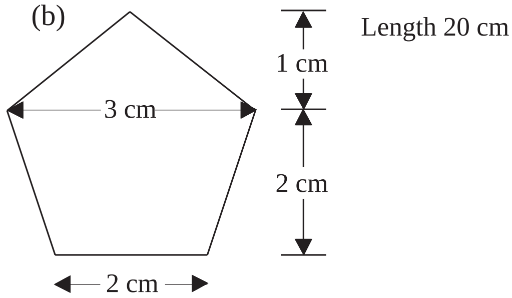


7. Each diagram below shows the cross section of a prism. Find the volume of the prism, given the length specified.

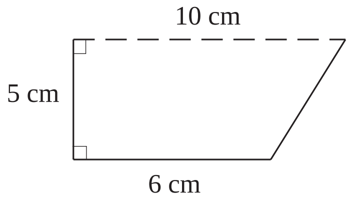
(a)



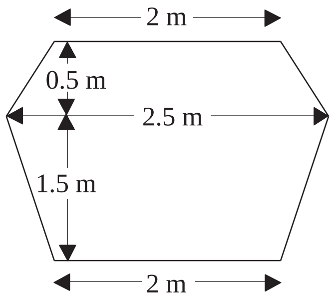
(b)



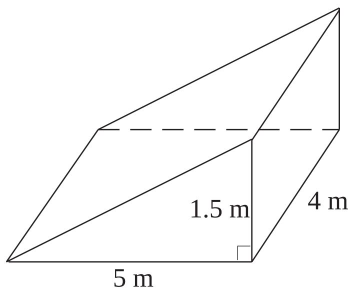
8. The diagram shows the cross section of a length of guttering. Find the maximum volume of water that a 5 m length of guttering could hold.



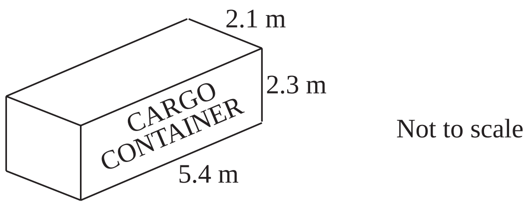
9. The diagram shows the cross section of a skip that is 15 m in length and is used to deliver sand to building sites. Find the volume of sand in the skip when it is filled level to the top.



10. A ramp is constructed out of concrete. Find the volume of concrete contained in the ramp.



11. The diagram shows a cargo container.



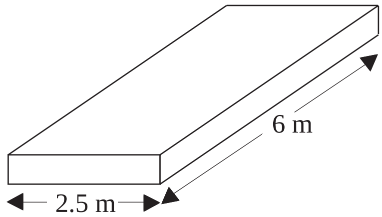
Calculate the volume of the container.

(SEG)

Just for Fun

A man wishes to take 4 litres of water out of a big tank of water, but he has only one 5-litre and one 3-litre jar. How can he do it?

12.



Not to scale

A garage has a rectangular concrete base 6 m long and 2.5 m wide.

The base is shown in the diagram.

(a) Calculate the area of the garage floor.

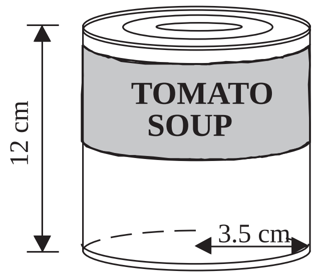
(b) The concrete is 0.2 m thick.
Calculate the volume of the concrete base.

(c) Write 0.2 m in millimetres.

(MEG)

13. Tomato soup is sold in cylindrical tins.

Each tin has a base radius of 3.5 cm and a height of 12 cm.

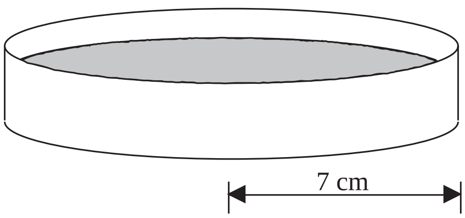


Not to scale

(a) Calculate the volume of soup in a full tin.

Take π to be 3.14 or use the π key on your calculator.

(b) Mark has a full tin of tomato soup for dinner. He pours the soup into a cylindrical bowl of radius 7 cm.

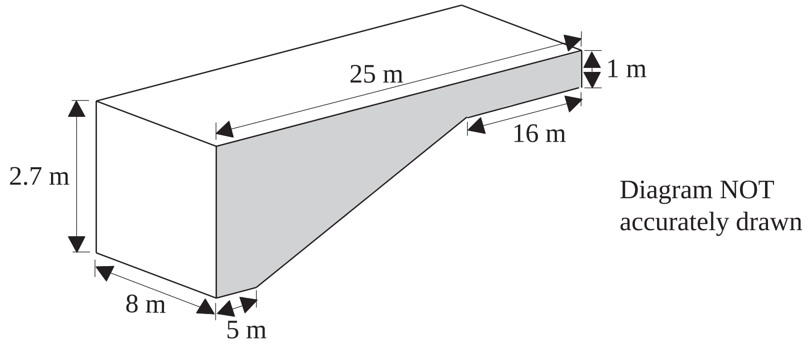


Not to scale

What is the depth of the soup in the bowl?

(SEG)

14.



The diagram represents a swimming pool.
The pool has vertical sides.
The pool is 8 m wide.

(a) Calculate the area of the shaded cross section.

The swimming pool is completely filled with water.

(b) Calculate the volume of water in the pool.

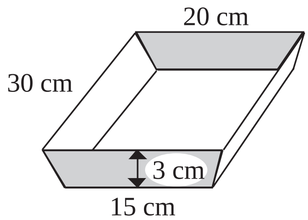
64 m^3 leaks out of the pool.

(c) Calculate the distance by which the water level falls.

(LON)

15. The diagram shows a paint trough in the shape of a prism.

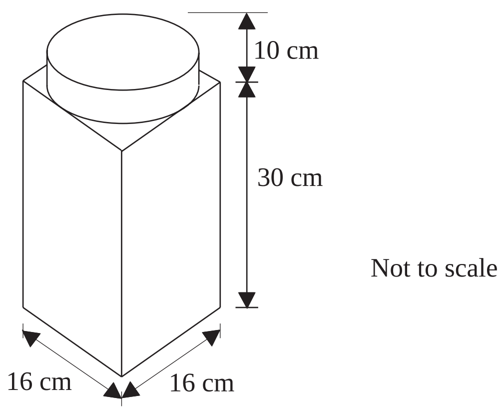
Each shaded end of the trough is a vertical trapezium.



Calculate the *volume* of paint which the trough can hold when it is full.

(SEG)

16. The diagram shows a lamp.



(a) The base of the lamp is a cuboid.

Calculate the volume of the base.

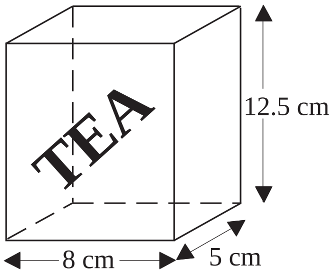
(b) The top of the lamp is a cylinder.

(i) Calculate the circumference of the cylinder.
Take π to be 3.14 or use the π key on your calculator.

(ii) Calculate the volume of the cylinder.

(SEG)

17. The diagram represents a tea packet in the shape of a cuboid.



(a) Calculate the volume of the packet.

There are 125 grams of tea in a full packet.

Jason has to design a new packet that will contain 100 grams of tea when it is full.

(b) (i) Work out the volume of the new packet.

(ii) Express the weight of the new tea packet as a percentage of the weight of the packet shown.

The new packet of tea is in the shape of a cuboid.
The base of the new packet measures 7 cm by 6 cm.

(c) (i) Work out the area of the base of the new packet.

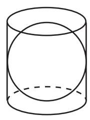
(ii) Calculate the height of the new packet.

(LON)

Information

Archimedes (287BC-212BC), a Greek Mathematician, was once entrusted with the task of finding out whether the King's crown was made of pure gold. While taking his bath, he came up with a solution and was so excited that he dashed out into the street naked shouting "Eureka" (I have found it). The container that you use in the Science laboratory to measure the volume of an irregular object is known as an Eureka can (named after this incident). Archimedes was so engrossed in his work that when his country was conquered by the Romans, he was still working hard at his mathematics. When a Roman soldier ordered him to leave his desk, Archimedes replied, "Don't disturb my circles." He was killed by that soldier for disobeying orders.

Archimedes' greatest contribution was the discovery that the volume of a sphere is $\frac{2}{3}$ *that of a cylinder whose diameter is the same as the diameter of the sphere. At his request, the sphere in the cylinder diagram was engraved on his tombstone.*



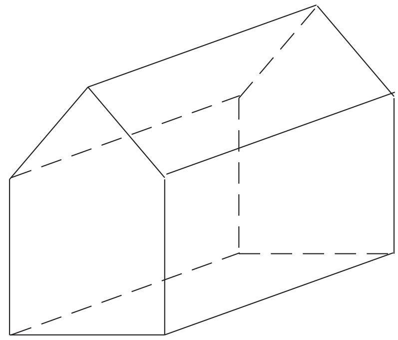
7.9 Plans and Elevations

The *plan* of a solid is the view from *above*.

The *elevation* of a solid is the view from a *side*. We often use the terms *front elevation* and *side elevation*.

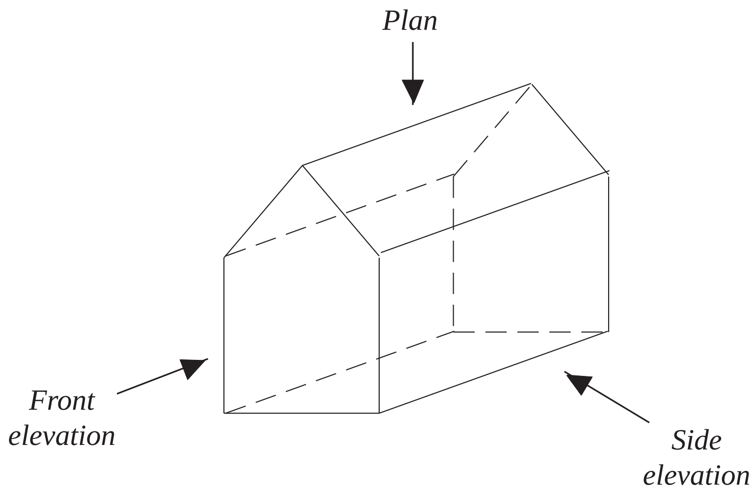
Worked Example 1

Draw the plan, front elevation and side elevation of the shed in the diagram.



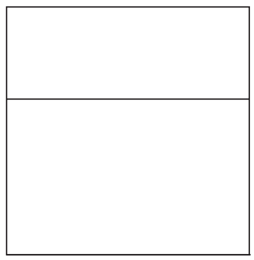
Solution

To draw the plan and elevation, look at the shed as shown below.



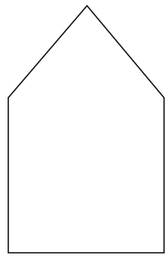
Looking from the side gives the

Side elevation



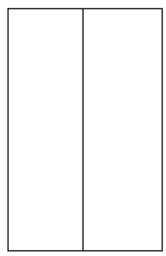
Looking from the front gives the

Front elevation



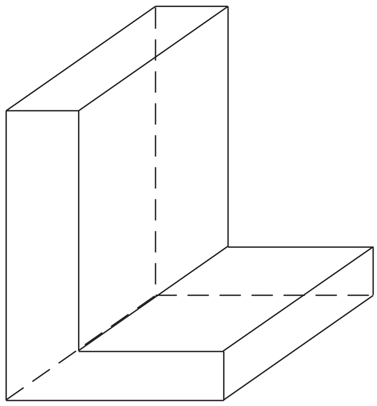
Looking from above gives the

Plan



Worked Example 2

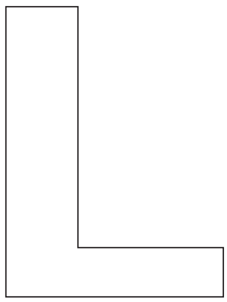
Draw the front elevation, side elevation and plan of this shape.

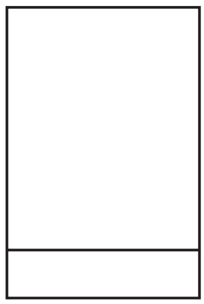


Solution

Looking from the front gives the

Front elevation



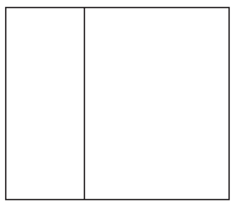


Looking from the side gives the

Side elevation

Looking from above gives the

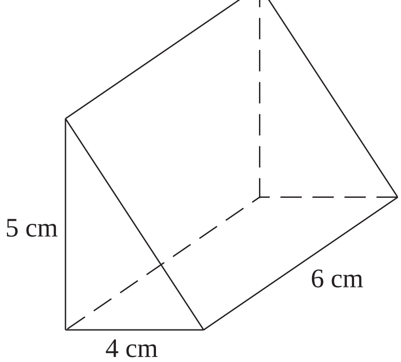
Plan



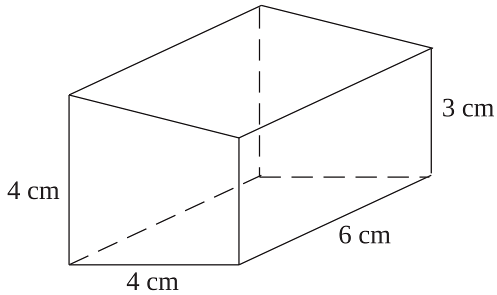
Exercises

1. Draw the front elevation, plan and side elevation for each solid shown below.

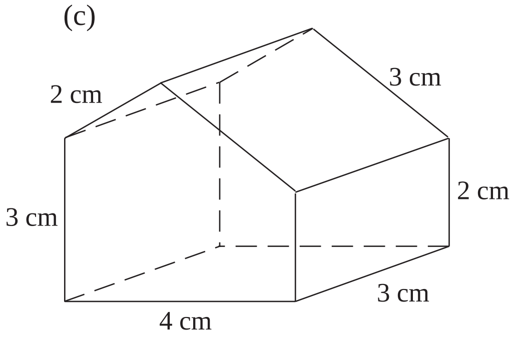
(a)



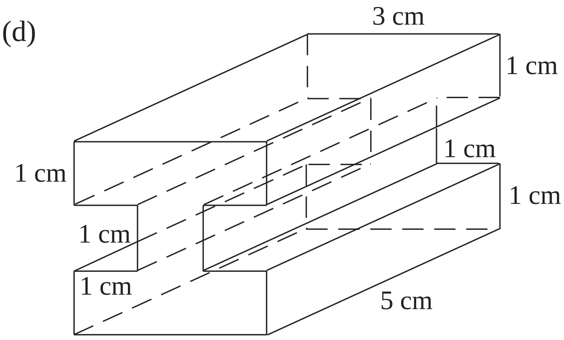
(b)



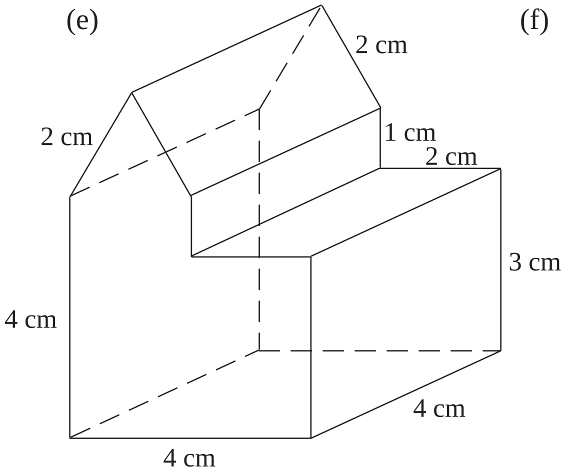
(c)



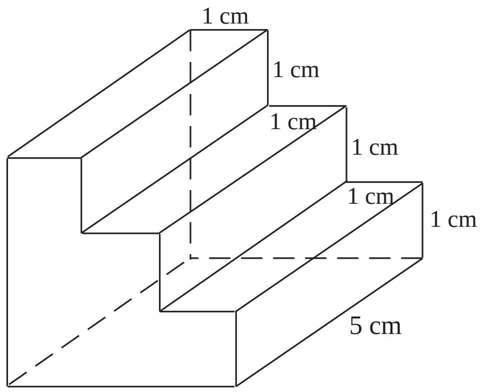
(d)



(e)



(f)



2. Draw the plan and front elevation of a square based pyramid that has a height of 6 cm and base with sides of 5 cm.

3. Draw a plan and front elevation for:

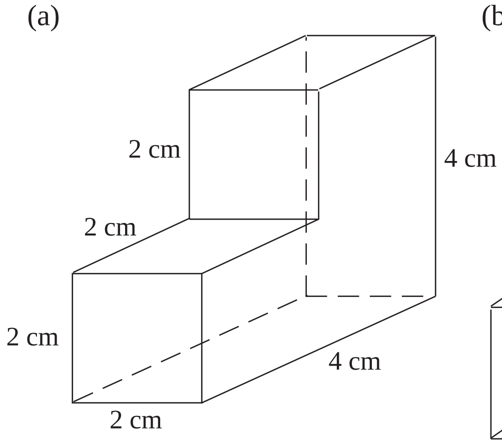
(a) a tin of baked beans, (b) a letter box,

(c) a roll of sellotape (d) a ball.

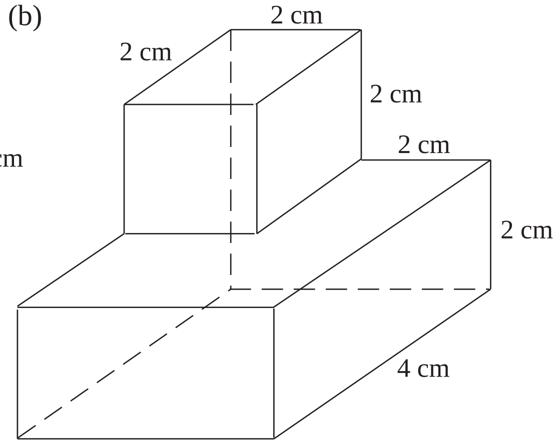
4. A pencil with a hexagonal cross section stands on one end with its point up. Draw a plan, front elevation and side elevation of the pencil.

5. Draw the front elevation, side elevation and plan of the solids below.

(a)



(b)



7.10 Using Isometric Paper

The spots on isometric paper are arranged at the corners of equilateral triangles.

Isometric paper

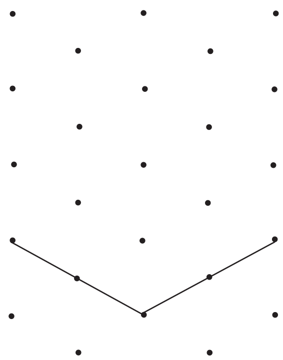
Worked Example 1

Draw a cube with sides of 2 cm on isometric paper.

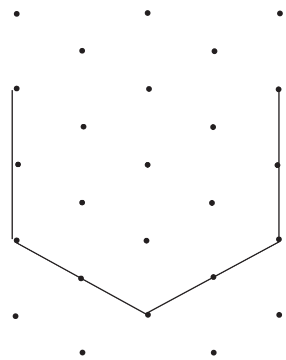
Solution

The diagrams show the three stages needed to draw the cube.

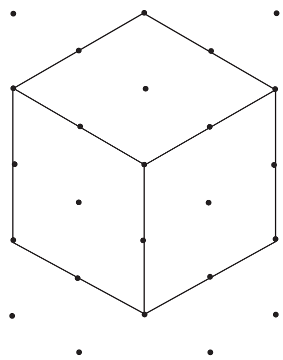
First draw the base.



Then add the upright edges.

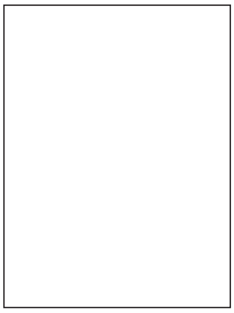


Finally add the top of the cube.

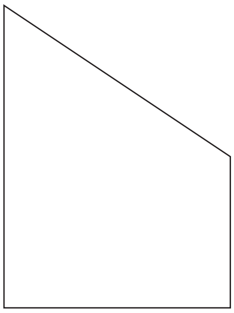


Worked Example 2

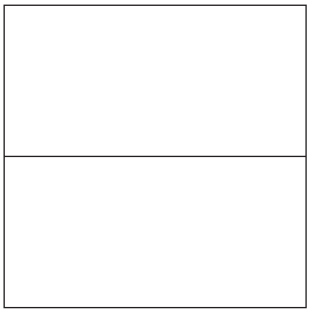
The diagrams below show the plan and elevations of a solid object.



Plan



Front elevation

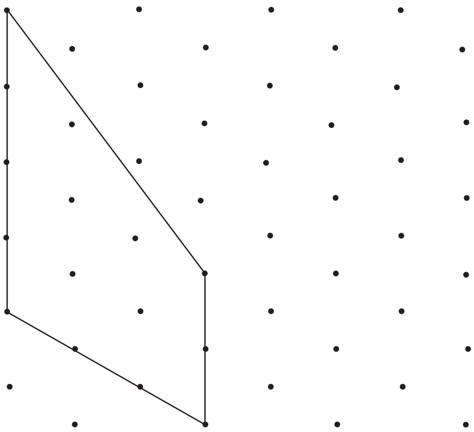


Side elevation

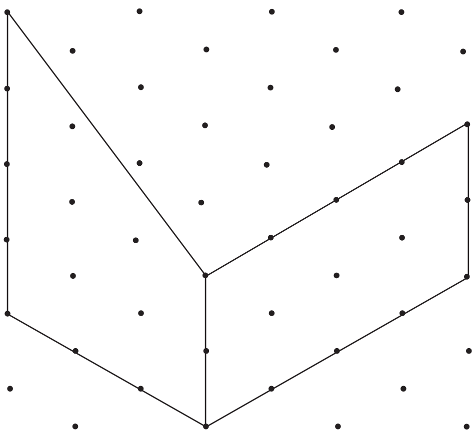
Draw the object on isometric paper.

Solution

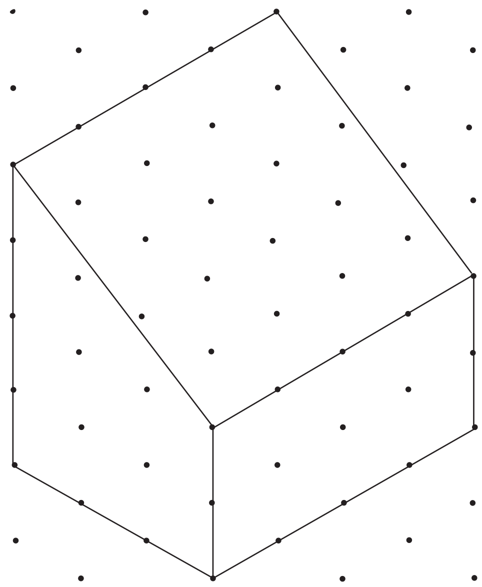
From the front elevation the front can be drawn.



Then using the side elevation the lower part of the side can be drawn.

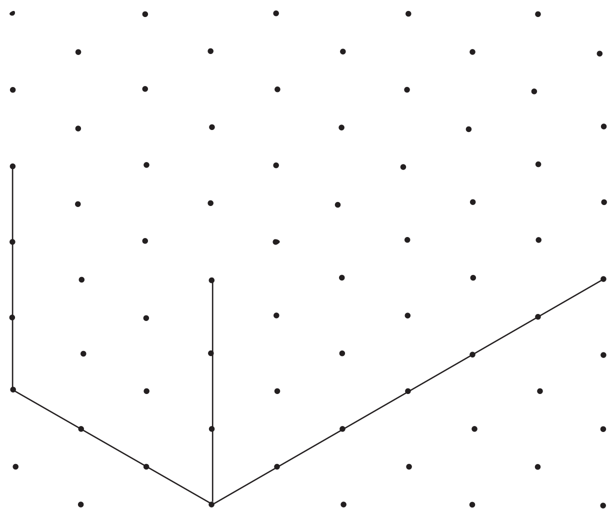


Finally the drawing can be completed.



Exercises

1. The diagram shows part of the drawing of a cuboid. Copy and complete the cuboid.

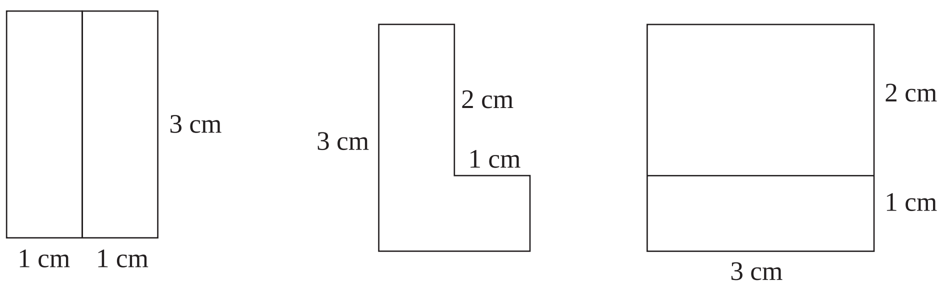


2. On isometric paper draw cubes with sides of lengths 1 cm and 3 cm.

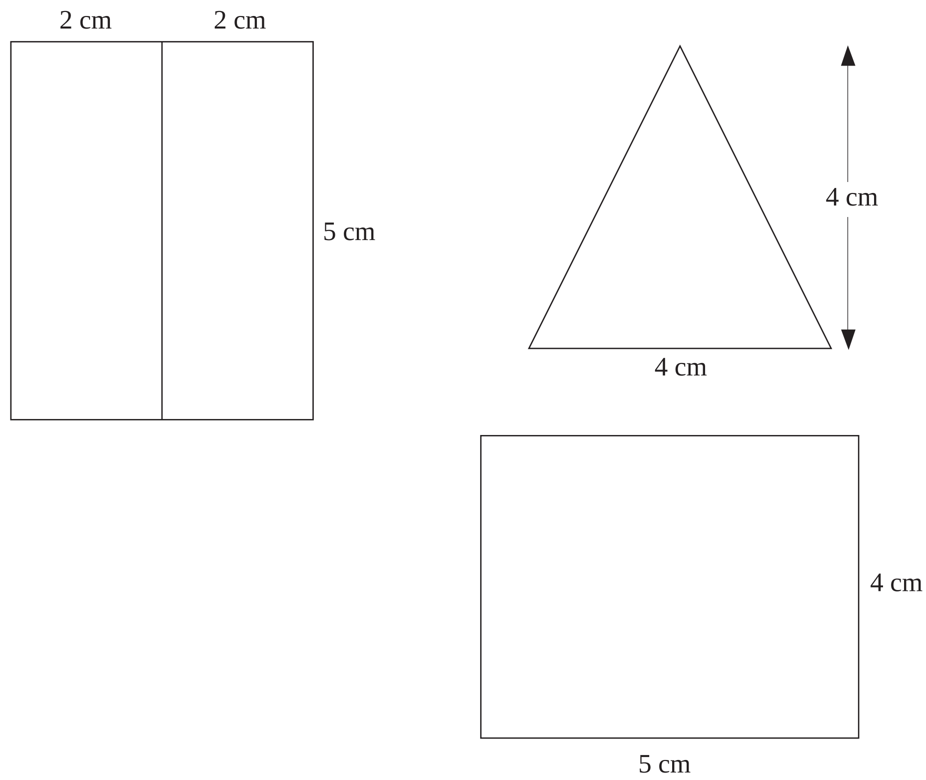
3. A cuboid has sides of length 3 cm, 4 cm and 5 cm. Draw the cuboid in three different ways on isometric paper.

4. In each case below the plan and two elevations of a solid are given. Draw an isometric drawing of each solid.

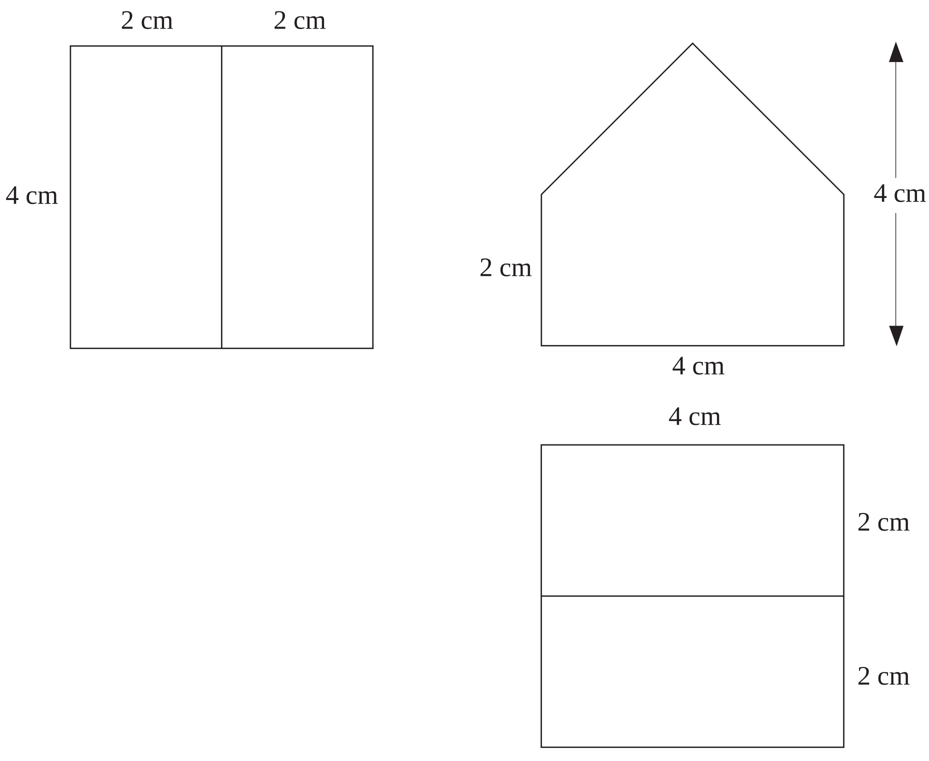
(a)



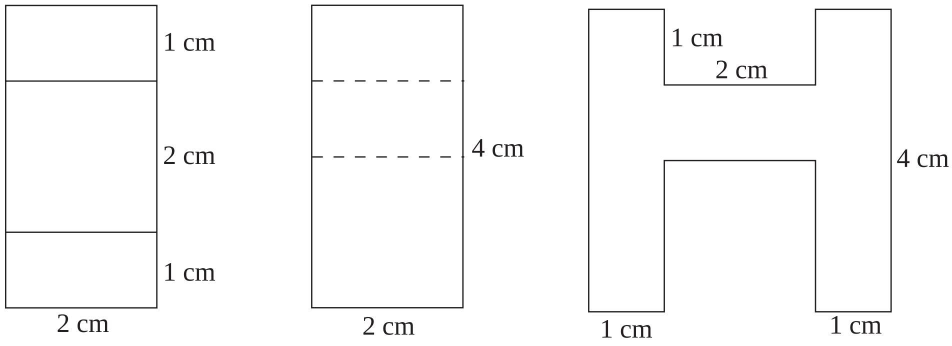
(b)



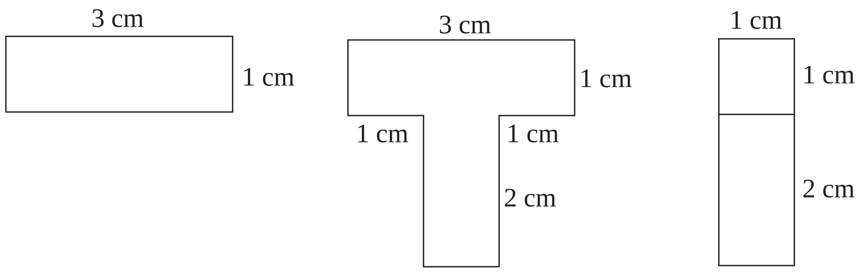
(c)



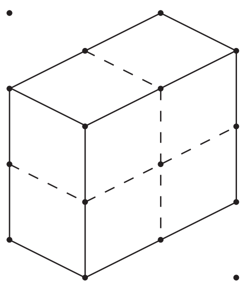
(d)



(e)

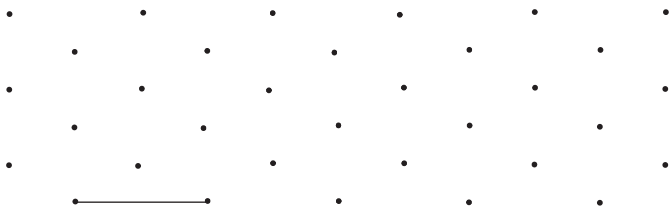


5. Four cubes with sides of length 1 cm can be joined together in different ways. One way is shown below.



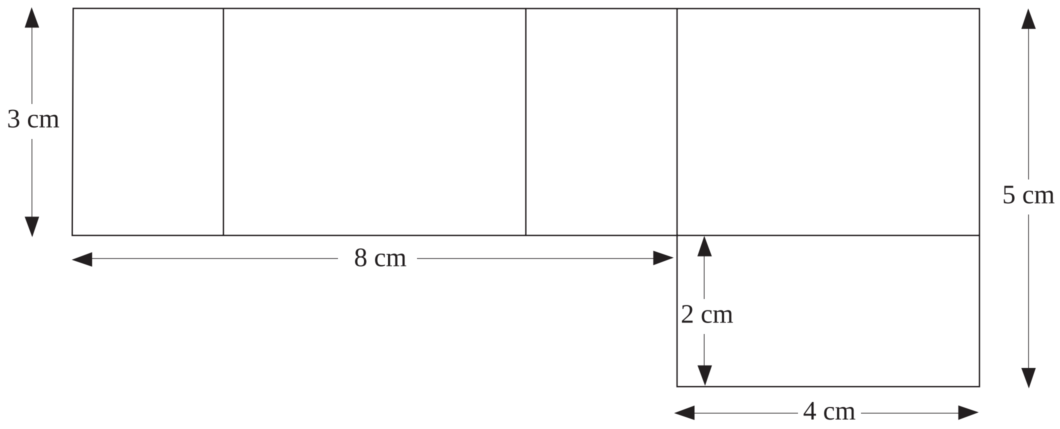
Find other ways in which the four cubes could be joined together.

6. Complete the drawing of a cuboid. One edge is drawn for you.



(SEG)

7. The diagram below is the net of a small open box, with no top face.



(a) Find the perimeter of the net.

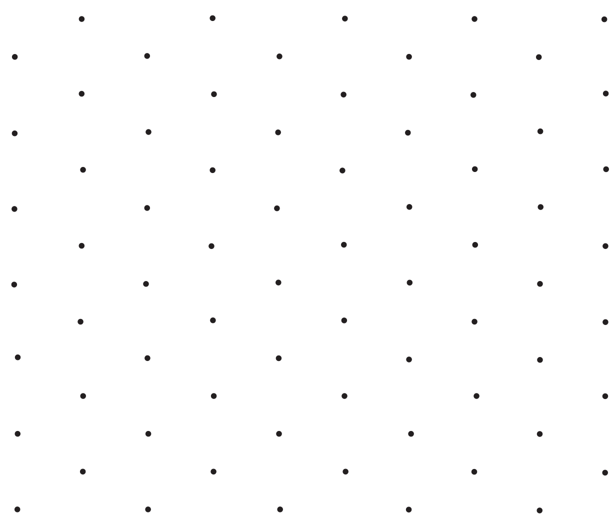
(b) Calculate the area of the net.

(c) Copy the diagram, and add one more rectangle in a suitable position to change the diagram to the net of a closed box.

(d) Write down the length, width and height of the box (in any order).

(e) Calculate the volume of the box.

(f) Draw an isometric view of the closed box on a grid like the one below.



(MEG)

Just for Fun

A man travels 5 km north, 4 km west and then 5 km south and discovers that he is back to where he begins his journey. Can you identify the place where he begins his journey?

7.11 Discrete and Continuous Measures

Quantities are said to be *discrete* if they can only take particular values.

For example,

Number of children in a class	28, 29, 30, 31, 32, ...
Shoe size	$6\frac{1}{2}$, 7, $7\frac{1}{2}$, 8, $8\frac{1}{2}$, 9, ...
Bottles of milk in a fridge	0, 1, 2, 3, ...

Quantities that can take any value within a range are said to be *continuous.*

For example,

Height

Weight

Time

Worked Example 1

Which of the following are *discrete* and which are *continuous*? For those that are discrete give an example of an impossible value.

(a) Temperature in a classroom.

(b) Votes cast in an election.

(c) Number of cars parked in a car park.

(d) Length of a piece of paper.

Solution

(a) The temperature in a classroom can take any value, and so this is continuous.

(b) Only single votes can be cast in an election and so the number is discrete. For example, it would be impossible to have $32\frac{1}{2}$ votes.

(c) Only complete cars can be parked and so this is discrete. It is impossible to have $22\frac{1}{2}$ cars parked in a car park.

(d) The length of a piece of paper is a continuous quantity as it can take any value.

Worked Example 2

(a) The length of a piece of rope is given as 21.4 m. What range of possible values could the length of rope lie within?

(b) The number of cars parked in a car park is said to be 43. Is this number exact or does it represent a range?

Solution

(a) As the length of the rope is a continuous quantity it could take any value. If it is quoted as being 21.4 m it must lie in the range

$$21.35 \leq \text{length} < 21.45$$

(this was covered in *Unit 6*)

(b) As the number of cars is a discrete quantity, the number is exact and does not represent a range.

Exercises

1. State whether each of the following is discrete or continuous.

   (a) Volume of water in a glass.
   (b) Number of fish in a tank.
   (c) The population of France.
   (d) The length of a phone call.
   (e) The lengths of plants.
   (f) The number of words in an essay.
   (g) The time spent on homework.
   (h) The number of computers in a school.
   (i) The time it takes to get to school.
   (j) The weight of a cake.
   (k) The number of pupils in a school.
   (l) The distance pupils travel to school.

2. Give an example of a continuous quantity and a discrete quantity.

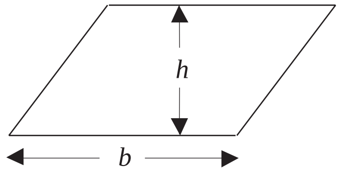
3. In each case state whether the value given is exact or give the range of values in which it could lie.

   (a) Distance to school is 4.63 miles.
   (b) Shirt size is $12\frac{1}{2}$.
   (c) Weight of an apple is 125 grams.
   (d) Height is 162 cm.
   (e) Number of pages in a book is 264.
   (f) Volume of drink in a glass is 52.2 cm^3.
   (g) 22.2 cm of rain fell in a month.
   (h) The attendance at a football match was 24 731.
   (i) The weight of a letter was 54 grams.
   (j) A total of £42.63 was raised at a cake stall.

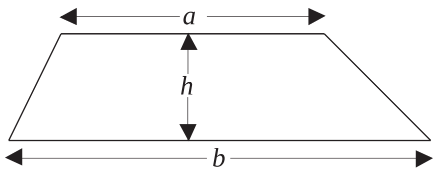
7.12 Areas of Parallelograms, Trapeziums, Kites and Rhombuses

The formulae for calculating the areas of these shapes are:

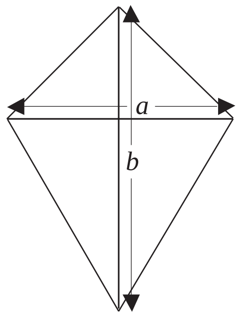
Parallelogram $\boxed{A = bh}$



Trapezium $\boxed{A = \frac{1}{2}(a+b)h}$



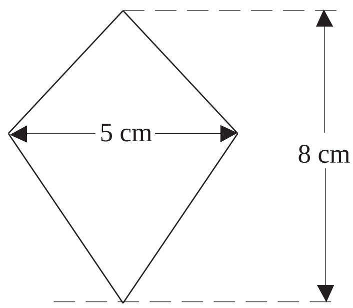
Kite $\boxed{A = \frac{1}{2}ab}$



The area of a *rhombus* can be found using either the formula for a kite or the formula for a parallelogram.

Worked Example 1

Find the area of this kite.



Solution

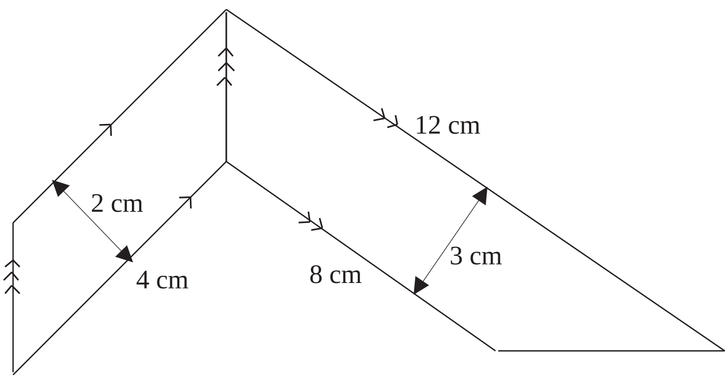
Using the formula $A = \frac{1}{2}ab$

with $a = 5$ and $b = 8$ gives

$$\begin{aligned} A &= \frac{1}{2} \times 5 \times 8 \\ &= 20 \text{ cm}^2 \end{aligned}$$

Worked Example 2

Find the area of this shape.



Solution

The shape is made up of a parallelogram and a trapezium.

$$\begin{aligned}\text{Area of parallelogram} &= 2 \times 4 \\ &= 8 \text{ cm}^2\end{aligned}$$

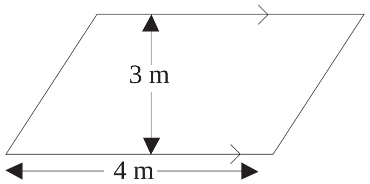
$$\begin{aligned}\text{Area of trapezium} &= \frac{1}{2}(8 + 12) \times 3 \\ &= 30 \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Total area} &= 8 + 30 \\ &= 38 \text{ cm}^2\end{aligned}$$

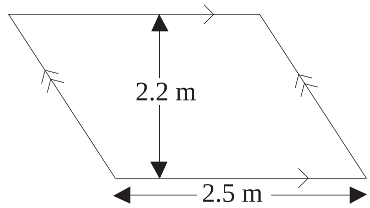
Exercises

Find the area of each of the following shapes.

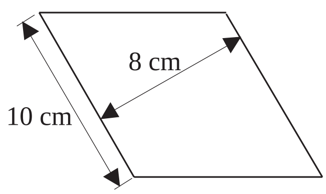
1. (a)



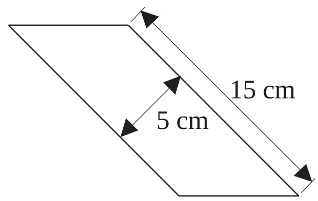
(b)



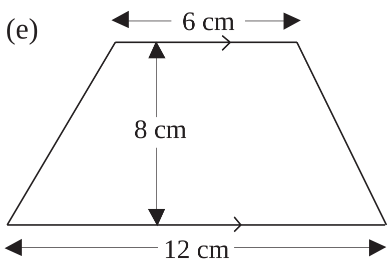
(c)



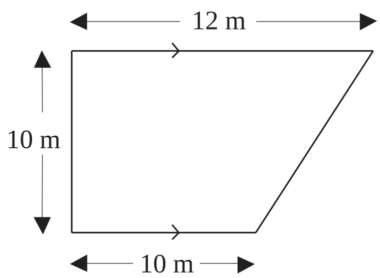
(d)



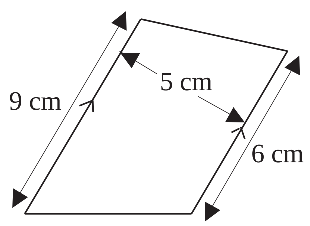
(e)



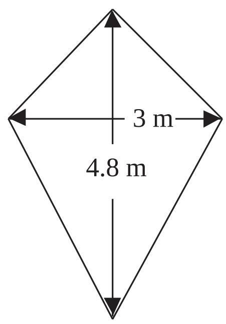
(f)



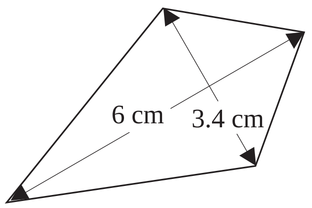
(g)



(h)

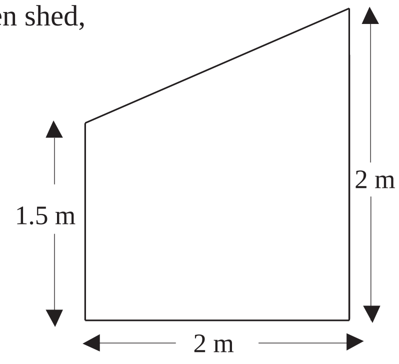


(i)



2. The diagram shown the end wall of a wooden garden shed,

(a) Find the area of this end of the shed.

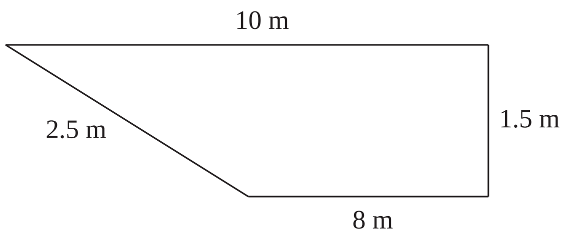


The other end of the shed is identical. The sides are made up of two rectangles of length 3 m.

(b) Find the area of each side of the shed.

(c) Find the total area of the walls of the shed.

3. The diagram shows the vertical side of a swimming pool.



(a) Find the area of the side of the pool.

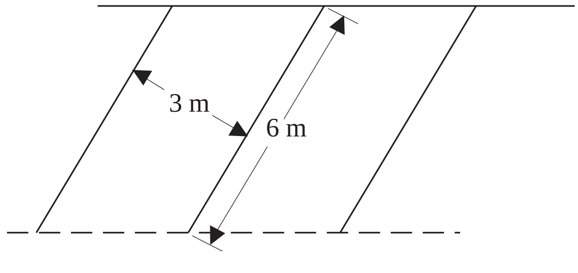
The w
idth of the swimming pool is 4 m.

(b) Find the area of the rectangular end of the swimming pool.

(c) Find the area of the horizontal base of the pool.

(d) Find the total area of the sides and horizontal base of the pool.

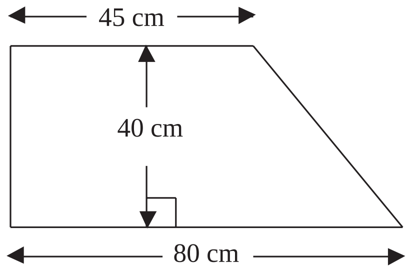
4. In a car park, spaces are marked out in parallelograms.



Find the area of each parking space.

5. The diagram shows a window of a car.

   Find the area of the window.

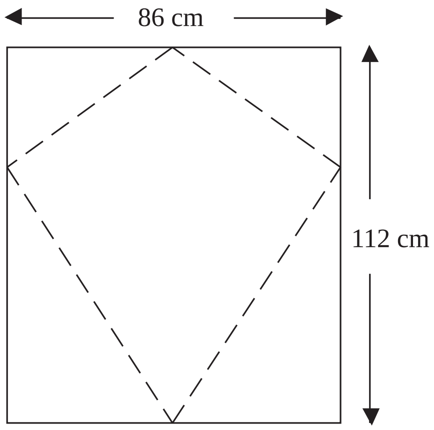


6. A kite is cut out of a sheet of plastic as shown.

   (a) Find the area of the kite.

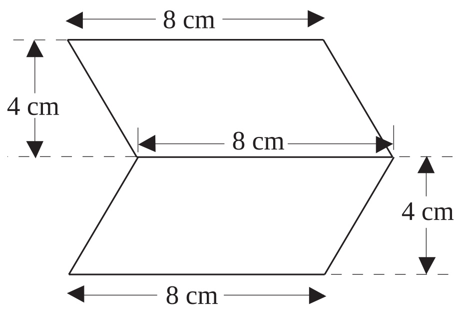
   (b) Find the area of the plastic that would be wasted.

   (c) Would you obtain similar results if you cut a kite out of a rectangle of plastic with dimensions 140 cm by 80 cm?

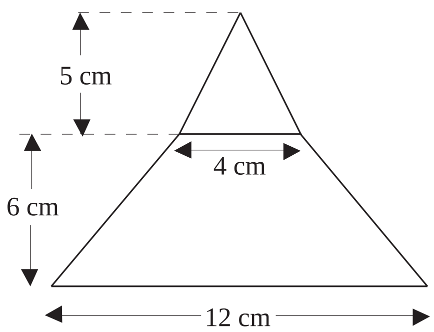


7. Find the area of each of the following shapes.

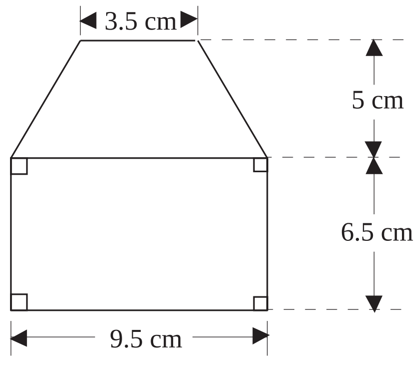
(a)



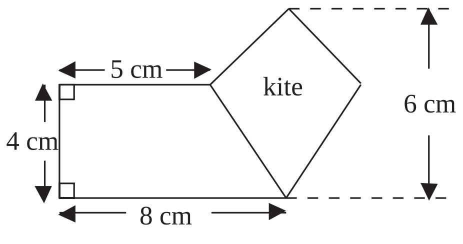
(b)



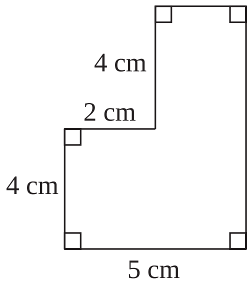
(c)



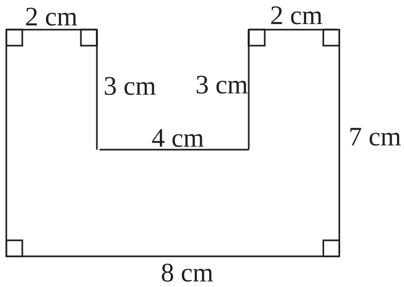
(d)



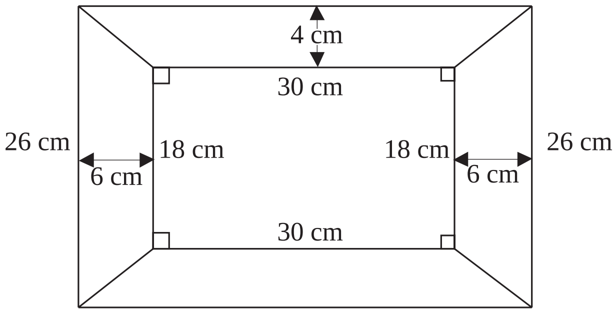
(e)



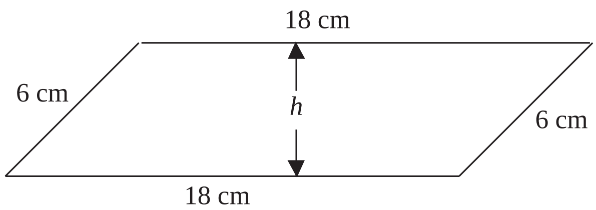
(f)



8. A simple picture frame is made by joining four trapezium shaped strips of wood. Find the area of each trapezium and the total area of the frame.



9. Four rods are joined together to form a parallelogram.



(a) Find the area of the parallelogram if:

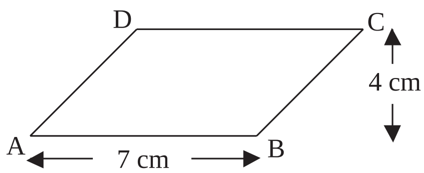
(i) $h = 2$ cm (ii) $h = 4$ cm (iii) $h = 5$ cm

(b) Can h be higher than 6 cm?

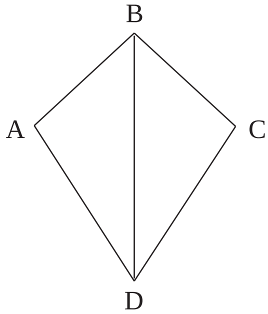
(c) What is the maximum possible area of the parallelogram?

10. (a) Find the area of parallelogram ABCD.

(b) Find the area of the triangle ABC.



11.



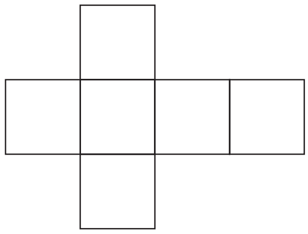
Not to scale

Why is the area of the kite ABCD equal to twice the area of the triangle ABD?

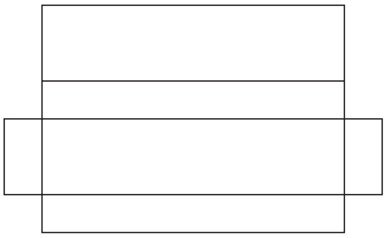
(MEG)

7.13 Surface Area

The *net* of a cube can be used to find its surface area.

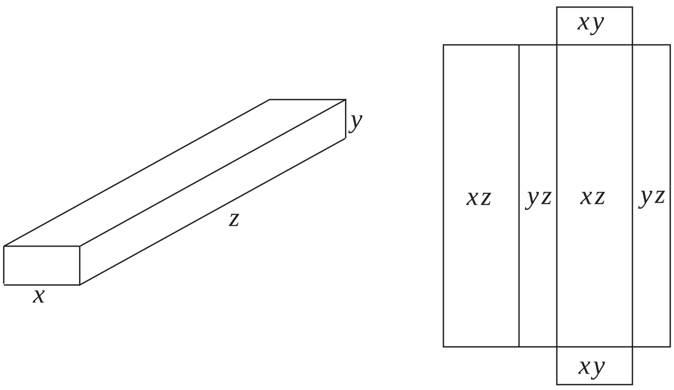


The net is made up of 6 squares, so the surface area will be 6 times the area of one square. If x is the length of the sides of the cube its surface area will be $6x^2$.



This diagram shows the net for a cuboid. To find the surface area the area of each of the 6 rectangles must be found and then added to give the total.

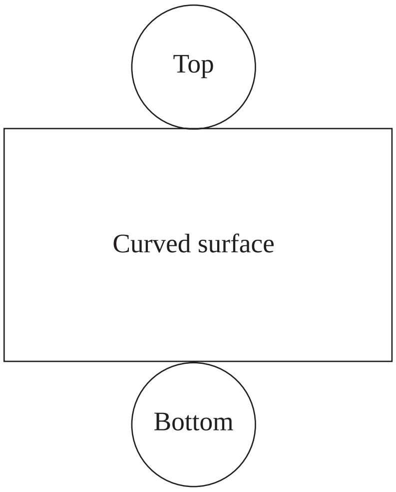
If x, y and z are the lengths of the sides of the cuboid, then the area of the rectangles in the net are as shown here.



The total surface area of the cuboid is then given by

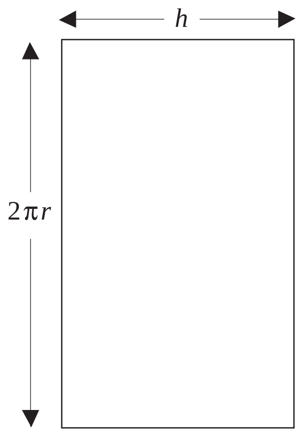
$$\boxed{A = 2xy + 2xz + 2yz}$$

To find the surface area of a cylinder, consider how a cylinder can be broken up into three parts, the top, bottom and curved surface.



The areas of the top and bottom are the same and each is given by πr^2.

The curved surface is a rectangle. The length of one side is the same as the circumference of the circles, $2\pi r$, and the other side is simply the height of the cylinder, *h*. So the area is $2\pi r h$.

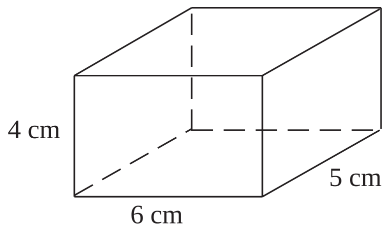


The total surface area of the cylinder is

$$\boxed{2\pi r^2 + 2\pi r h}$$

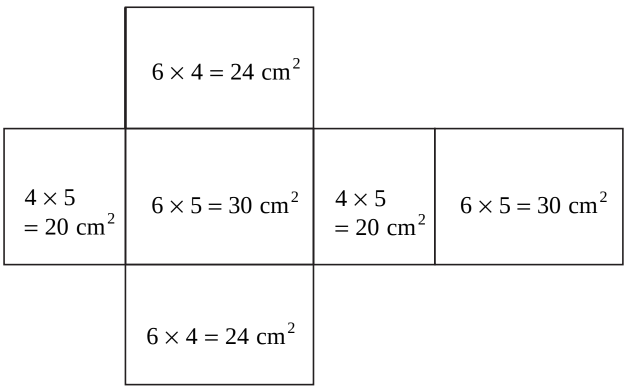
Worked Example 1

Find the surface area of the cuboid shown in the diagram.



Solution

The diagram shows the net of the cuboid and the areas of the rectangles that it contains.



Using the net, the total surface area is given by

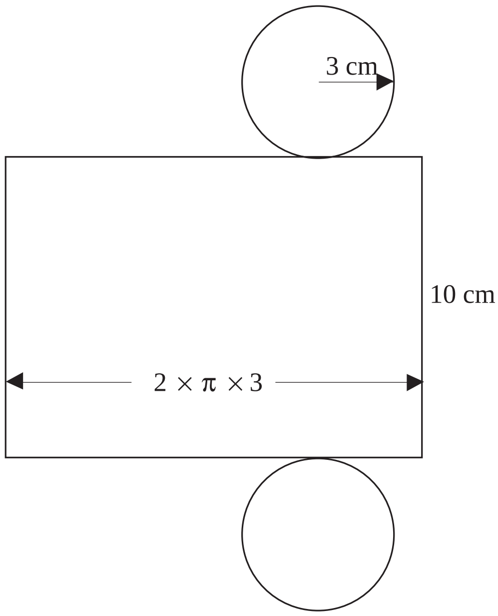
$$\begin{aligned} A &= 2 \times 20 + 2 \times 30 + 2 \times 24 \\ &= 148 \text{ cm}^2 \end{aligned}$$

Worked Example 2

Cans are made out of aluminium sheets, and are cylinders of radius 3 cm and height 10 cm. Find the area of aluminium needed to make one can.

Solution

The diagram shows the two circles and the rectangle from which cans will be made.



The rectangle has one side as 10 cm, the height of the cylinder and the other side is $2 \times \pi \times 3$ cm, the circumference of the top and bottom.

The area of the rectangle is $\quad 10 \times 2 \times \pi \times 3$

The area of each circle is $\quad \pi \times 3^2$

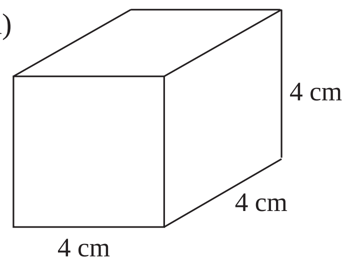
So the total surface area is $\quad A = 10 \times 2 \times \pi \times 3 + 2 \times \pi \times 3^2$

$$= 245.04 \text{ cm}^2 \text{ (to 2 d.p.)}$$

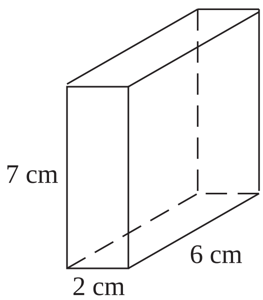
Exercises

1. Find the surface area of each of the following cubes or cuboids.

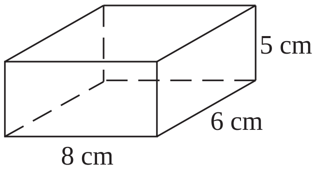
(a)



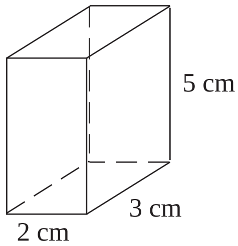
(b)



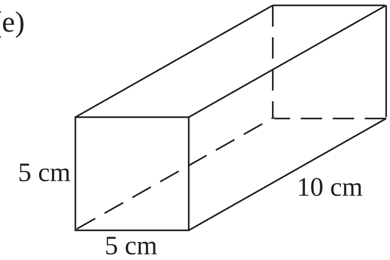
(c)



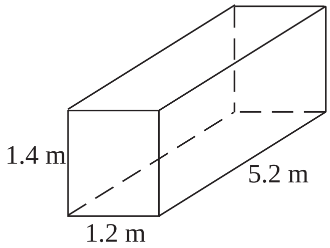
(d)



(e)

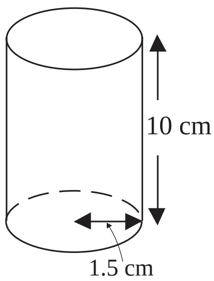


(f)

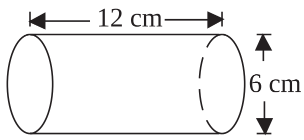


2. Find the total surface area of each cylinder shown below.

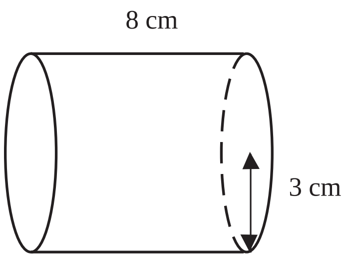
(a)



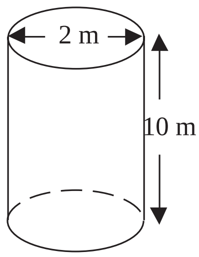
(b)



(c)



(d)



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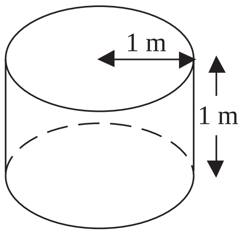

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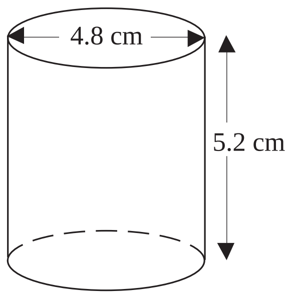
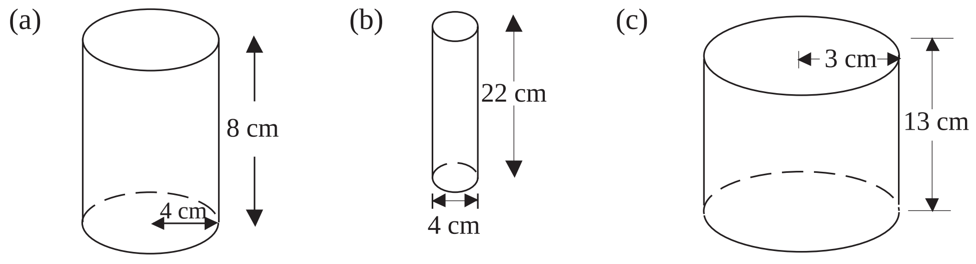


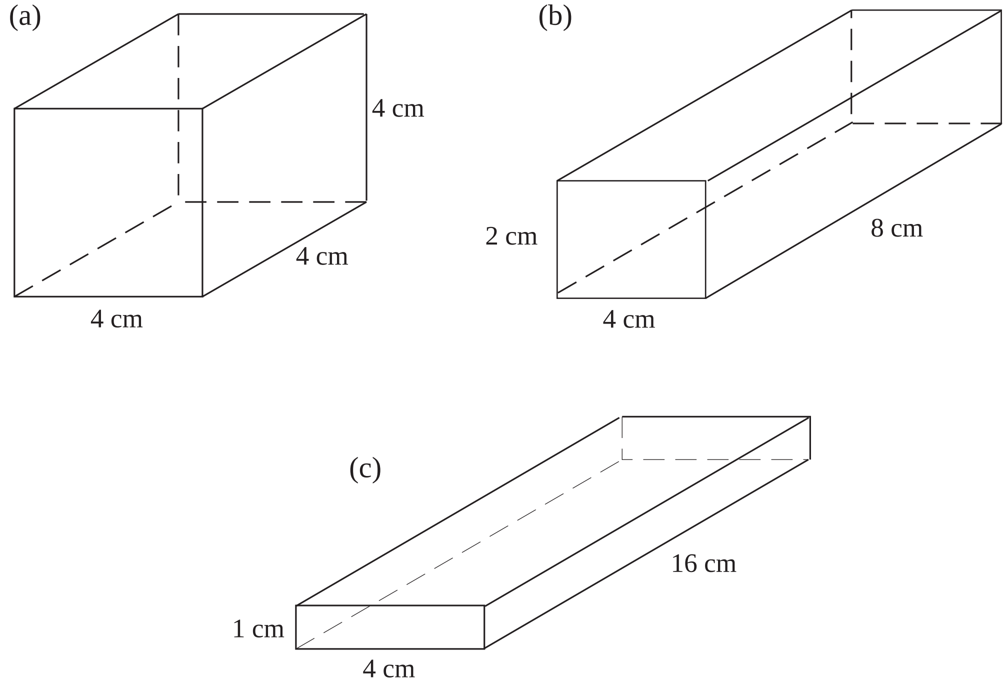
Information

Smaller animals have more surface area compared to their volume than larger animals. Because of this, smaller animals tend to lose water and body heat more easily than larger animals. Children have $2\frac{1}{2}$ *times as much surface area compared to volume as adults. Thus children are more prone to dehydration and hypothermia.*

3. Show that each of the cylinders below has the same surface area and find which has the biggest volume.



4. Show that each of the three cuboids below has the same volume. Which has the smallest surface area?



5. A gardener uses a roller to flatten the grass on a lawn. The roller consists of a cylinder of radius 30 cm and width 70 cm.

   (a) Find the area of grass that the roller covers as the cylinder completes 1 rotation.

   (b) If the roller is pulled 5 m, what area of grass does the roller flatten?

6. The volume of a cube is 343 cm^3. Find the surface area of the cube.

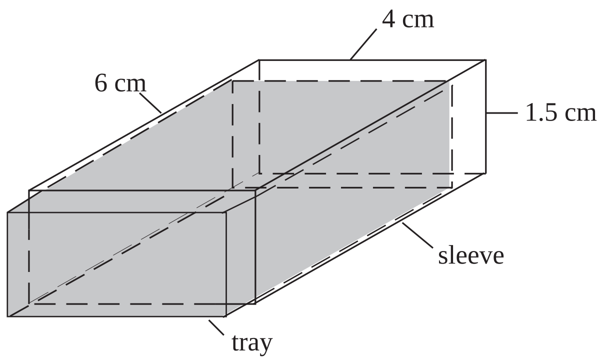
7. The surface area of a cube is 150 cm^2. Find the volume of the cube.

8. A matchbox consists of a tray that slides into a sleeve. If the tray and sleeve have the same dimensions and no material is used up in joins, find:

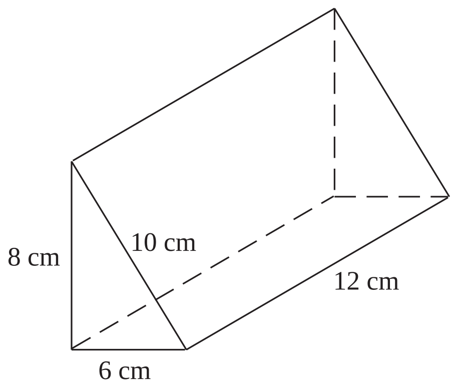
   (a) the area of card needed to make the tray,

   (b) the area of card needed to make the sleeve,

   (c) the total area of the card needed to make the matchbox.

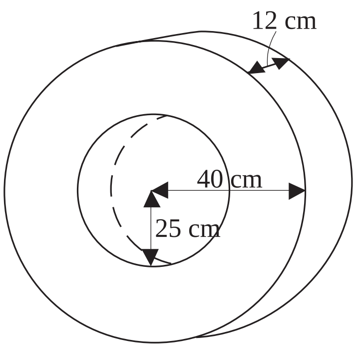


9. Draw a net of the prism shown in the diagram and use it to find the surface area of the prism.

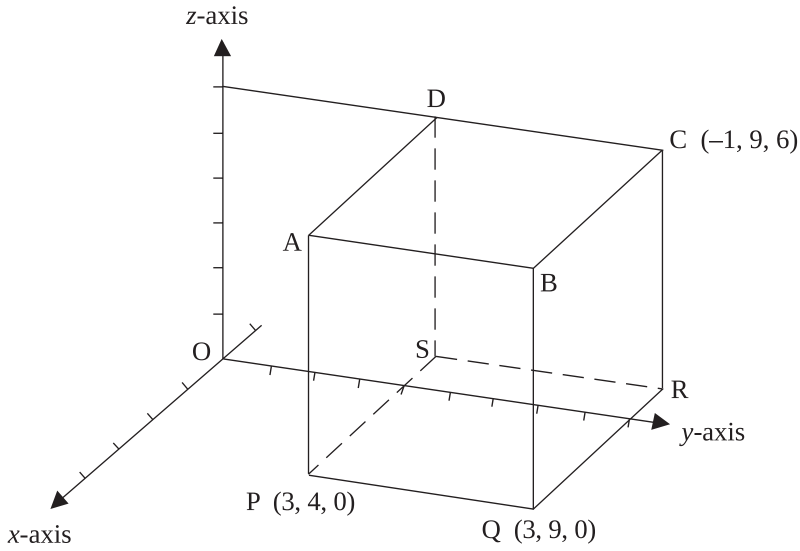


10. A car tyre can be thought of as a hollow cylinder with a hole cut out of the centre.

    Find the surface area of the outside of the tyre.



11.



The diagram shows a cuboid.

The co-ordinates of P are (3, 4, 0).
The co-ordinates of Q are (3, 9, 0).
The co-ordinates of C are (–1, 9, 6).

(a) Write down the (x, y, z) co-ordinates

(i) of R,

(ii) of B.

(b) Write down the lengths of each of the following edges of the cuboid.

(i) PQ,

(ii) QR.

(c) Calculate the total surface area of the cuboid.

(NEAB)

7.14 Mass, Volume and Density

Density is a term used to describe the mass of a unit of volume of a substance. For example, if the density of a metal is 2000 kg/m^3, then 1 m^3 of the substance has a mass of 2000 kg.

Mass, volume and *density* are related by the following equations.

$$\text{Mass} = \text{Volume} \times \text{Density}$$

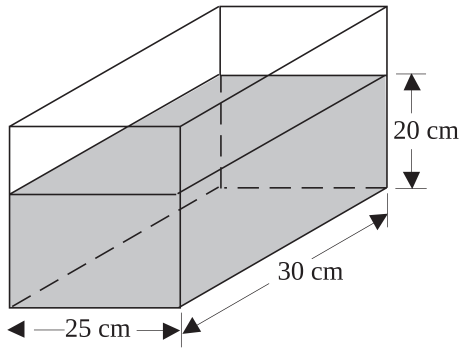
$$\text{Volume} = \frac{\text{Mass}}{\text{Density}}$$

or $\boxed{\text{Density} = \dfrac{\text{Mass}}{\text{Volume}}}$

The density of water is 1 gram / cm^3 (or 1 g cm^{-3}) or 1000 kg / m^3 (or 1000 kg m^{-3}).

Worked Example 1

Find the mass of water in the fish tank shown in the diagram.



Solution

First calculate the volume of water.

$$\begin{aligned} V &= 25 \times 30 \times 20 \\ &= 15000 \text{ cm}^3 \end{aligned}$$

Now use

$$\begin{aligned} \text{Mass} &= \text{Volume} \times \text{Density} \\ &= 15000 \times 1 \qquad \text{(as density of water = 1g / cm}^3\text{)} \\ &= 15000 \text{ grams} \\ &= 15 \text{ kg} \end{aligned}$$

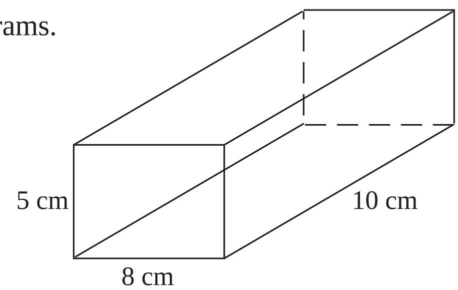
Worked Example 2

The block of metal shown has a mass of 500 grams.

Find its density in

(a) g / cm^3,

(b) kg / m^3.



Solution

(a) First find the volume.

$$\begin{aligned} \text{Volume} &= 5 \times 8 \times 10 \\ &= 400 \text{ cm}^3 \end{aligned}$$

Then use

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

$$\text{Density} = \frac{500}{400}$$

$$= 1.25 \text{ g /cm}^3$$

(b) The process can then be repeated working in kg and m.

$$\text{Volume} = 0.05 \times 0.08 \times 0.1$$

$$= 0.0004 \text{ m}^3$$

$$\text{Density} = \frac{0.5}{0.0004}$$

$$= \frac{5000}{4}$$

$$= 1250 \text{ kg / m}^3$$

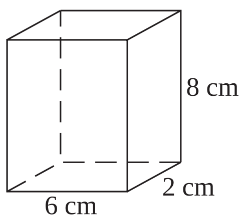
Exercises

1. A drinks carton is a cuboid with size as shown.

   (a) Find the volume of the carton.

   (b) If it contains 8 cm^3 of air, find the volume of the drink.

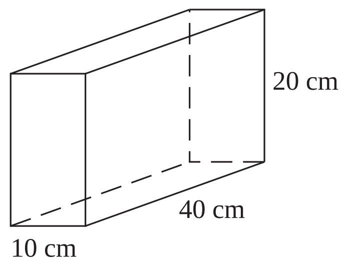
   (c) Find the mass of the drink if it has a density of 1 gram / cm^3.



2. The diagram shows a concrete block of mass 6 kg.

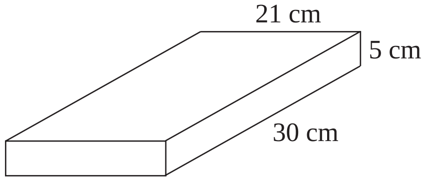
   (a) Find the volume of the block.

   (b) What is the density of the concrete?



3. A ream (500 sheets) of paper is shown in the diagram.

   If the mass of the ream is 2.5 kg, find the density of the paper.



4. A barrel is a cylinder with radius 40 cm and height 80 cm. It is full of water.

   (a) Find the volume of the barrel.

   (b) Find the mass of the water in the barrel.

5. A metal bar has a cross section with an area of 3 cm^2 and a length of 40 cm. Its mass is 300 grams.

   (a) Find the volume of the bar.

   (b) Find the density of the bar.

   (c) Find the mass of another bar with the same cross section and length 50 cm.

   (d) Find the mass of a bar made from the same material, but with a cross section of area 5 cm^2 and length 80 cm.

6. A bottle which holds 450 cm^3 of water has a mass of 530 grams. What is the mass of the empty bottle?

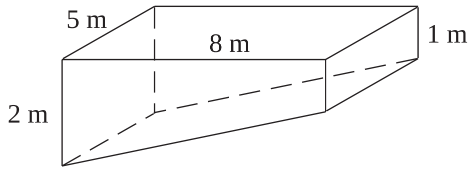
7. The diagram shows the dimensions of a swimming pool.

   (a) Find the volume of the swimming pool.

   (b) Find the mass of water in the pool if it is completely full.

   (c) In practice, the level of the water is 20 cm below the top of the pool.

   Find the volume and mass of the water in this case.

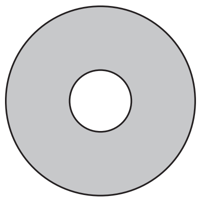


8. The density of a metal is 3 grams / cm^3. It is used to make a pipe with external radius of 1.5 cm and an internal radius of 0.5 cm.

   (a) Find the area of the cross section of the pipe.

   (b) If the length of the pipe is 75 cm, find its mass.

   (c) Find the length, to the nearest cm, of a pipe that has a mass of 750 grams.



9. A foam ball has a mass of 200 grams and a radius of 10 cm.

   (a) Use the formula $V = \dfrac{4\pi r^3}{3}$ to find the volume of the ball.

   (b) Find the density of the ball.

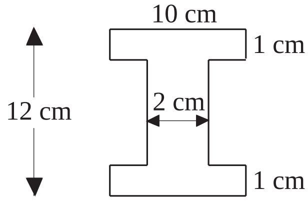
   (c) The same type of foam is used to make a cube with sides of length 12 cm. Find the mass of the cube.

10. The diagram shows the cross section of a metal beam. A 2 m length of the beam has a mass of 48 kg.

    (a) Find the density of the metal.

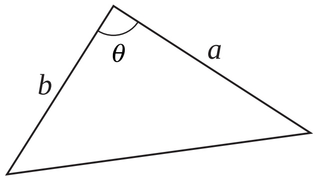
    A second type of beam uses the same type of metal, but all the dimensions of the cross section are increased by 50%.

    (b) Find the length of a beam of this type that has the same mass as the first beam.



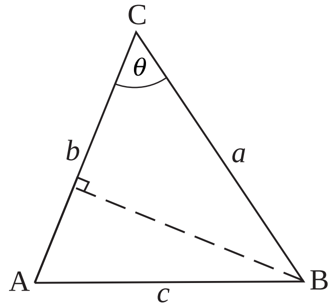
7.15 Volumes, Areas and Lengths

The area of a triangle with side lengths a and b and included angle θ is given by $\frac{1}{2}ab\sin\theta$ °.



$$\boxed{A = \frac{1}{2}ab\sin\theta}$$

Proof



If you construct the perpendicular from the vertex B to AC, then its length, p, is given by

$$p = a\sin\theta$$

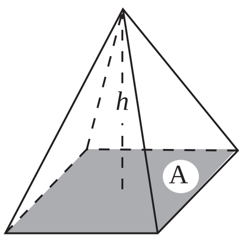
Thus the area of ABC is given by

$$\begin{aligned} \text{area} &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \times b \times p \\ &= \frac{1}{2} \times b \times (a\sin\theta) \\ &= \frac{1}{2}ab\sin\theta \end{aligned}$$

as required.

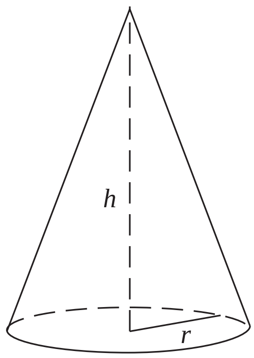
The volumes of a pyramid, a cone and a sphere are found using the following formulae.

Pyramid



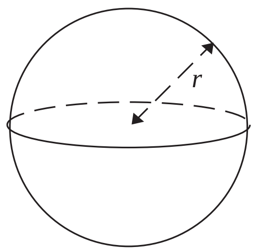
$$V = \frac{1}{3}Ah$$

Cone



$$V = \frac{1}{3}\pi r^2 h$$

Sphere

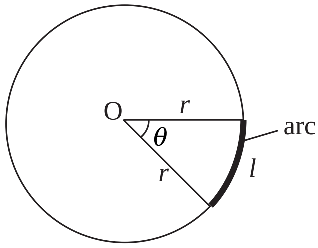


$$V = \frac{4}{3}\pi r^3$$

The proofs of these results are rather more complex and require mathematical analysis beyond the scope of this text.

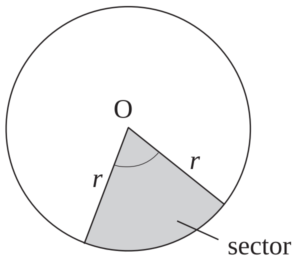
A part of the circumference of a circle is called an *arc*. If the angle subtended by the arc at the centre of the circle is θ then the arc length l is given by

$$\boxed{l = \frac{\theta}{360°} \times 2\pi r}$$



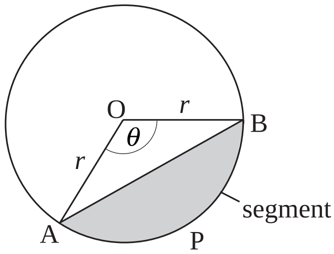
The region between the two radii and the arc is called a *sector of the circle*. The area of the sector of the circle is

$$\boxed{A = \frac{\theta}{360°} \times \pi r^2}$$



The region between the chord AB and the arc APB is called a *segment of the circle*. The area of the segment of the circle is

$$\boxed{A = \frac{\theta}{360°} \times \pi r^2 - \frac{1}{2} r^2 \sin\theta}$$



Worked Example 1

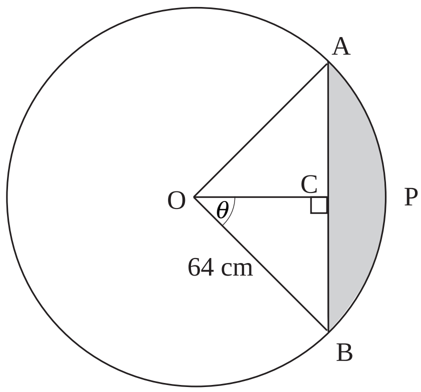
The shaded area shows a segment of a circle of radius 64 cm.
The length of the chord AB is 100 cm.

(a) Find the angle θ, to 2 d.p.

(b) Find the area of triangle OAB.

(c) Find the area of the sector of the circle with angle 2θ.

(d) Find the area of the segment shaded in the figure.

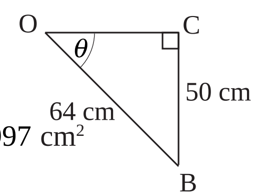


Solution

(a) If $AB = 100$ cm then, by symmetry, $BC = 50$ cm.

$$\sin\theta = \frac{50}{64}$$

$$\theta = 51.38°$$



(b) The area of the triangle OAB is $\frac{1}{2} \times 64^2 \times \sin 2\theta = 1997 \text{ cm}^2$

(c) The sector has area $\frac{(2 \times 51.38°)}{360°} \times \pi \times 64^2 = 3673 \text{ cm}^2$

(d) The segment has area $3673 - 1997 = 1676 \text{ cm}^2$.

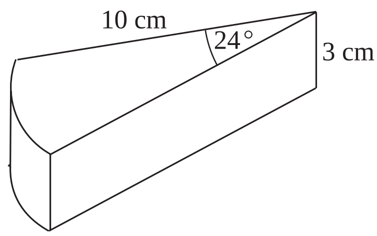
Worked Example 2

A wooden door wedge is in the shape of a sector of a circle of radius 10 cm with angle $24°$ and constant thickness 3 cm.

Find the volume of wood used in making the wedge.

Solution

The area of the top face of the wedge is the area of a sector of radius 10 cm and angle $24°$.



$$\text{Area} = \frac{24°}{360°} \times \pi \times 10^2 = \frac{20\pi}{3} = 20.94 \text{ cm}^2$$

The volume of the wedge

$$= \text{Area} \times 3 = 20\pi = 62.83 \text{ cm}^3$$

Worked Example 3

A cone and sphere have the same radius of 12 cm. Find the height of the cone if the cone and sphere have the same volume.

Solution

Suppose that the height of the cone is h cm.

$$\text{Volume of cone} = \frac{1}{3}\pi \times 12^2 \times h = 48\pi h \text{ cm}^3$$

$$\text{Volume of sphere} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(12)^3 = 2304\pi \text{ cm}^3$$

Since the volumes are equal

$$48\pi h = 2304\pi$$

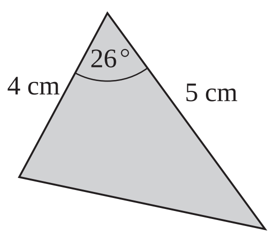
Solving for h,

$$h = \frac{2304\pi}{48\pi} = \frac{2304}{48} = 48 \text{ cm}$$

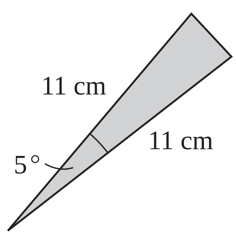
Exercises

1. Find the area of the shaded regions in the following figures.

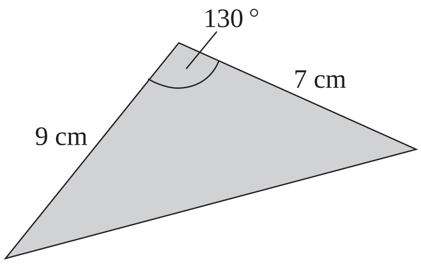
(a)



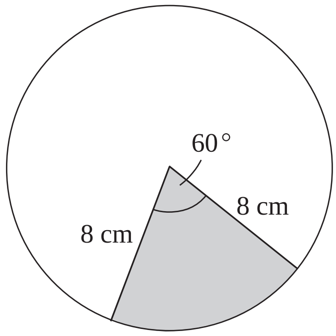
(b)



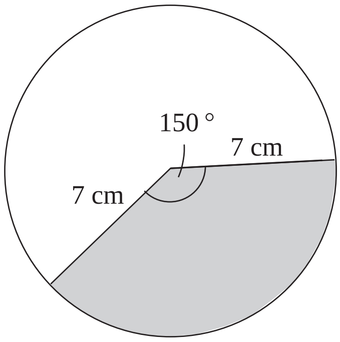
(c)



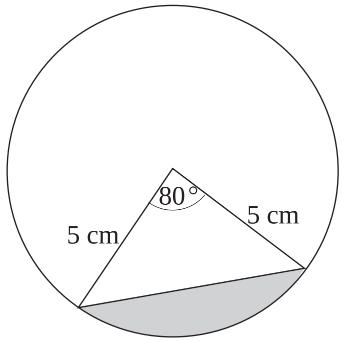
(d)



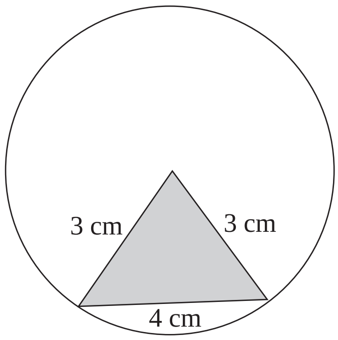
(e)



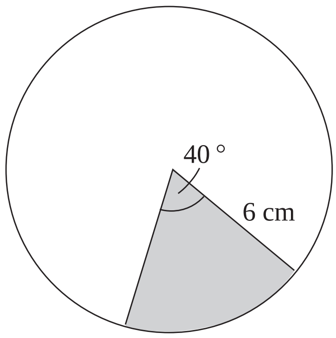
(f)



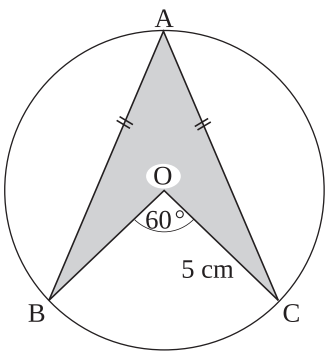
(g)



(h)

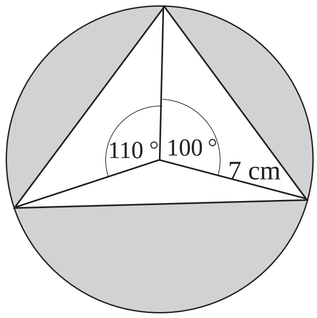


(i)



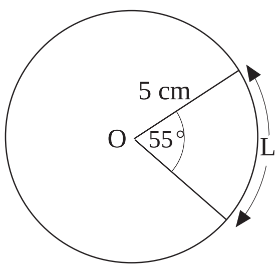
AB = AC

(j)

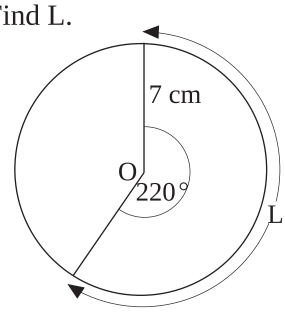


2. Find the values of the unknowns marked in the following.

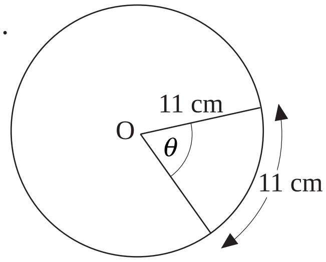
(a) Find L.



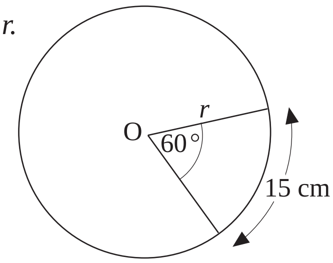
(b) Find L.



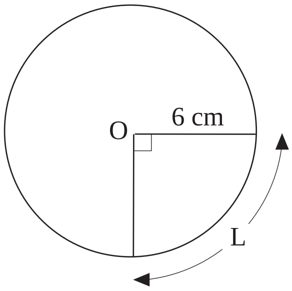
(c) Find θ.



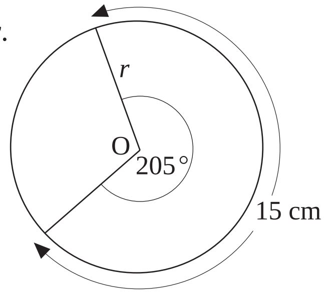
(d) Find *r*.



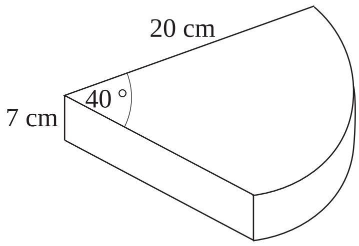
(e) Find L



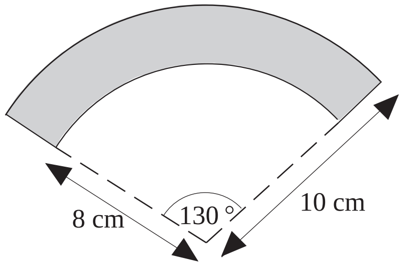
(f) Find *r*.



3. A cake is made in the shape of a sector of a circle with size shown. The thickness of the cake is 7 cm. The top and edges are to be covered with marzipan of thickness $\frac{1}{2}$ cm. Find the volume of marzipan needed.



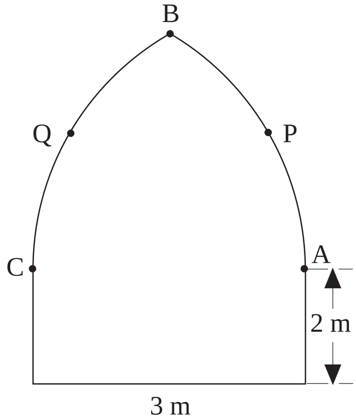
4. Find the area of the shaded region shown.



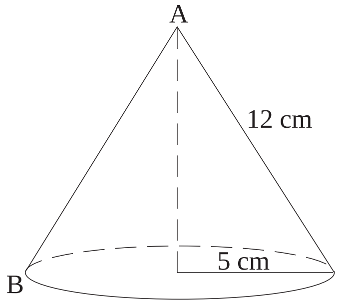
5. The shape represents a stained glass window. The frame consists of two arcs above a rectangular section. APB has centre C and BQC has centre A.

   (a) Find the length of material used to make the window frame.

   (b) Find the area of glass needed to make the window.



6. A cone of slant height 12 cm and radius 5 cm is made out of cardboard. The cone is cut along the edge AB and opens to form a sector of a circle.



   (a) Draw the sector and find the angle within the sector.

   (b) Find the area of cardboard used in making the cone. (Ignore the base.)

   (c) The original cone is turned upside down and filled with water. Find the volume of water inside the cone.

7. Find the volume of the following containers:

   (a) a cylinder of base radius 3 cm and height 10 cm,

   (b) a cone of base radius 5 cm and height 12 cm,

   (c) a cone of base radius 7 cm and height 5 cm,

   (d) a cone of base radius 1 m and height 1.5 m,

   (e) a cone of base radius 9 cm and slant height 15 cm,

   (f) a cone of base diameter 20 cm and slant height 25 cm,

   (g) a sphere of radius 6 cm,

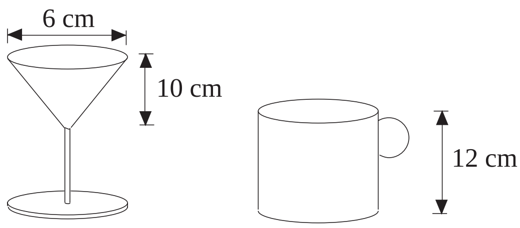
   (h) a hemisphere of radius 5 cm,

   (i) a pyramid of square base 10 cm and height 15 cm,

   (j) a pyramid of rectangular base 8 cm by 6 cm and height 9 cm.

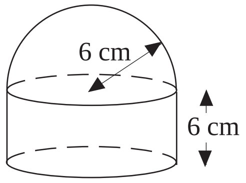
8. Calculate the radius of a sphere which has the same volume as a solid cylinder of base radius 5 cm and height 12 cm.

9. A wine glass is in the shape of a cone on a stem. The cylindrical tumbler is used to fill the wine glass. Find the diameter of the tumbler so that it has the same volume as the wine glass.

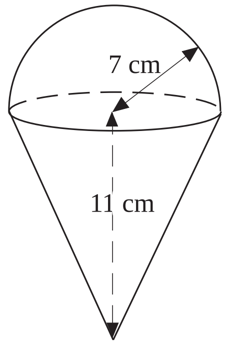


10. Find the volume of the following shapes made up of cones, hemispheres and cylinders.

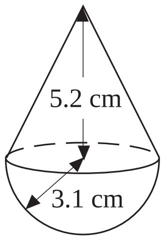
   (a)



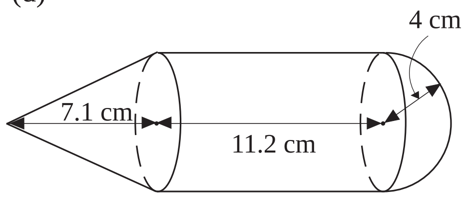
   (b)



(c)

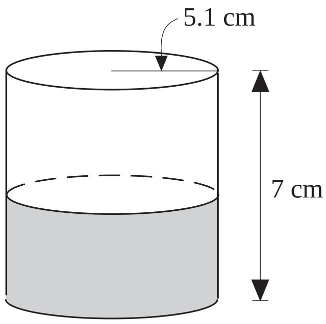


(d)

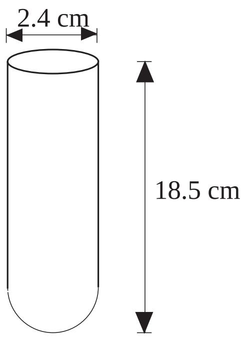


11. A cylinder is half filled with water as shown. A heavy sphere of diameter 2.5 cm is placed in the cylinder and sinks to the bottom.

    By how much does the water rise in the cylinder?



12. A test tube is in the shape of a hollow cylinder and hollow hemisphere. Calculate the volume of a liquid that can be held in the test tube.

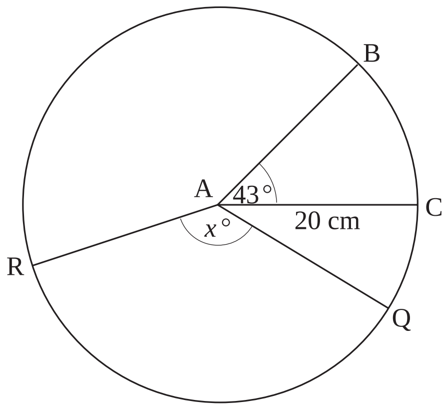


13. The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

    Calculate the volume of a sphere of radius 1.7 cm, giving your answer correct to 1 decimal place.

(MEG)

14.



BAC is a sector of a circle, radius 20 cm, whose centre is at A.

Angle BAC = 43°.

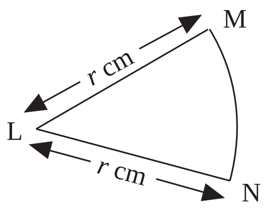
(a) Calculate the area of the sector BAC.

(b) The area of sector QAR is 450 cm^2.

Angle QAR is $x°$.

Calculate the value of x.

(c)



The area of the sector MLN of another circle, centre L, is 600 cm^2.

The total perimeter of the sector is 100 cm.

It can be shown that the radius, r cm, of the sector satisfies the equation

$$r^2 - 50r + 600 = 0.$$

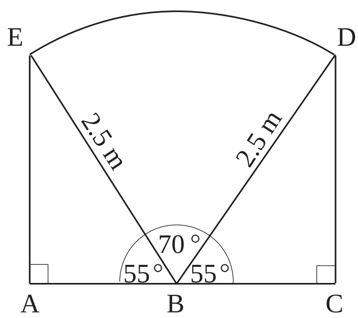
Find the values of r which satisfy this equation.

(NEAB)

15. Paul is painting a wall. He can reach an area ABCDE as shown.

The area consists of three parts:

a sector BDE of a circle of radius 2.5 m and angle $70°$ and two right angled congruent triangles EAB and DCB.



(a) Calculate the distance AB.

(b) Calculate the area of the sector BDE.

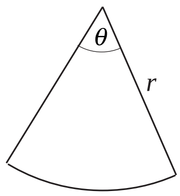
Take π to be 3.14 or use the π key on your calculator.

(c) Calculate the area ABCDE Paul can reach.

(d) Calculate the perimeter of the shape ABCDE.

(SEG)

16. Silver pendants are made in the shape of a sector of a circle with radius r cm and angle θ.



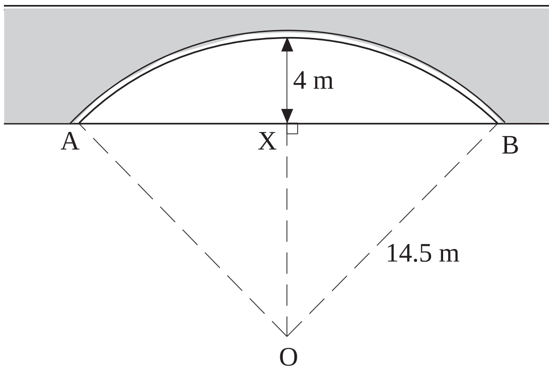
(a) Calculate the total perimeter of the pendant when $r = 3$ and $\theta = 30°$.

(b) Another pendant has the same perimeter but $r = 2.5$ cm.

Calculate, to the nearest degree, the angle θ required for this pendant.

(SEG)

17.



to scale

The diagram shows the cross section of a tunnel which has a maximum height of 4 m above its horizontal base AB. The roof of the tunnel is part of a circle, centre O and radius 14.5 m.

The mid point of AB is X.

(a) (i) Write down the length of OX. (ii) Calculate the length of AB.

(b) Calculate the size of angle BOX.

(c) Calculate the area of cross section of the tunnel.

(MEG)

7.16 Dimensions

Most quantities are based on three fundamental quantities, mass, length and time. The letters M, L and T are used to describe the dimensions of these quantities.

If m is the mass of an object, then we write

$$[m] = M$$

to show that M is the dimension of m.

If x is a length, then, similarly,

$$[x] = L$$

$$[x^2] = L^2$$

$$[x^3] = L^3$$

If t is a time, then

$$[t] = T$$

Here the emphasis will be on quantities involving lengths.

As areas are calculated by multiplying two lengths,

$$[\text{Area}] = L^2 \quad \text{or} \quad \text{Dimensions of Area} = L^2$$

Similarly,

$$[\text{Volume}] = L^3 \quad \text{or} \quad \text{Dimensions of Volume} = L^3$$

The ideas of dimensions can be used to check formulae.

Worked Example 1

The area of a circle is given by

$$A = \pi r^2$$

(a) What are the dimensions of A and r?

(b) What are the dimensions of π?

Solution

(a) A is an area, so $[A] = L^2$

r is a length, so $[r] = L$ and $\left[\mathrm{r}^2\right] = L^2$

(b) The dimension of the formula $A = \pi r^2$ must be consistent, so

$$[A] = [\pi] \times \left[r^2\right]$$

But using the results from (a),

$$L^2 = [\pi] \times L^2$$

$$[\pi] = \frac{L^2}{L^2} = L^0$$

$$= 1$$

The number π has no dimension.

Note

Numbers that are not lengths or other quantities have no dimension.

Worked Example 2

If x, y and z are all lengths, state whether each expression below could be for a length, an area or a volume.

(a) $x + y + z$ (b) $x y z$ (c) $\dfrac{x}{z} + y$

(d) $\dfrac{x y}{z}$ (e) $xy + xz + yz$

Solution

(a) $[x + y + z] = [x] + [y] + [z]$

$= L + L + L$

As the formula contains 3 terms, all of dimension L, this expression gives a length.

(b) $$\begin{aligned}[xyz] &= [x]\times[y]\times[z] \\ &= L \times L \times L \\ &= L^3\end{aligned}$$

As the dimension of xyz is L^3 this expression gives a volume.

(c) $$\begin{aligned}\frac{[x]}{[z]} + [y] &= \frac{L}{L} + L \\ &= 1 + L\end{aligned}$$

The first term has no dimension and the second has a dimension, L, so the expression is not dimensionally consistent.

(d) $$\begin{aligned}\frac{[xy]}{[z]} &= \frac{[x]\times[y]}{[z]} \\ &= \frac{L \times L}{L} \\ &= \frac{L^2}{L} \\ &= L\end{aligned}$$

As the dimension of this quantity is L, this expression would give a length.

(e) $$\begin{aligned}[xy + xz + yz] &= [xy]+[xz]+[yz] \\ &= L^2 + L^2 + L^2\end{aligned}$$

So the expression contains three terms, all of which have dimension L^2, and so could represent an area.

Exercises

1. If x, y and z all represent lengths, consider each expression and decide if it could be for a length, an area, a volume or none of these.

(a) xy (b) $\frac{x}{y}$ (c) $\frac{x^2 y^2}{z}$

(d) $\frac{x^3}{z}$ (e) $\frac{xy^2}{z}$ (f) $\frac{x}{z} + \frac{y}{x}$

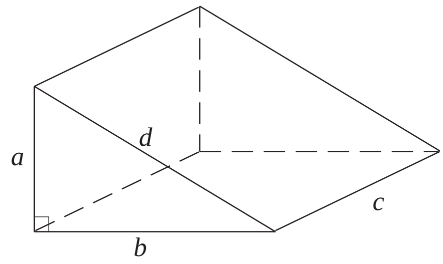
(g) $\frac{y^2}{x} + y^3$ (h) $\frac{yz^3}{x}$ (i) $2xyz$

2. Which of the formulae listed below are not dimensionally consistent, if *p*, *q* and *r* are lengths and *A* is an area?

(a) $p = q + r$ (b) $p = \frac{q}{r} + r$ (c) $A = pqr$

(d) $A = \frac{p^2 q}{r}$ (e) $p = \frac{q^2}{r}$ (f) $q^2 = p^2 + r^2$

3. The diagram shows a triangular wedge. Use dimensions to select the formula from the list below that could give the surface area of the wedge.



$A = abc + bd + ac$

$A = ab + ac + bc + cd$

$A = a + b + c + d$

$A = abc + bcd + abd$

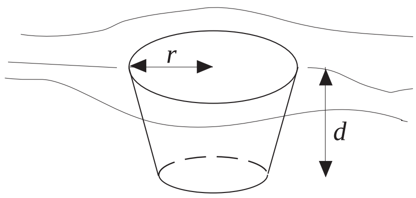
4. Consider $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$

What are the dimensions of $\sin\theta$?

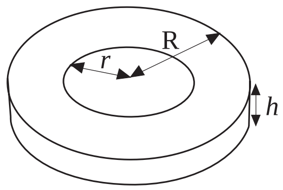
5. The diagram shows a hole, which has a radius of *r* at the surface and a depth *d*. Which of the following could give the volume of the hole?

$V = \frac{13}{17}\pi r d$ $V = \frac{13}{17}\pi r^2 d$

$V = \frac{13}{17}\pi^2 r d$ $V = \frac{13}{17}\pi r^3 d$



6. Jared tries to find the volume of sellotape on a roll. Use dimensions to find the formula from the list below that could be correct.



$V = \pi(R - r)h$

$V = \pi^2 Rh - \pi r^2 h$

$V = \pi Rh^2 - \pi rh$

$V = \pi R^2 h - \pi r^2 h$

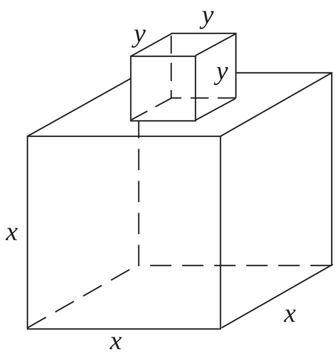
7. Alex has forgotten the formula for the volume of a sphere. He writes down the following formulae to try and jog his memory.

$$\frac{4}{3}\pi^2 r, \quad \frac{4}{3}\pi r^3, \quad \frac{3}{4}\pi^2 r, \quad \frac{4}{3}\pi r^2, \quad \frac{3}{4}\pi r^3$$

(a) Which of these formulae could give volumes?

(b) Which of these formulae could give areas?

8. Amil has the two expressions below; one is for the volume of the shape and the other is for the surface area.

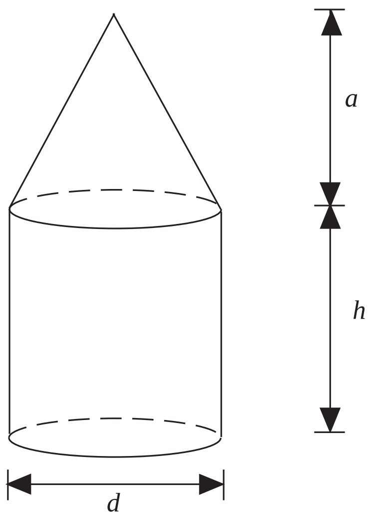


$$6x^2 + 4y^2$$

$$x^3 + y^3$$

Explain what each expression gives.

9. Which of the formulae below could give the volume of this shape?



$$V = \frac{\pi dh}{4} + \frac{\pi da}{12}$$

$$V = \frac{\pi d^2 h}{4} + \frac{\pi da}{12}$$

$$V = \frac{\pi dh}{4} + \frac{\pi da^2}{12}$$

$$V = \frac{\pi d^2 h}{4} + \frac{\pi d^2 a}{12}$$

10. (a) If *a*, *b* and *c* are lengths and $s = \frac{a + b + c}{2}$, find the dimension of *s*.

(b) If $X = \sqrt{s(s-a)(s-b)(s-c)}$, find the dimension of *X* and state what type of quantity it could be.

11. By considering dimensions, decide whether the following expressions could be a formula for

perimeter, area, or volume.

In the expressions below, *a*, *b* and *c* are all lengths.

(a) $a + b + c$

(b) $\frac{2}{3}\pi a^3 + \pi a^2 b$

(SEG)

12. The expressions shown in the table below can be used to calculate lengths, areas or volumes of various shapes.

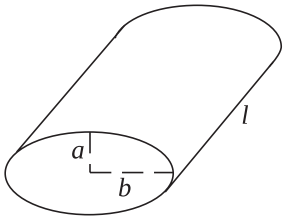
π, 2, 4 and $\frac{1}{2}$ are numbers which have no dimensions. The letters *r, l, b* and *h* represent lengths.

Put a tick in the box underneath those expressions that can be used to calculate a volume.

$2\pi r$	$4\pi r^2$	$\pi r^2 h$	πr^2	lbh	$\frac{1}{2}bh$

(LON)

13. The diagram shows a child's play brick in the shape of a prism.



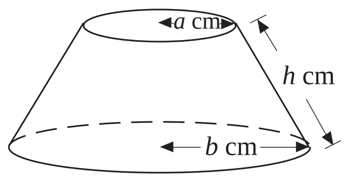
The following formulae represent certain quantities connected with this prism.

$$\pi ab, \quad \pi(a+b), \quad \pi abl \quad \pi(a+b)l$$

Which of these formulae represent areas?

(SEG)

14. One of the formulae in the list below can be used to calculate the area of material needed to make the curved surface of the lampshade in the diagram.



(i) $\pi h(a+b)^2$ (ii) $\pi h^2(a+b)$,

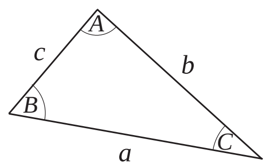
(iii) $\pi h(a+b)$, (iv) $\pi h^2(a+b)^2$.

State which formula is correct.
Give a reason for your answer.

(MEG)

7.17 Areas of Triangles

There are two useful formulae for the area of a triangle when



(i) the lengths of two sides and the included angle are known,

(ii) the lengths of three sides are known.

For case (i), the formula is

$$\boxed{A = \frac{1}{2} a b \sin C}$$

and this was proved at the start of section 7.15

Note

Similarly, $A = \frac{1}{2} bc \sin A = \frac{1}{2} ab \sin B$.

For case (ii), the formula, known as *Heron's formula*, is given by

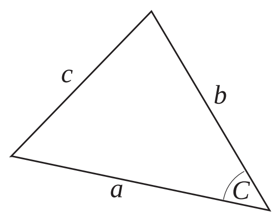
$$\boxed{A = \sqrt{s(s-a)(s-b)(s-c)}}$$

where $s = \frac{1}{2}(a+b+c)$.

The proof is given below, although it does involve rather complex algebraic manipulation.

Proof

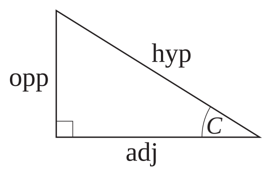
You start with the formula



$$A = \frac{1}{2} ab \sin C$$

By definition, in a right angled triangle,

$$\sin C = \frac{\text{opp}}{\text{hyp}}$$



$$\Rightarrow \quad (\sin C)^2 = \frac{(\text{opp})^2}{(\text{hyp})^2}$$

$$= \frac{(\text{hyp})^2 - (\text{adj})^2}{(\text{hyp})^2} \qquad \text{(using Pythagoras' theorem)}$$

$$= 1 - \frac{(\text{adj})^2}{(\text{hyp})^2}$$

$$= 1 - (\cos C)^2$$

Thus

$$A = \frac{1}{2} ab \sqrt{1 - (\cos C)^2}$$

$$= \frac{1}{2} ab \sqrt{(1 - \cos C)(1 + \cos C)}$$

But, from the cosine rule (*Unit 4*),

$$c^2 = a^2 + b^2 - 2ab\cos C$$

i.e. $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Substituting into the formula for A,

$$A = \frac{1}{2}ab\sqrt{\left(1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)\right)\left(1 + \left(\frac{a^2 + b^2 - c^2}{2ab}\right)\right)}$$

$$= \frac{1}{2}ab \, . \, \frac{1}{2ab}\sqrt{\left(2ab - a^2 - b^2 + c^2\right)\left(2ab + a^2 + b^2 - c^2\right)}$$

$$= \frac{1}{4}\sqrt{\left(c^2 - (a-b)^2\right)\left((a+b)^2 - c^2\right)}$$

$$= \frac{1}{4}\sqrt{\left(c - (a-b)\right)\left(c + (a-b)\right)\left((a+b) - c\right)\left((a+b) + c\right)}$$

$$= \frac{1}{4}\sqrt{(c + b - a)(c + a - b)(a + b - c)(a + b + c)}$$

But $$s = \frac{1}{2}(a + b + c)$$

and

$$s - a = \frac{1}{2}(a + b + c) - a = \frac{1}{2}(b + c - a)$$

$$s - b = \frac{1}{2}(a + c - b)$$

$$s - c = \frac{1}{2}(a + b - c)$$

giving

$$A = \frac{1}{4}\sqrt{2(s-a)2(s-b)2(s-c)2s}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

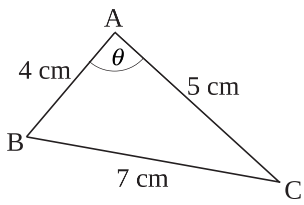
as required.

Worked Example 1

For the triangle shown, find

(a) the area of the triangle,

(b) angle θ.



Solution

(a) Using Heron's formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $$s = \frac{1}{2}(4+5+7) = 8 \text{ cm}$$

$$A = \sqrt{8 \times 4 \times 3 \times 1} = \sqrt{96} = 9.80 \text{ cm}^2$$

(b) Using the formula $A = \frac{1}{2}ab\sin\theta°$

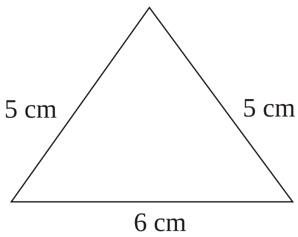
$$\sin\theta = \frac{2 \times A}{ab} = \frac{2 \times 9.8}{4 \times 5} = 0.98$$

$$\theta = 78.5°$$

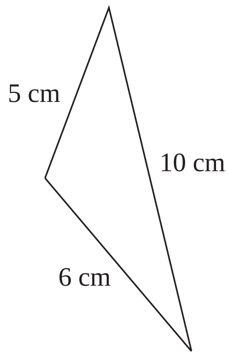
Exercises

1. Calculate the areas of the triangles shown.

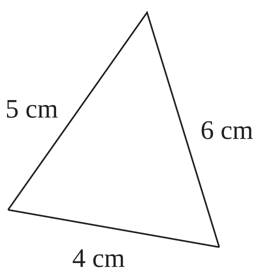
(a)



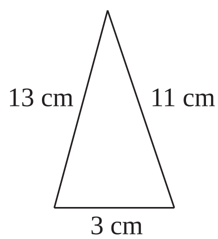
(b)



(c)



(d)

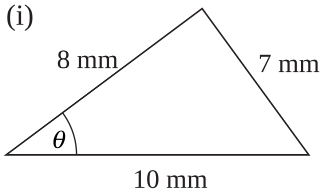


2. For each of the triangles shown find

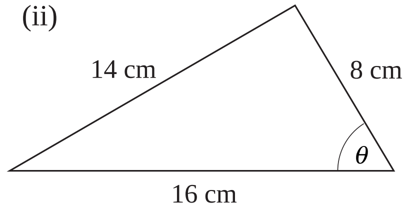
   (a) the area of the triangle,

   (b) the angle shown by θ.

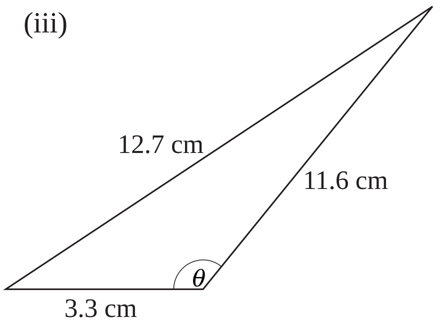
(i)



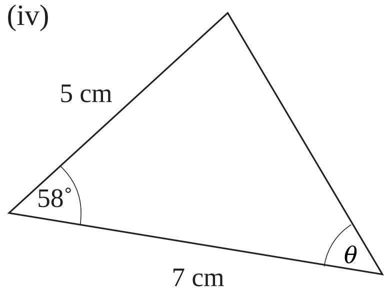
(ii)



(iii)



(iv)



Just for Fun

A cylindrical plastic bucket is approximately half-filled with water. Find out how you can determine with certainty whether the amount of water is greater or less than half the capacity of the pail.

Just for Fun

Two spherical water-melons of diameter 30 cm and radius 20 cm are sold for £5 and £10 respectively. Which is the better buy? Explain your answer.

Just for Fun

You are given 4 rectangular pieces of wood. Two of these measure 4 cm by 3 cm while the other two measure 13 cm by 1 cm. Use these 4 pieces of wood to enclose an area as large as you possibly can.