UNIT 12 *Number Patterns and Sequences* Activities

Activities

ACTIVITY 12.1 *Lines*

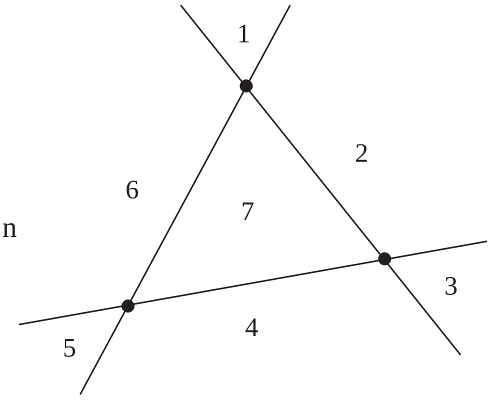
If three lines are arranged as in the diagram, there are seven regions formed, with three crossover points.

This investigation looks at the relationship between

- the number of lines *(n)*

and the maximum number of

- crossover points
- regions.



$n = 3$

1. Draw similar diagrams to find the maximum number of crossover points and regions for:

   (a) 2 lines (b) 4 lines (c) 5 lines.

2. Predict the result for:

   (a) 6 lines (b) 7 lines.

3. (a) Generalise your results and write down formulae for the maximum number of crossovers and regions.

   (b) Use the formulae to predict the maximum number of crossover points and regions for:

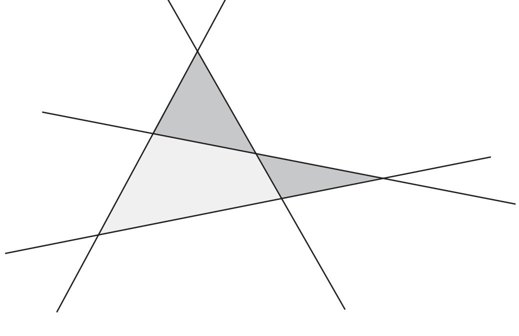
   (i) 20 lines (ii) 100 lines.

Extension

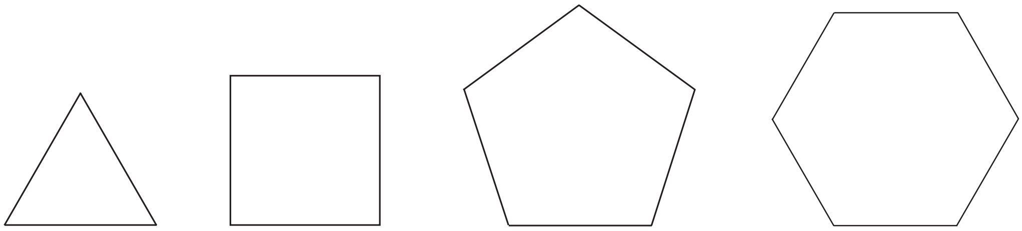
1. What are the conditions which must be fulfilled to ensure that you get the maximum number of crossovers?

2. Investigate the number and type of enclosed regions, without necessarily satisfying the conditions above, for $n = 3$, 4, 5 and 6.

   (e.g. in the diagram, $n = 4$, and the enclosed regions are 2 triangles and one quadrilateral)



ACTIVITY 12.2 *Regular Polygons*



$u_1 = 3$ $u_2 = 4$ $u_3 = 5$ $u_4 = 6$

Here is a sequence of regular polygons. Let u_n be the number of sides of the *n*th shape in the sequence. You can see that:

$$u_1 = 3, \quad u_2 = 4, \quad . . .$$

1. What is the value of u_5 and u_6?

2. What is the general formula for u_n? Check your answer for u_7.

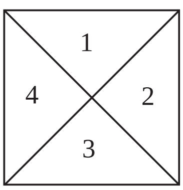
3. How many diagonals can be drawn from a single vertex in each of the shapes above?

4. How many diagonals can be drawn from a vertex of the *n*th shape in the sequence?

5. How many diagonals in total can be drawn in each of the shapes above?

6. How many diagonals in total can be drawn in the *n*th shape?

Extension

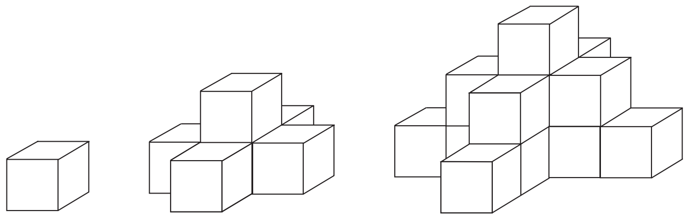


For the second shape, when all the diagonals have been drawn, there are 4 regions enclosed.

1. Repeat this process for the third and fourth shapes

2. Generalise your results to the *n*th shape.

ACTIVITY 12.3 *Towers*



How many cubes are needed to build a tower which has 100 steps?

At first sight, this might seem daunting but here are two ways of tackling this kind of problem. Both methods demonstrate how powerful mathematical analysis can be.

1 *Iteration*

Here we find the formula which fits the data.

1. Complete the table opposite.

No. of steps	1	2	3	4	5	6
No. of cubes	1	6				

2. Find the first and second differences of the number of cubes. What does this show?

3. Assuming the formula for u_n, the number of cubes in the nth tower is

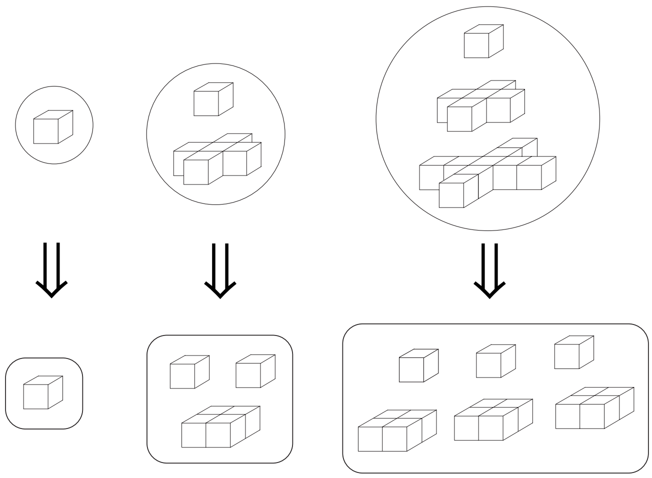
$$u_n = an^2 + bn + c .$$

Using $n = 1$, 2 and 3, write down three equations satisfied by a, b and c. Solve these equations for a, b and c and use your formula to check u_6.

4. What is the value of u_{100}?

2 *Summing the Series*

Treat each of the towers in the following way .



1. Show that:

(i) $u_4 = (4 \times 1) + (6 \times 2^2)$

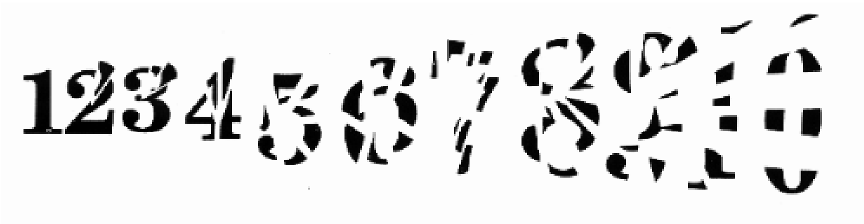
(ii) $u_5 = (5 \times 1) + (10 \times 2^2)$.

2. Deduce that:

$$u_n = n \times 1 + \frac{n(n-1)}{2} \times 2^2 .$$

3. Simplify the formula for u_n and find u_{100}.

ACTIVITY 12.4 *Ulam's Sequence*



A sequence can be made for many reasons and it can be very interesting to try to invent the rules for one and investigate what happens. Here is a sequence for which the rules were invented by *Stanislaw Ulam*, an American mathematician.

Step 1	Start with the two numbers, 1 and 2.
Step 2	Look at all other numbers in turn, starting with 3.
Step 3	If a new number can be made by: (a) adding two different numbers which are already in the sequence, and (b) this can be done in *only one way*, then that new number belongs to the sequence.

For example:

3 is in *Ulam's Sequence* because $3 = 1 + 2$, and both these numbers are already in the sequence. Also, 3 cannot be made in any other way by adding two *different* numbers.

5 is *NOT* in *Ulam's Sequence* because although $5 = 2 + 3$, (both of which are already in the sequence), 5 also equals $1 + 4$, so there is *more than one* way of satisfying the part (a) of *Step 3* above.

1. Write down the first 10 terms of *Ulam's Sequence*.
2. What are the next two numbers (after 3 and 5) which must be left out and why?
3. What is the first number left out which can be made in *three* ways?
4. Continue the sequence for another four terms.
5. What is the next consecutive pair of numbers after (3, 4)?

Extension

Investigate what would happen if:

(a) you used another pair of numbers as a starting point,
(e.g. change the order, use a zero as one of the pair, use a negative number).

(b) the sequence required *three* numbers to be added.

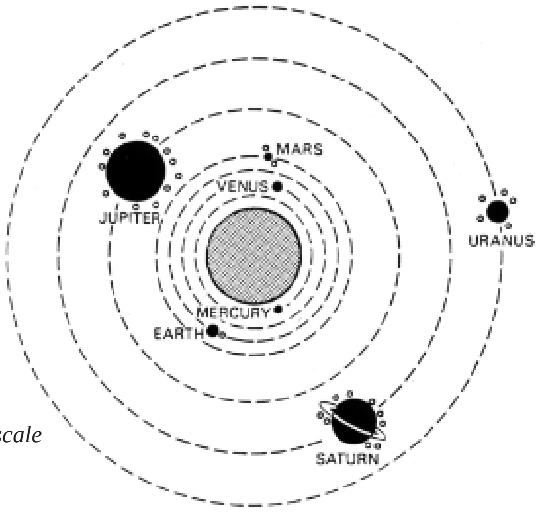
ACTIVITY 12.5 *Bode's Law*

In 1772, the German astronomer, *Johann Bode*, published his law which relates the distance ratio:

$$x_n = \frac{\text{distance of the planet from the Sun}}{\text{distance of the Earth from the Sun}}$$

to n, the number which Bode used to specify each planet, as shown in the table below.

$n = 1$	Venus
$n = 2$	Earth
$n = 3$	Mars
$n = 4$	
$n = 5$	Jupiter
$n = 6$	Saturn
$n = 7$	Uranus
$n = 8$	Neptune
$n = 9$	Pluto



Not to scale

He stated his law as:

$$x_n = 0.4 + 0.3 \times 2^{n-1}$$

1. Use this formula to find $x_1, x_2, ..., x_9$.

2. Find the first and second differences of this sequence. What do you notice?

3. The actual distances are given in the table opposite. Find the actual values of

$$x_1 \left(= \frac{108.2}{149.6} \right), \quad x_3 \left(= \frac{227.9}{149.6} \right), \quad x_5, \text{ etc.}$$

and compare with the predicted values from *Bode's Law*.

Planet	*Distance from Sun* (in millions of km)
Mercury	57.9
Venus	108.2
Earth	149.6
Mars	227.9
Jupiter	778.3
Saturn	1427.0
Uranus	2870.0
Neptune	4497.0
Pluto	5907.0

4. A large number of asteroids are found at about 433.8×10^6 km from the Sun.

Does *Bode's Law* provide confirmation that there was once a single planet at this position? [*Hint*: consider x_4.]

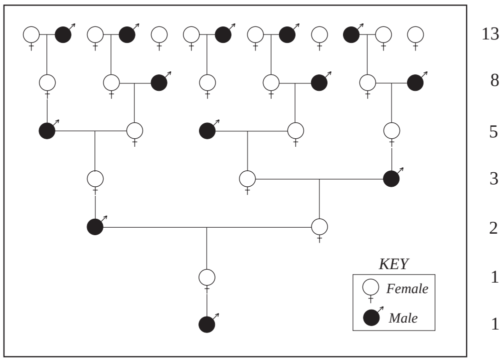
5. Do the data support the view that *Neptune* and *Pluto* were once a single planet?

ACTIVITY 12.6 *Fibonacci's Sequence*

The Italian mathematician, *Leonardo de Pisa* (nicknamed *Fibonacci*) lived from about AD1170 to 1250. He devoted much of his time and effort to the study of the so-called *Fibonacci numbers*:

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

This sequence appears frequently in the natural world, such as the pattern of reproduction in bees (shown opposite), the arrangement of leaves on stems, petals on flowers and spirals on cones.



Pattern of bee reproduction

1. Write down the next seven numbers in the sequence.

2. What is the general formula which generates the next number?

3. Consider the numbers:

   2, 3, 5, 8

   Multiply the two outside numbers and then the two inside numbers and note the difference.

   Try this with other sets of four consecutive Fibonacci numbers. *What do you notice?*

4. Square each of the first five consecutive Fibonacci numbers and add the results. Then multiply the fifth and sixth terms. *What do you notice?*

   Now square and add the first six numbers and multiply the 6th and 7th numbers. *Generalise your results.*

5. Consider the ratio of consecutive Fibonacci numbers.

   $$\frac{1}{1} = 1, \quad \frac{1}{2} = 0.5, \quad \frac{2}{3} = 0.\dot{6}, \quad \frac{3}{5} = 0.6, \ . \ . \ .$$

   To what number does this sequence tend? (This is called the *limit* of the sequence.)

   [Hint: Check the value of $\frac{-1+\sqrt{5}}{2}$.]

Extension

Investigate sequences obeying the Fibonacci relationship but with different starting values, e.g. replace (1, 1) by (1, 3).