UNIT 10 Equations *NC: Algebra 3c, 3d*

		St	Ac	Ex	Sp
TOPICS (Text and Practice Books)					
10.1	*Negative Numbers*	✓	-	-	-
10.2	*Arithmetic with Negative Numbers*	✓	✓	-	-
10.3	*Simplifying Expressions*	✓	✓	✓	-
10.4	*Simple Equations*	✓	✓	✓	-
10.5	*Solving Equations*	✓	✓	✓	✓
10.6	*Trial and Improvement Method*	✓	✓	✓	✓
10.7	*Expanding Brackets*	✓	✓	✓	✓
10.8	*Simultaneous Linear Equations*	✗	✓	✓	✓
10.9	*Factorisation 1*	✗	✗	✓	✓
10.10	*Factorisation 2*	✗	✗	✓	✓
10.11	*Solving Quadratic Equations by Factorisation*	✗	✗	✓	✓
10.12	*Solving Quadratic Equations Using the Formula*	✗	✗	✓	✓
10.13	*Algebraic Fractions*	✗	✗	✓	✓
10.14	*Completing the Square*	✗	✗	✓	✓
10.15	*Algebraic Fractions and Quadratic Equations*	✗	✗	✓	✓
Activities					
10.1	*Solving Equations*	✓	✓	✓	✓
10.2	*Magic Squares*	✓	✓	✓	✓
10.3	*Solving Simultaneous Equations*	✗	✓	✓	✓
10.4	*Hill Walking*	✗	✓	✓	✓
10.5	*Diophantine Equations*	✗	✓	✓	✓
OH Slides					
10.1	*Using Negative Numbers*	✓	-	-	-
10.2	*Brackets*	✓	✓	✓	✓
10.3	*Equations*	✓	✓	✓	✓
10.4	*Trial and Improvement*	✓	✓	✓	✓
10.5	*Factorisation of Quadratic Equations*	✗	✗	✓	✓
10.6	*Quadratic Equations: Formula*	✗	✗	✓	✓
10.7	*Quadratic Equations: Number of Roots*	✗	✗	✓	✓
Mental Tests					
10.1		✓	✓	✓	✓
10.2		✓	✓	✓	✓
10.3		✗	✗	✓	✓
10.4		✗	✗	✓	✓
Revision Tests					
10.1		✓	-	-	-
10.2		✗	✓	-	-
10.3		✗	✗	✓	-

UNIT 10 *Equations* Teaching Notes

Background and Preparatory Work

Primitive algebraic methods began to emerge about 1700 BC in Babylon and in Egypt. However, these methods remained largely dependent on words and so developed very slowly until about 1600 AD, when a suitably flexible symbolism was developed. This flexible use of symbols then gave birth, during the next 70 years, to a truly remarkable flowering of new mathematics – first coordinate geometry (about 1630), leading to the representation of functions and the differential and integral calculus (around 1670).

These few historical details should suffice to underline the fact that

algebra is the natural language of mathematics

a fact that is understood in most countries. In contrast, developments in England in recent years have tended to give the impression that algebra is an 'optional extra' which most students will never need. In reality, algebra holds the key to subsequent progress for anyone who may one day need to use their mathematics – on no matter how lowly a level. The present unit is important because solving equations provides a very natural way of introducing students to this key skill.

The wording of the National Curriculum, the kind of problems set at KS3 and on GCSE papers, and the associated mark schemes have all conspired to obscure what 'solving equations' is really about. Thus a few words of explanation may be needed before we get down to details.

Suppose a carpenter needs to prepare and fix a strut of a certain (as yet unknown) length. He could 'judge by eye', cut a piece of wood which is roughly the right size, and then test whether it does the job – cutting off a bit more if the strut is too long, or throwing it away and trying again if it is too short. However, it would be better to use the proven method of first carefully measuring the required length, and then cutting the exact length needed. In one sense, there are two possible approaches; *but they are not genuine alternatives*, and one would never *teach* students to use the first method. It is not just that the first method is likely to be very wasteful (of both time and materials). The real difference is that, in this setting, the first (trial-and-improvement) method ignores the underlying 'discipline' of proven methods of carpentry in the hope that 'seat-of-the pants' guesswork is just as good.

In the present unit, trial-and-improvement can help beginners to get a feel for what is meant by a 'solution' of an equation. However, although guesswork may sometimes succeed in finding one solution, it can never guarantee that all solutions have been found. Moreover, the approach works only in the simplest cases, and is no substitute for a general, systematic method.

Mathematics is *the science of exact calculation*. Its methods are not just the result of accumulated wisdom: they are dictated by the exact, inescapable nature of numbers, symbols and equations. For many youngsters, the art of solving equations should give them their first real glimpse of the power of this 'science of exact calculation'. There should be no guesswork! Also students should understand that there is no approximation involved; thus, for example, fractions should never be replaced by decimals, and when learning the principles of solving equations all calculation should be exact (hence *no calculators*!). It is also important to present successive equations in a calculation in a standard way, with the corresponding justification given in brackets (on the right, say).

If the problem to be solved is given in the form of a 'word problem', then before any of this can be done it is necessary first to translate, or interpret, the information given in the 'word problem' into the form of an *equation*. There are all sorts of potential pitfalls here – not least the fact that one is at the mercy of the students' skill at *comprehension*. We restrict ourselves here to just two comments on this stage of the equation solving process.

(1) There are good reasons for keeping word problems short, and for being aware of the danger that students often respond to particular 'cue' words, rather than thinking about what the problem actually says.

(2) When choosing an unknown (say x) it is important to insist that x should always represent a *pure number* (rather than a length, or a weight, or a distance). Thus the solution should always start:

'Let x be the unknown number of marbles', or
'let the unknown length be x cm', or
'let the unknown weight by x g', or
'let the unknown distance be x km'.

One can then safely calculate with expressions such as '$3x - 4$', since these represent *numbers* (though one should remember to re-insert the units at the end of the solution).

Once the unknown has been given a name, and the given information has been written in the form of an initial equation, one is faced with a quite different problem – namely how to 'solve' the equation to find the unknown.

The art of solving equations is rooted in a single fundamental property of real numbers.

> *The Product Property*
> The only way that a product, $p \, . \, q \, . \, r \, . \, . \, . \, . \, . \, z$ can be (exactly!) equal to 0 is if one of the factors p, q, r, ..., z is (exactly!) equal to zero.

Examples:

(a) If the unknown number x satisfies the equation

$$(x-1)(x+2) = 0,$$

then either $(x-1) = 0$, so $x = 1$,

or $(x+2) = 0$, so $x = -2$.

Similarly,

(b) if the unknown number x satisfies the equation

$$(3x+2)(7-4x)(x-2) = 0,$$

then one of the factors must equal zero: so

$3x+2 = 0$, or $7-4x = 0$, or $x-2 = 0$.

T 10.2

Of course, if in the original equation the unknown x represented the number of children in a family, or the width (in cm) of a rectangle, then negative solutions would be discarded.

The *Product Property* implies that our general strategy for solving equations must have two parts:

(1) Given an equation in any form, we must first collect all terms (by doing all sorts of elementary algebra – carefully and correctly) on *one side of the equation* (say LHS), with ZERO on the other side.

(2) We must then write the expression on the LHS as a product; in other words we need to be able to *factorise*.

It is important for the teacher to note that the usual approach to the solution of *linear* equations appears to ignore the *Product Property*! Instead, it focuses on 'collecting all the xs on one side of the equation and all the numbers on the other side'.

Example:

Solve	$5-4x = 3x-2$	(Add $4x$ to both sides)
$\therefore$	$5 = 3x-2+4x$	
$\therefore$	$5 = 7x-2$	(Add 2 to both sides)
$\therefore$	$7 = 7x$	(Divide both sides by 7)
$\therefore$	$x = 1$	

Though it is unfortunate that the first method used does not automatically use (and thus reinforce the importance of) the general *Product Property*, the above approach still has several important advantages.

(1) It gives the student a clear sense of what it means to 'solve' an equation, moving systematically from opaque beginnings $(5-4x = 3x-2)$ towards the final *coup de grâce* $(x = 1)$.

(2) It emphasises, and allows the student to practise, the simplest instances of 'doing the same to both sides of an equation'.

Despite these advantages, it is important to recognise that this approach to solving linear equations often tempts students to 'generalise' in the *wrong* way to solve quadratic equations.

Example:

Solve $x^2 = 6 - x$ (Add x to both sides)

$\therefore$ $x^2 + x = 6$ (Factorise the LHS)

$\therefore$ $x(x+1) = 6$

'$\therefore$' $x = 2.$

Thus it is important to ensure that students understand that:

- the final '$\therefore$' in the above example is wrong since it overlooks the solution $x = -3$;
- the *two-sided* strategy (collect the xs on one side and the numbers on the other) which worked so well for linear equations *cannot* be used to solve quadratic equations;
- the two-sided approach for solving linear equations can be rewritten in the form of the *one-sided* approach (collecting everything on one side with ZERO on the other side, and then factorising before using the *Product Property*) –

  'To solve $5 - 4x = 3x - 2$, add $4x - 5$ to both sides to obtain $7(x-1) = 0$, $\therefore$ $x = 1$';

and finally,

- the *Product Property* is the only valid general method we have.

Once one accepts the centrality of the *Product Property*, the rest of the material to be taught fits neatly into place.

- To understand the arithmetic which underlies the algebra, students have to be comfortable with the arithmetic of negative numbers. T10.4 and 10.5
- To manipulate even the simplest algebraic expressions confidently, they need practice at simplifying expressions. T10.3
- Students repeatedly need to use, and so must understand, what it means to 'do the same to both sides of an equation'. T10.4 and 10.5
- To simplify and transform the initial equation, they must be able to multiply out brackets confidently and correctly and to clear denominators in algebraic fractions. T10.7 T10.17 and 10.13
- For the final step they need to be able to *factorise* the expression on the LHS of their final equation. T10.9, T10.10 and T10.11

Teaching Points

Introduction

This is an important unit which is of fundamental importance for further mathematical study. Before students can continue successfully on to A-Level Mathematics they *must* become competent in all aspects of this unit.

Unfortunately, many, if not most, students find algebra difficult, confusing or intimidating. This is completely unnecessary, as all algebra is based on putting into practice a set of logical steps. Provided you know the rules and stick to them, you cannot go wrong – the key is knowing these rules well.

Language / Notation

At all levels students need to be familiar with the use of x as an unknown. Although this may cause problems at the *Standard* level, it is important that *all* students become familiar and happy with this notation.

As a motivator for the use of algebraic letters, you might find the *Magic Square* activity sheet useful. Whilst the algebraic notation might cause problems, the language used is relatively straightforward. A10.2

- Expanding brackets; e.g.

  $$(x+1)(x+2) = x(x+2)+1(x+2) = x^2+3x+2$$

- Factorisation (the reverse of expanding brackets);
- Difference of two squares; e.g. $x^2 - a^2 = (x-a)(x+a)$
- Quadratic equation; in general, $ax^2 + bx + c = 0$
- Completing the square; e.g. $x^2 + 4x + 9 = (x+2)^2 + 5$

Key Points

- It is absolutely crucial that at all times equations must balance; that is, the right-hand-side must equal the left-hand-side.
- All manipulation must be logical, for example:

  $$x + 5 = 9 \qquad \text{(Take 5 from both sides)}$$
  $$\therefore \quad x + 5 - 5 = 9 - 5$$
  $$\therefore \quad x = 4.$$

  The middle step can be omitted when students are confident but it is important that they realise the real reason why 'taking +5 to the other side, so it becomes – 5' gives a correct answer.

  Similarly, for example,

  $$4x = 12 \qquad \text{(Divide both sides by 4)}$$
  $$\therefore \quad \frac{4x}{4} = \frac{12}{4}$$
  $$\therefore \quad x = 3.$$

  A 10.1

  OS 10.3

- *Trial and improvement* is a method which is used in place of algebraic manipulation where there is *not* a precise method to solve the equation. Linear and quadratic equations are exceptions to this as they can always be solved algebraically. T 10.6

  OS 10.4

Misconceptions

In this unit there are numerous errors which are made by students, usually because they do not understand the logic of the manipulation being undertaken.

So look out for examples, such as the ones below, which often are carried out incorrectly.

$$6 - (-4) = 10$$

$$-5 - (-10) = 5$$

$$a(bx + c) = abx + ac \quad \text{not } abx + c$$

$$(a + b)^2 = a^2 + 2ab + b^2 \quad \text{not } a^2 + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2 \quad \text{not } a^2 - b^2$$

$$\frac{1}{a} + \frac{1}{b} = \frac{b + a}{ab} \quad \text{not } \frac{1}{ab}$$

$$\frac{1}{a + b} \neq \frac{1}{a} + \frac{1}{b}$$

$$\frac{1}{a - b} \neq \frac{1}{a} - \frac{1}{b}$$

Many students will also have difficulty in understanding that

$$\frac{ax}{b} = \left(\frac{a}{b}\right)x = \frac{x}{\left(\frac{b}{a}\right)}.$$

Two such examples are,

$$\frac{2x}{3} = \left(\frac{2}{3}\right)x = \frac{x}{\left(\frac{3}{2}\right)},$$

and $$\frac{x}{\left(\frac{1}{2}\right)} = 2x.$$

The reason that this last result is true can readily be seen by multiplying top and bottom by 2.

Key Concepts

		St	A	E	Sp	
1.	Rules for multiplication and division:					T 10.2
	$+ \times + \equiv +$	✓	✓	-	-	
	$+ \times - \equiv -$	✓	✓	-	-	
	$- \times + \equiv -$	✓	✓	-	-	
	$- \times - \equiv +$	✓	✓	-	-	
	$+ \div + \equiv +$	✓	✓	-	-	
	$+ \div - \equiv -$	✓	✓	-	-	
	$- \div + \equiv -$	✓	✓	-	-	
	$- \div - \equiv +$	✓	✓	-	-	
2.	$(a+b)(c+d) = ac + ad + bc + bd$	✓	✓	✓	✓	T 10.7
3.	Quadratic equations: $ax^2 + bx + c = 0$					
	(i) factorisation	✗	✗	✓	✓	T 10.11
	(ii) formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	✗	✗	✓	✓	T 10 .12
	(iii) completing the square.	✗	✗	✓	✓	T 10.14