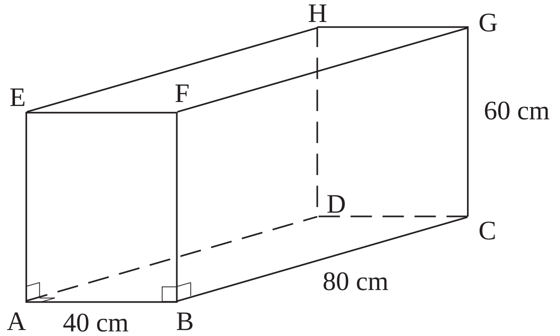
18 3-D Geometry

18.1 Using Pythagoras' Theorem and Trigonometry in Three Dimensions

Pythagoras' theorem and the trigonometry used in earlier units can be applied in three dimensional problems. The main technique is to reduce the problem to a two dimensional situation by identifying suitable triangles to work with. The standard results can then be applied to these triangles.

Worked Example 1

A box has the dimensions shown in the diagram. A game is to be packed into this box. Part of the game is a rod that will just fit into the box. Find the length of this rod.



Solution

The longest rod that can fit into the box will have one end at A and the other at G, or lie along a similar diagonal. The problem is to find the length AG.

The first stage is to find the length of AC, the diagonal in the base directly below AG.

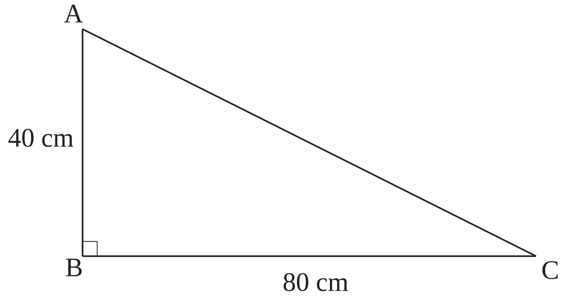
From the triangle ABC as shown:

$$AC^2 = AB^2 + BC^2$$

$$= 40^2 + 80^2$$

$$= 8000$$

$$= AC = \sqrt{8000} \text{ cm}$$



Now, AG can be found by considering the triangle ACG.

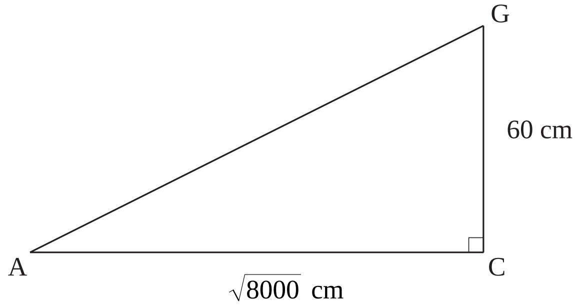
$$AG^2 = AC^2 + CG^2$$

$$= 8000 + 60^2$$

$$= 11\,600$$

$$AG = \sqrt{11\,600}$$

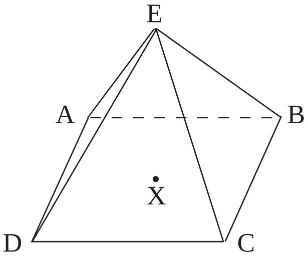
$$\approx 107.7 \text{ cm}$$



Worked Example 2

A pyramid is made up of a square and four equilateral triangles with sides of length 10 cm. Find the height of the pyramid.

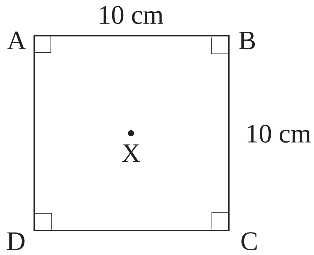
Solution



The point X marked on the diagram is at the centre of the base directly below E.

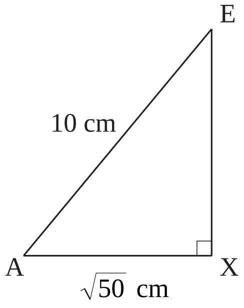
The distance XE is the required height.

Consider first the base of the pyramid.



$$AC^2 = AB^2 + BC^2$$
$$= 10^2 + 10^2$$
$$= 200$$
$$AC = \sqrt{200}$$
$$AX = \frac{1}{2}AC$$
$$= \frac{\sqrt{200}}{2}$$
$$= \frac{\sqrt{4 \times 50}}{2} = \frac{2 \times \sqrt{50}}{2}$$
$$= \sqrt{50}$$

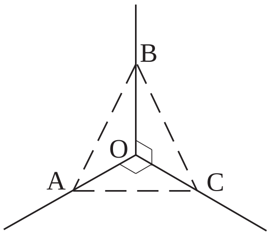
Next work in the triangle AXE.



$$AE^2 = AX^2 + XE^2$$
$$10^2 = 50 + XE^2$$
$$XE^2 = 100 - 50$$
$$= 50$$
$$XE = \sqrt{50}$$
$$\approx 7.07 \text{ cm}$$

Worked Example 3

A carpenter is to cut an isosceles triangle out of a sheet of wood. The triangle is to fit into a corner as shown in the diagram.



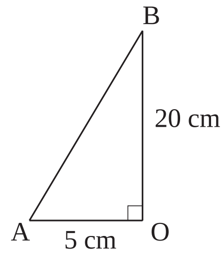
The length OA = OC = 5 cm and OB = 20 cm.

Find the size of each angle in the triangle.

Solution

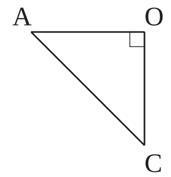
First find the lengths AC and AB.

From the triangle OAB.



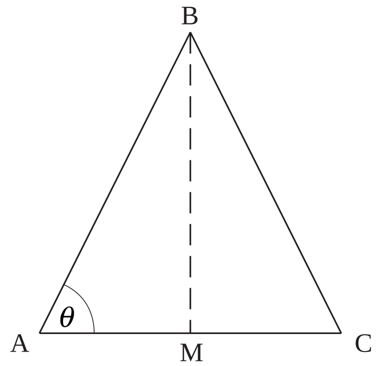
$$AB^2 = AO^2 + OB^2$$
$$= 5^2 + 20^2$$
$$= 42.5$$
$$AB = \sqrt{425}$$

From the triangle AOC:



$$AC^2 = OA^2 + OC^2$$
$$= 5^2 + 5^2$$
$$= 50$$
$$AC = \sqrt{50}$$

Next consider the triangle ABC. This can be cut in half to form two right angled triangles.



$$AM = \frac{1}{2}AC$$
$$= \frac{\sqrt{50}}{2}$$
$$= \sqrt{12.5}$$

Then using trigonometry:

$$\cos\theta = \frac{AM}{AB}$$
$$= \frac{\sqrt{12.5}}{\sqrt{425}}$$

so

$$\theta \approx 80.1°$$

So $B\hat{A}C = 80.1°$

As it is an isosceles triangle $A\hat{C}B = B\hat{A}C$

$$= 80.1°$$

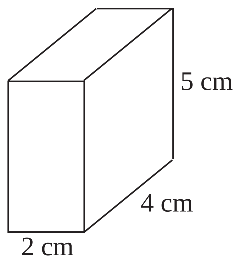
Finally, $A\hat{C}B = 180 - (80.1 + 80.1)$

$$= 19.8°$$

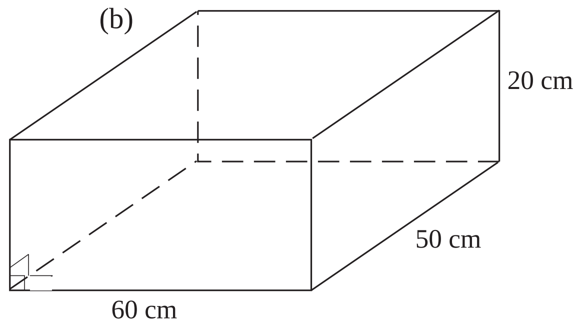
Exercises

1. Find the length of the longest rod that could be placed in each box shown below.

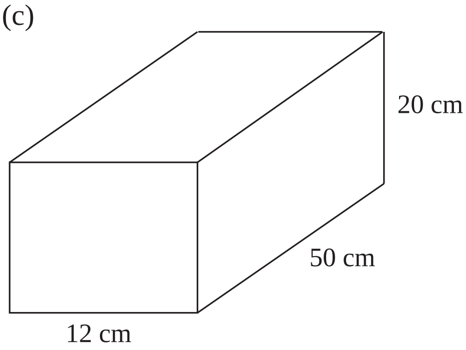
(a)



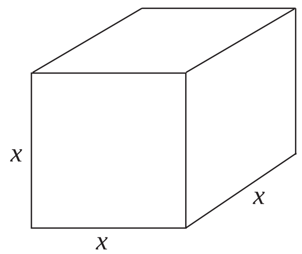
(b)



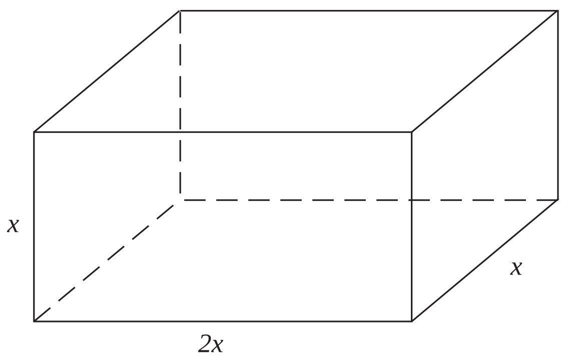
(c)



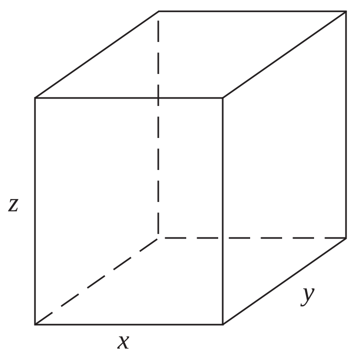
(d)



(e)



(f)



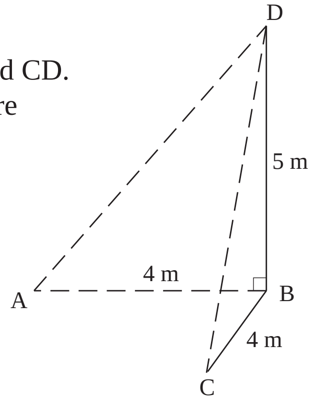
2. A square-based pyramid is made up of a square and four isosceles triangles with sides of lengths 8 cm, 8 cm and 4 cm. Find the height of the pyramid.

3. A telegraph pole is supported by two cables AD and CD. The cables are fixed to the points A and C which are both 4 m from the base of the pole B.

   Find the angle ADC if:

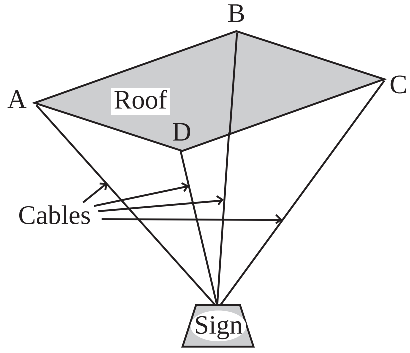
   (a) $A\hat{B}C = 90°$

   (b) $A\hat{B}C = 120°$



4. An illuminated sign is suspended by four cables in a shopping centre. Each cable is 4 m long. The cables are attached to points A, B, C and D which form a rectangle on the roof of the shopping centre.

   If AB = 6 m and BC = 4 m find the distance of the top of the sign below the roof.

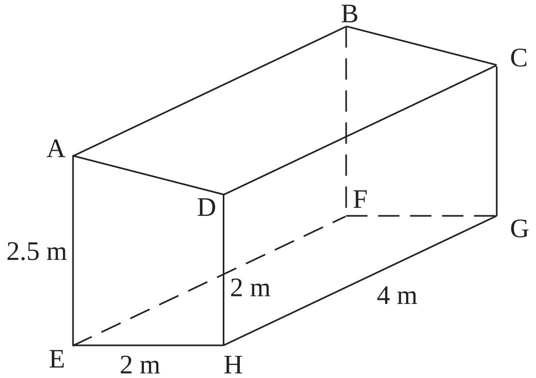


5. A gardener has to clean some leaves off the roof of the conservatory, shown in the diagram. He hopes to be able to reach the whole roof by putting his ladder in one place.

   Find the maximum distance that the gardener has to reach to cover the whole roof if he climbs his ladder so that he is at:

   (a) the point D,

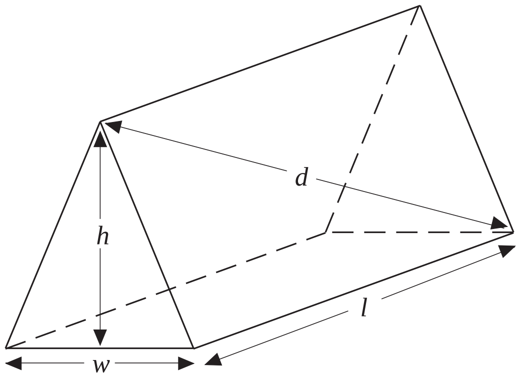
   (b) the mid-point of DC.



6. The diagram shows a simple ridge tent. The tent has length l, height h and width w. To check that the tent has been assembled correctly the distance d is measured.

   (a) Find d if $h = 1.2$ m, $w = 1.4$ m and $l = 2$ m.

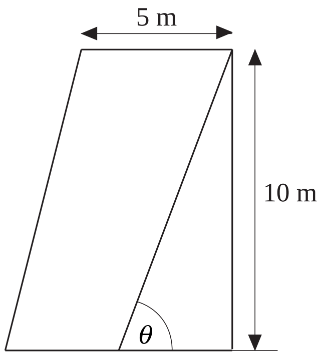
   (b) Find a formula for d in terms of w, h and l.



7. A ski slope has width 5 m and the difference in height between the top and bottom is 10 m.

   (a) A skier travels in a straight line on the slope. If the angle of the slope, θ, is 20°, find the maximum distance that she can travel.

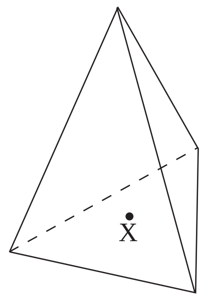
   (b) If the maximum distance that a skier can travel in a straight line on the slope is 20 m, find the angle θ.



8. A tetrahedron is made of an equilateral triangle and three isosceles triangles.

The lengths of the identical sides of the isosceles triangles are 15 cm and the height of the tetrahedron is 12 cm.

Find the lengths of the sides of the equilateral triangle.



9. Isosceles triangles with sides of lengths 12 cm, 12 cm and 8 cm are used to form a square-based pyramid and a tetrahedron.

For each shape find:

(a) the height,

(b) the angle between the slanting edges and the vertical.

10.

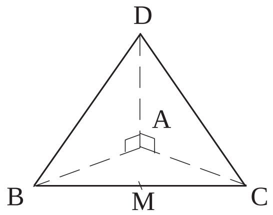


Diagram NOT accurately drawn

The diagram represents a pyramid ABCD.

ABC is an isosceles triangle with AB = AC = 7 cm and BC = 10 cm.

BCD is an isosceles triangle with BD = CD = 9 cm.

D is vertically above A and angle BAD = angle CAD = $90°$.

M is the mid point of BC.

(a) Calculate the length of AM.

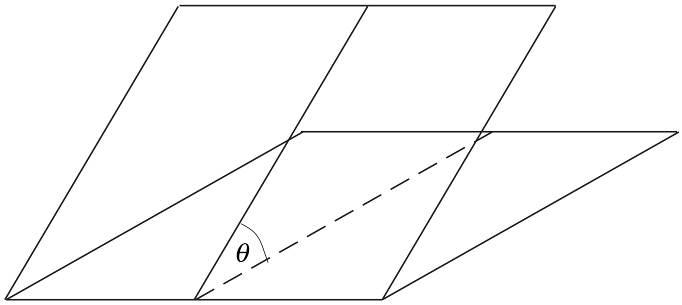
(b) Calculate the size of angle BCD.

(c) Calculate the size of angle DMA.

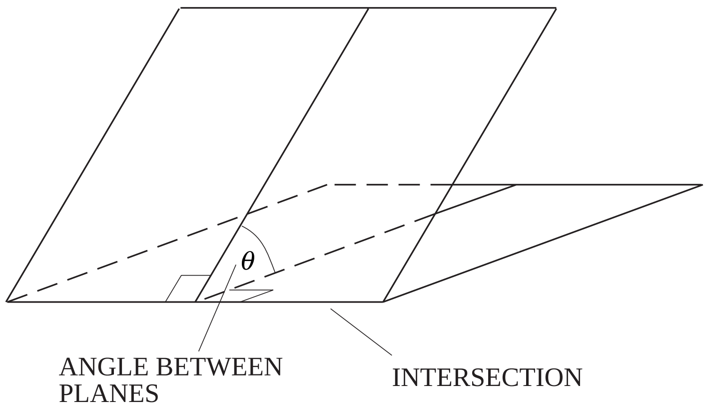
(LON)

18.2 Angles and Planes

When calculating the angle between a line and a plane you should always find the smallest angle between the line and the plane. Consider a skier travelling in a straight line down a slope. The angle between the path of the skier and a horizontal plane is the angle between the path and a line directly below the path in the horizontal plane.



When finding the angle between two planes it is important to consider where the planes intersect and the line that this forms. The angle between the two planes is equal to the angle between lines in each plane that are perpendicular to the line formed by the intersection.



Worked Example 1

The diagram shows a wedge.

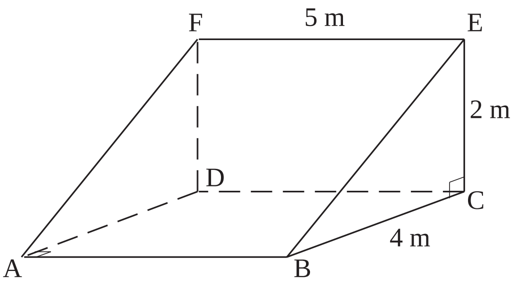
Find the angles between:

(a) the line BE at the plane ABCD,

(b) the line BF and the plane ABCD,

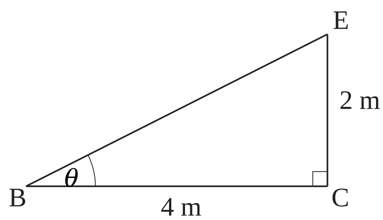
(c) planes ABCD and the plane ABEF,

(d) the lines BD and BE.



Solution

(a) Consider the triangle BCE.
The angle between the line and the plane has been labelled θ on the diagram.

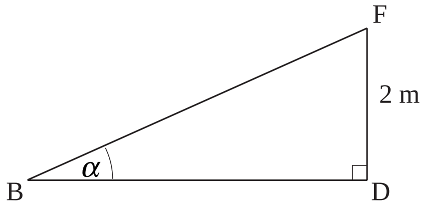


$$\tan\theta = \frac{2}{4}$$

$$\theta \approx 26.6°$$

(b) The required angle can be found from the triangle BDF because the line BD is directly below BF. This is labelled α on the diagram.

The length BD can be found by considering the base of the wedge.



$$BD^2 = 4^2 + 5^2$$
$$= 41$$
$$BD = \sqrt{41}$$

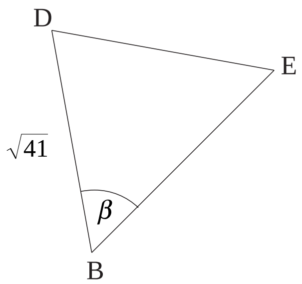
Next the angle θ can be found.

$$\tan\alpha = \frac{2}{\sqrt{41}}$$
$$\alpha \approx 17.3°$$

(c) The angle between the planes ABCD and ABEF is the same as $C\hat{B}E$ and $D\hat{A}F$.

In part (a) $C\hat{B}E$ was found as $26.6°$.

(d) Consider the triangle BDE.
The length BD is known as $\sqrt{41}$.
The other two sides must be calculated.



$$BE^2 = 4^2 + 2^2$$
$$= 20$$
$$BE = \sqrt{20}$$

and

$$DE^2 = 5^2 + 2^2$$
$$= 29$$
$$DE = \sqrt{29}$$

Using the cosine rule, because BDE is not a right angled triangle, gives,

$$\cos\beta = \frac{BD^2 + BE^2 - DE^2}{2 \times BD \times BE}$$
$$= \frac{41 + 20 - 29}{2 \times \sqrt{41} \times \sqrt{20}}$$
$$= \frac{32}{2\sqrt{820}}$$
$$\beta \approx 56.0°$$

Worked Example 2

A regular square base pyramid has a base with side of length 4 cm and height 10 cm.

Find

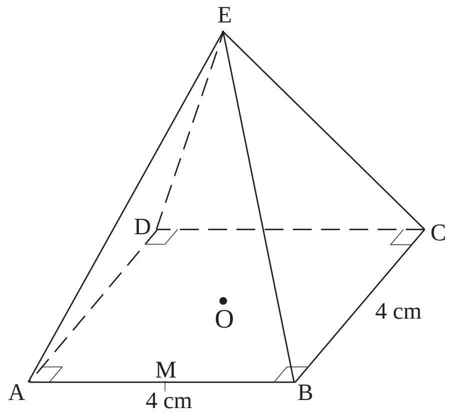
(a) the angle between a face and the base,

(b) the length of a sloping edge,

(c) the angle between an edge and the base.

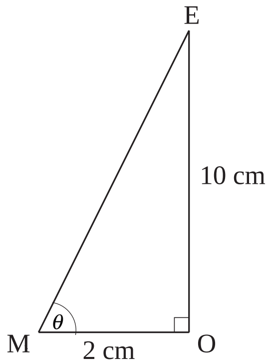
Solution

The diagram shows the pyramid.



(a) To find the angle between a face and the base introduce the points M, the mid point of AB and O, a point directly below E.

Then consider the triangle MOE. The angle between the two planes has been labelled θ.

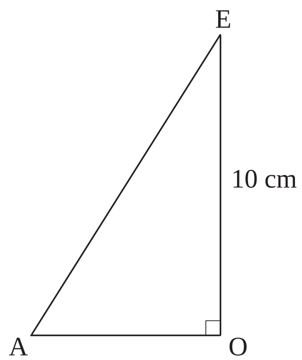


$$\tan\theta = \frac{10}{2}$$

$$\theta = 78.7°$$

(b) To find the length of a sloping edge consider the triangle AOE.

First find the length AC and then OA by considering the base.



$$AC^2 = AB^2 + BC^2$$

$$= 4^2 + 4^2$$

$$= 32$$

$$AC = \sqrt{32}$$

$$AO = \frac{1}{2}AC$$

$$= \frac{1}{2}\sqrt{32}$$

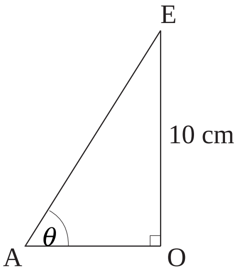
$$= \sqrt{8}$$

Now the length AE can be found:

$$AE^2 = AO^2 + DE^2$$
$$= 8 + 10^2$$
$$= 108$$
$$AE = \sqrt{108}$$
$$\approx 10.4 \text{ cm}$$

(c) The angle between the edge AE and the base is given by $O\hat{A}E$ because the line OA is directly below AE.

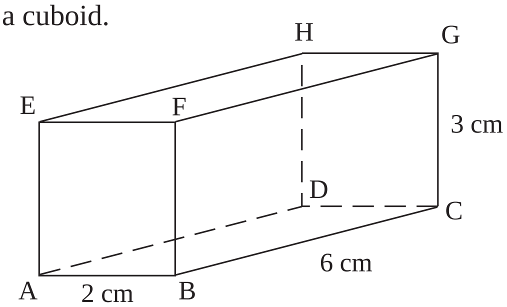
Using the triangle AOE gives:



$$\tan\theta = \frac{10}{\sqrt{8}}$$
$$\theta \approx 74.2°$$

Exercises

1. The diagram shows a cuboid.



Find the angle between;

(a) the line AG and the plane ABCD,

(b) the plane ABGH and the plane ABCD,

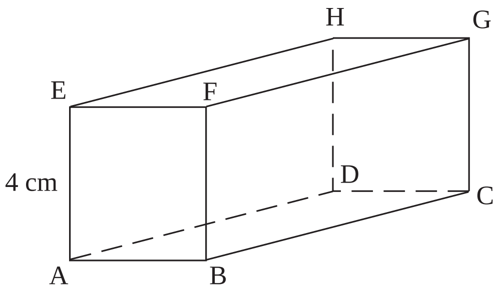
(c) the line AC and the plane ADEH,

(d) the line AG and the plane ADEH,

(e) the plane ACGE and the line AH,

(f) the plane ADGF and the plane BCHE.

2. The diagram shows a cuboid of height 4 cm.

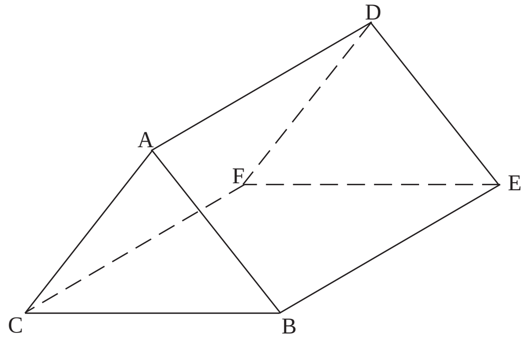


The angle between the planes ABCD and ABGH is $20°$.
The angle between the planes ADGF and BCHE is $45°$.

(a) Find the length of AB.

(b) Find the length of BC.

3. The diagram shows a prism which has a cross section that is an equilateral triangle with sides of length 5 cm. The length of the prism is 10 cm.



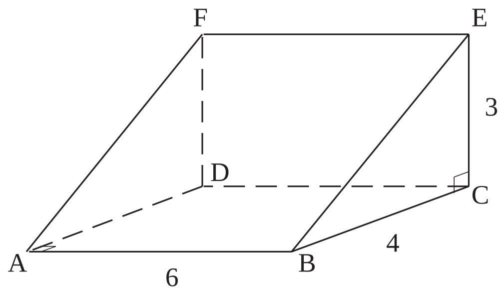
Find the angle between:

(a) the plane AEF and the base (BCFE),

(b) the line AF and the base,

(c) the line AF and the line AB.

4.



For the prism shown above find:

(a) the angle between the plane ABCD and the line BE,

(b) the angle between the plane ABCD and the line BF,

(c) the angle between the plane ABCD and the plane ABEF,

(d) the angle between the lines AF and AC.

5. A regular square-based pyramid has height 10 cm and the sides of its base are 4 cm. Find:

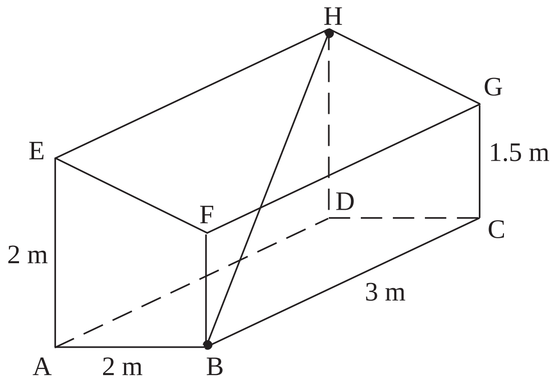
(a) the slant height of the pyramid,

(b) the angle between an edge and the base,

(c) the angle between a face and the base.

6. A pyramid has height 12 cm and base with sides of length 4 cm and 6 cm. Find the angle between each triangular face and the base.

7.



The diagram shows the dimensions of a garden shed and a long pole that just fits into the shed diagonally from B to H.

Find the angle between:

(a) the pole and the floor,

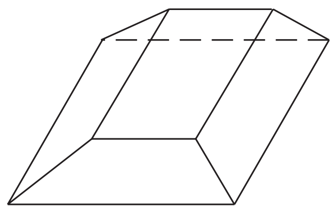
(b) the pole and the wall ADHE,

(c) the pole and the line EH,

(d) the pole and the line AB.

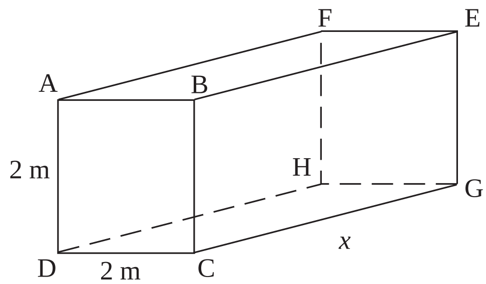
8. A pyramid has height of 6 cm and its base is a regular hexagon with sides of length 3 cm. Find the angle between each face and the base and the angle between each sloping edge and the base.

9. The diagram shows a regular square-based prism that has had its top removed. The height of the solid is 6 cm, the top is a square with sides of length 2 cm and the base is a square with sides of length 4 cm.



Find the angle between each edge and the base and the angle between each face and the base.

10. The angle between the lines DB and BG is θ.



Show that $\cos\theta = \dfrac{2}{\sqrt{8 + 2x^2}}$ and find x if $\theta = 60°$.