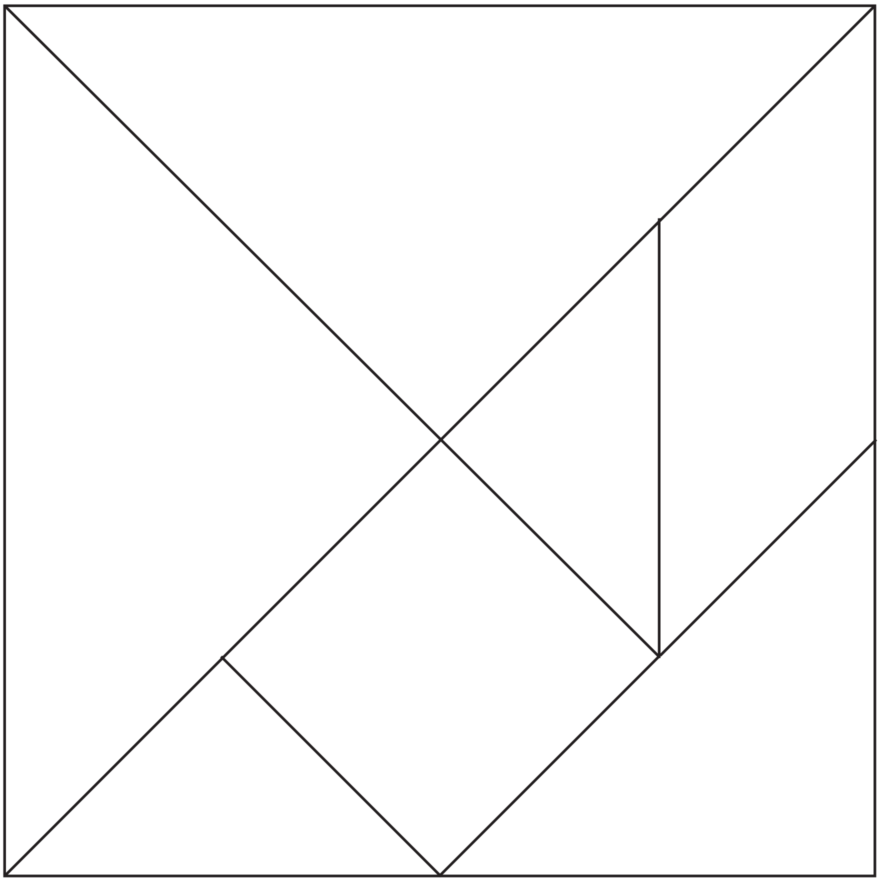
UNIT 7 *Mensuration* Activities

Activities

ACTIVITY 7.1 *Tangram*

Cut out the square below into 7 shapes.



This is a very old Chinese puzzle known as a *tangram*. Cut out the 7 shapes and rearrange them to form:

(a) a square from two triangles, and then change it to a parallelogram;

(b) a rectangle using three pieces, and then change it into a parallelogram;

(c) a trapezium with three pieces;

(d) a parallelogram with four pieces;

(e) a trapezium from the square, parallelogram and the two small triangles;

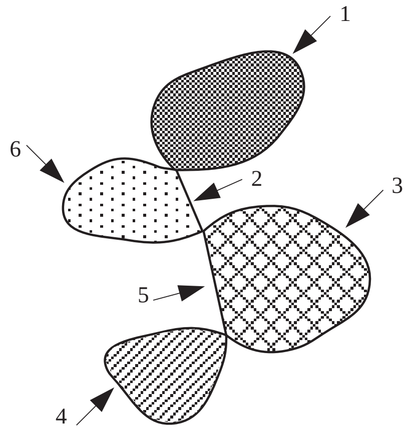
(f) a triangle with three pieces;

(g) a rectangle with all seven pieces.

Finally, put the pieces back together to form the original square.

ACTIVITY 7.2 *Closed Doodles*

The sketch opposite shows an example of a *closed doodle*, that is, it starts and finishes at the same place.



This particular doodle has

3 crossover points

4 internal regions

6 arcs (or branches).

The arcs are labelled 1 to 6 on the sketch.

There is a surprisingly simple relationship between crossovers, regions and arcs.

In drawing a closed doodle, note that the pen or pencil must never leave the paper, and must not go over any part of the doodle more than once.

1. Draw 6 closed doodles of your own.

   Copy and complete the table.

Crossovers	*Regions*	*Arcs*
3	4	6
. . .	. . .	. . .
. . .	. . .	. . .

2. From your regions, and writing

   c = no. of crossovers, r = no. of inside regions, a = no. of arcs

   deduce a simple formula which connects these 3 numbers.

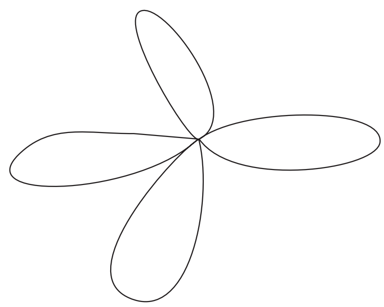
3. Draw 2 more complicated closed doodles and show that your formula still holds.

4. Does the *circle* obey your rule?

5. Does your rule hold for 'cloverleaf' type doodles, as shown opposite?

   Design a more complicated doodle of this type.

   Does it still obey the rules?



6. Can you construct closed doodles with

   (a) $c = 6$, $r = 7$, $a = 12$

   (b) $c = 8$, $r = 3$, $a = 9$

   (c) $c = 5$, $r = 9$, $a = 13$?

   If so, construct an example to illustrate it, but if not, explain why not.

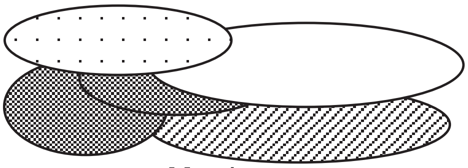
ACTIVITY 7.3 *Map Colouring*

After the introduction of colour printing presses there was much interest in *minimising* the number of colours used to distinguish different countries, in order to reduce costs.

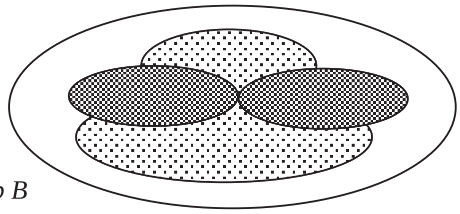
The printers had to be careful that no two countries with shared borders were coloured the same!

Map A below would not have been allowed. Two adjacent countries have the same colour.

Map B is allowed. Countries with the same colour can meet at a point.



Map A

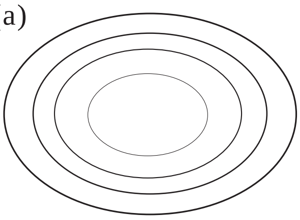


Map B

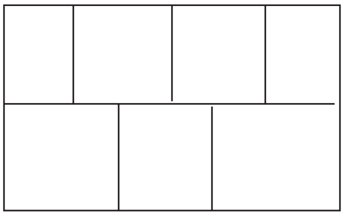
1. How many colours would be needed for *Map A*?

2. Colour the following maps, using a *minimum* number of colours.

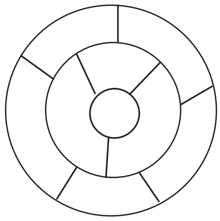
(a)



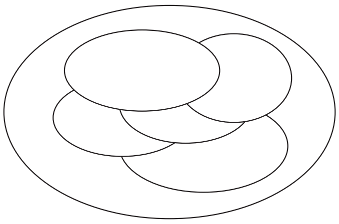
(b)



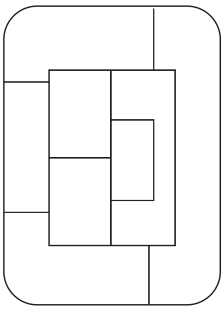
(c)



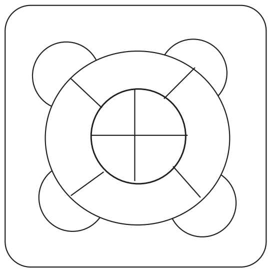
(d)



(e)



(f)



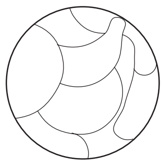
3. Draw some maps to your own design, making them as complicated as you like. Give them to a friend to find out the minimum numbers of colours needed.

Extensions

1. Find a *Map of Europe* which includes the new states which used to make up the USSR. What is the least number of colours needed:

(a) if you do *not* colour the sea (b) if one colour is used for the sea?

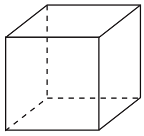
2. Map colouring on a sphere is more complicated. Draw patterns on a plain ball and investigate the minimum number of colours needed to colour *any* map.



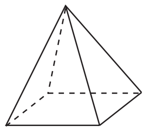
ACTIVITY 7.4 *Euler's Formula*

This particular result, named after its founder, the famous Swiss mathematician (1707-1783) is an example of a *topological invariant*; that is, something that remains constant for particular shapes.

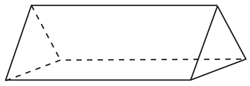
(i)



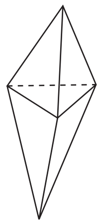
(ii)



(iii)



(iv)



1. For each of the shapes opposite, find
   (a) the number of edges, e
   (b) the number of vertices, v
   (c) the number of faces, f.

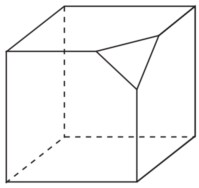
2. Show that

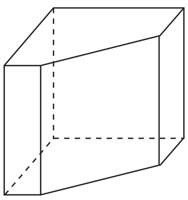
$$e + 2 = v + f$$

for each of these shapes.

Of course, verifying a formula for a few examples is no proof that the result is true for all such shapes. We will look at ways of trying to contradict the formula.

3. Suppose we cut off one corner of a cube.
   (a) How many more edges, vertices and faces are there?
   (b) Does the result still hold?



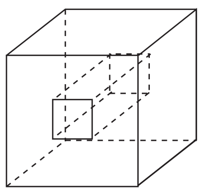


4. A slice is taken off a cube. Again, how many more edges, vertices and faces are there?

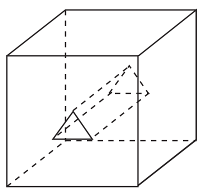
5. Try changing a cube in other ways and in each case check whether Euler's formula still holds.

Extensions

1. Suppose that a square hole is made right through the cube. Does Euler's formula now hold?



2. Try making a triangular hole right through a cube. Does Euler's formula now hold?



3. Put a similar hole in one of the other shapes used above. Does Euler's formula still hold?

ACTIVITY 7.5 *Square-based Oblique Pyramid*

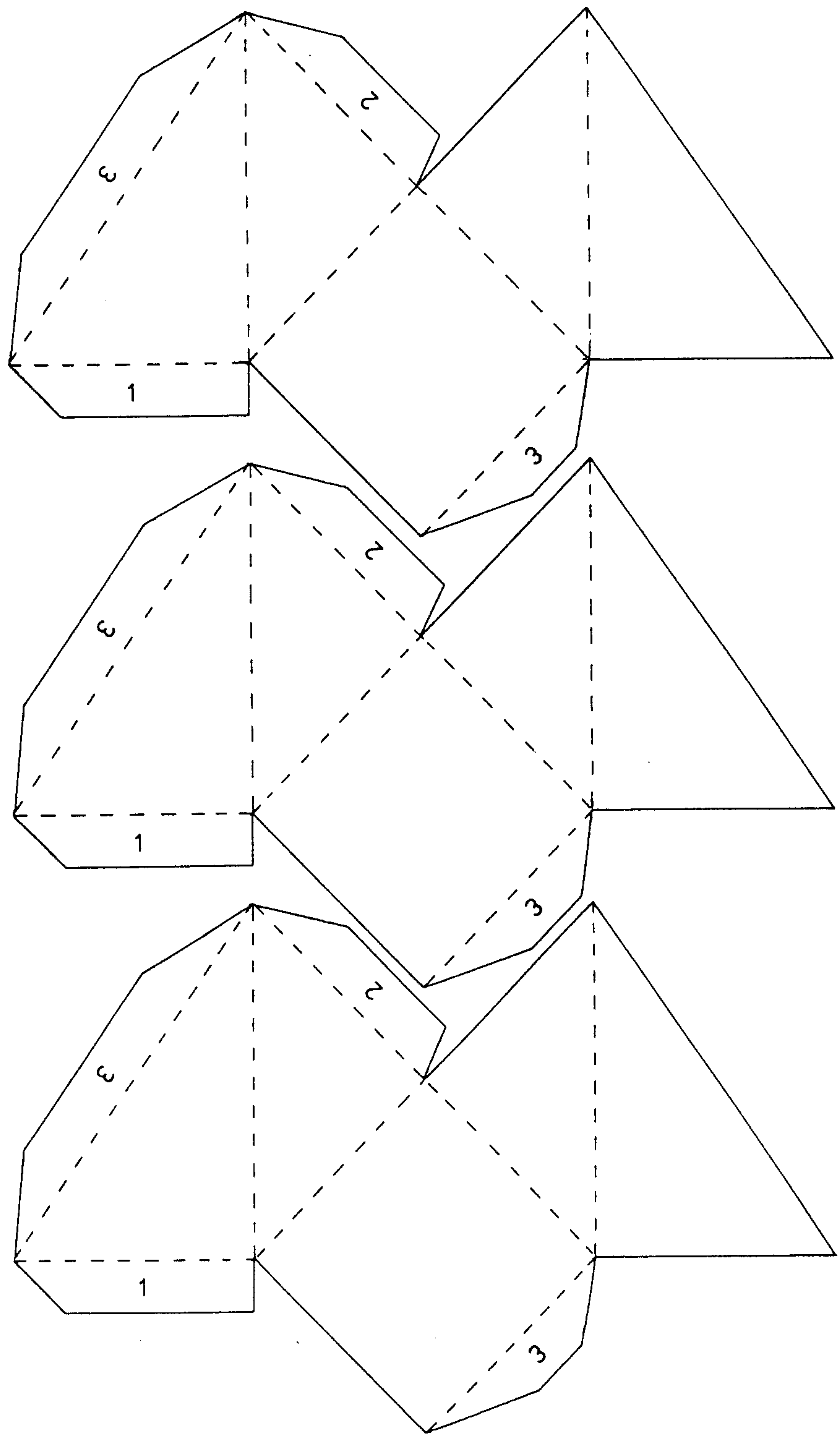
Following the instructions, make three square-based oblique pyramids from the identical shapes below.

Instructions *Cut out along the solid lines; score and fold along the broken lines.*

Roughly fold each shape into position to see how it will look.

Glue the tabs in the order they are numbered (where tabs have the same numbers, glue them at the same time).

Now put your three shapes together to form a cube.



ACTIVITY 7.6.1 *Klein Cube*

A *Klein Cube* is a three-dimensional version of a Mobius Strip (*August Mobius* was a pupil of the great mathematician *Carl Friedrich Gauss* (1777-1855)), and is named after its inventor, the German mathematician *Felix Klein* (1849-1928). He designed the *Klein Bottle* which is a three dimensional shape with only one surface; the model given here is based on the design of the Klein Bottle.

When you have constructed it, imagine that you start to paint the outside blue and continue painting along any joining surface. You will eventually find the whole shape (inside and outside) has been painted blue; hence it has only *one* surface.

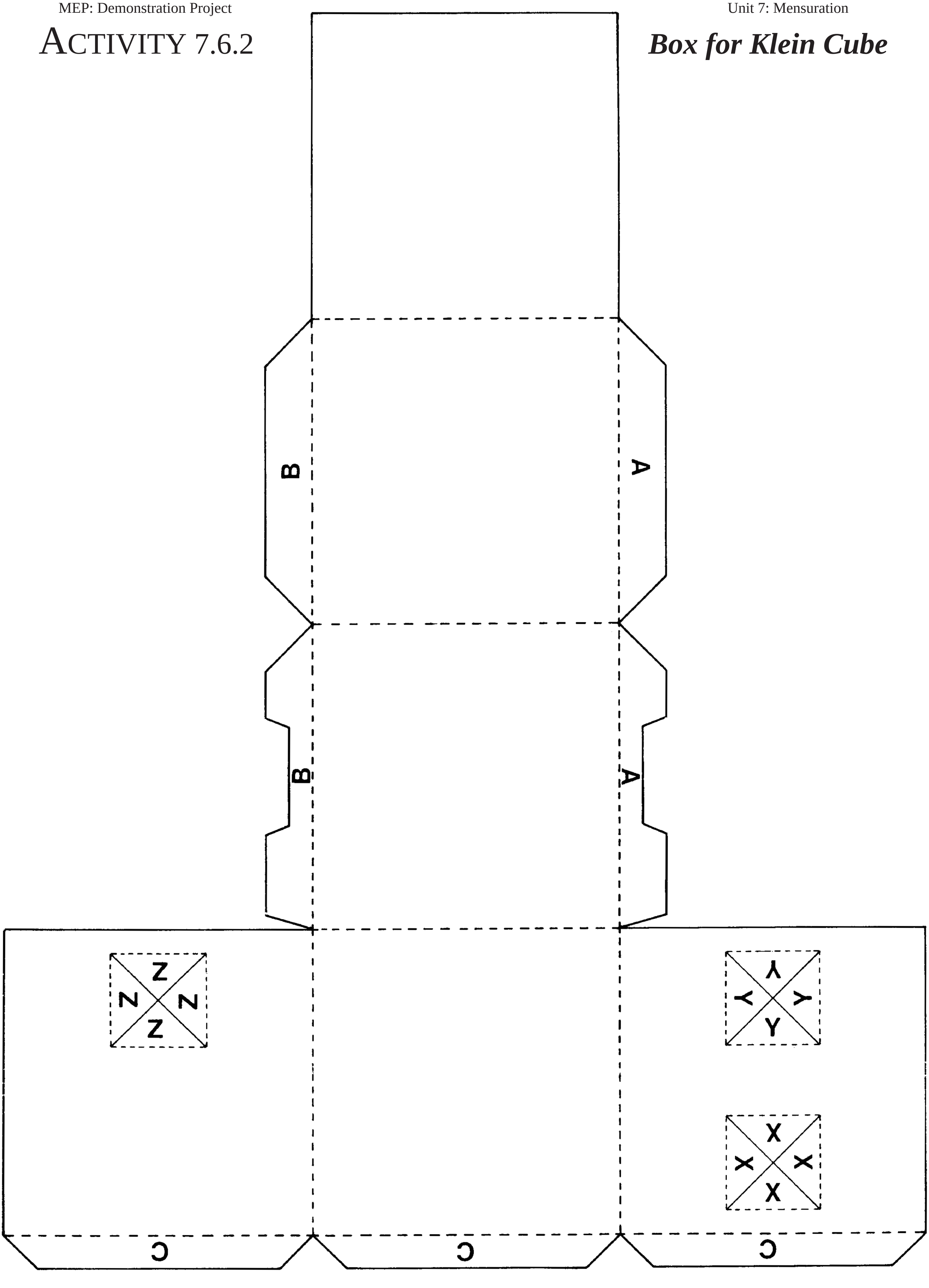
Instructions for making the Klein Cube

Activity pages 7.6.2 and7.6.3

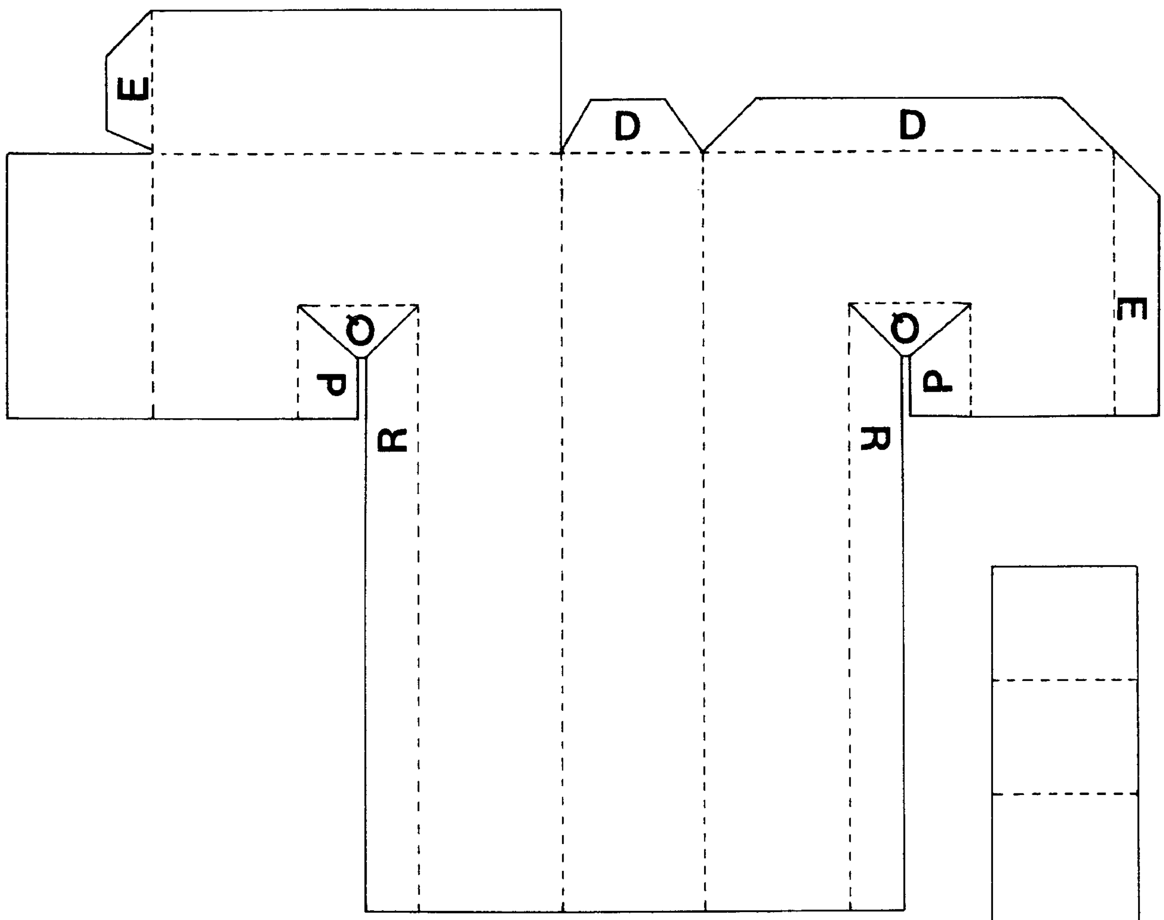
1. First cut out around the solid line of the edge of the net on page 7.6.2. Score and crease all the dotted lines to make tabs A, B and C.
2. Cut along solid lines in X, Y and Z and score along the dotted lines.
2. Fold up to make a box. Glue tabs A and B but NOT C.
3. Push tabs X and Y to outside of box and tab Z inside.
4. Next make the Tube for Klein Cube using the nets and instructions on page 7.6.3.
5. Now push the longer piece of the tube through the Y-opening of the box and into the Z-opening.
6. Glue the X, Y and Z tabs to the tube. The X and Y tabs go on the outside of the tube; the Z-tab goes inside the tube.
7. Glue tabs C to complete your Klein Cube.

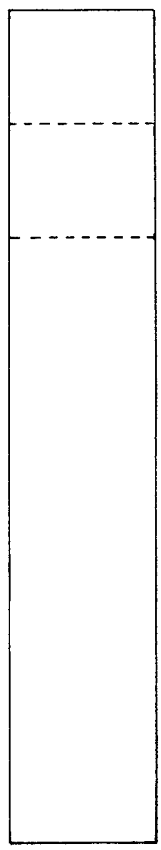
ACTIVITY 7.6.2

Box for Klein Cube



ACTIVITY 7.6.3 *Tube for Klein Cube*

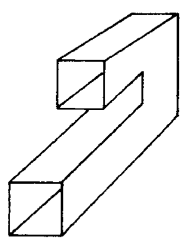




Tube for Klein Cube

Using the nets on this page,

1. Cut out around the solid line of the edge of the larger net.
2. Cut along solid lines PQ and QR.
3. Score and crease all dotted lines to make tabs D, E, P, Q and R.
4. Fold up and glue tabs D and then tabs E.
5. Cut out, score and crease the smaller net.
6. Fold it and glue tabs P, Q and R to complete the tube.
   (It is easiest to do P and Q first, then R.)



Finished Tube

ACTIVITY 7.7 *Transforming Polygons*

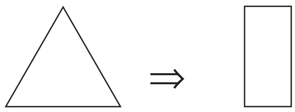
The American mathematician *David Hilbert* (1862-1942) was the first person to prove that

Any polygon can be transformed into any other polygon of equal area by cutting it into a finite number of pieces and rearranging.

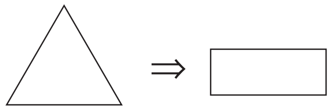
Unfortunately, the proof of this result does not tell you how to do it – just that it can be done!

We will first look at some easy examples and then show how any equilateral triangle can be made into a square of the same area.

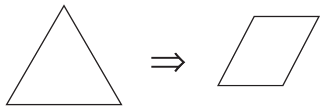
1. How can an equilateral triangle be transformed into a rectangle, which has one of its sides equal to the *height* of the triangle?



2. How can an equilateral triangle be transformed into a rectangle which has one of its sides equal to *the length of side* of the triangle.

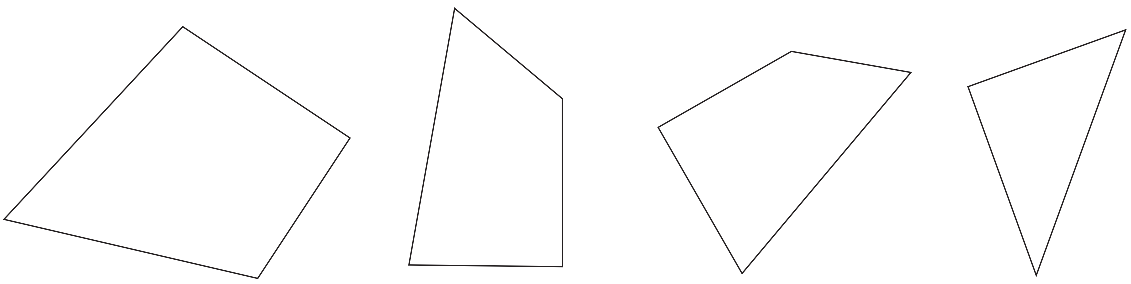


3. How can you transform the equilateral triangle into a parallelogram which has a base height equal to *three quarters* the length of a side of the triangle?



These problems are all quite straightforward in their construction. A much more difficult problem is to transform an equilateral triangle into a *square* of the same area.

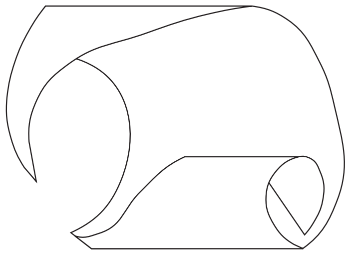
4. Cut out the pieces shown below and check that you can make both an *equilateral triangle* and a *square* from them. The pieces must all be kept the same way up.



Extension
Starting with an equilateral triangle, and by making suitable cuttings, see what shapes you can make.

ACTIVITY 7.8 *Tubes*

Tubes containing sweets, crisps, etc. and the inner tubes of various rolls (e.g. toilet rolls, cling film, foil) are made from cardboard. If you look carefully at the construction of these tubes you will often see that the net is made from a *parallelogram* rather than a rectangle.



1. Find as many different tubes as possible and undo the construction to find the parallelogram net used to make the tube. Measure the corner angle of each of these parallelograms.

2. (a) Why is the construction of these rolls made from a parallelogram net rather than a rectangle?

   (b) What difference does changing the angle make?

Although the tubes for toilet rolls do vary, we will look at average measurements.

radius = 2 cm, height = 11 cm

3. Design a parallelogram net which would make a toilet roll tube with these measurements. Cut it out in paper (or card) to check that it works.

4. If the parallelograms are constructed from cardboard $2 \text{ m} \times 2 \text{ m}$, how many cardboard tubes of your design could be made? What percentage of cardboard is wasted?

The reverse problem is to cut out parallelograms with given dimensions and see what size rolls are produced.

5. If the length and height of the parallelogram are 15 cm and 12 cm respectively, and the corner angle is $45°$, what size roll will be made?

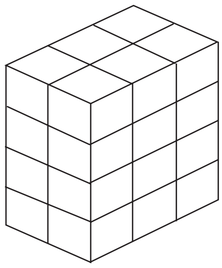
6. Does the size of the roll produced in problem 5 depend on the corner angle?

Extension
If the roll has dimensions

radius = r cm, height = l cm

what are the dimensions required for the parallelogram net?

ACTIVITY 7.9 *Minimum Wrapping*

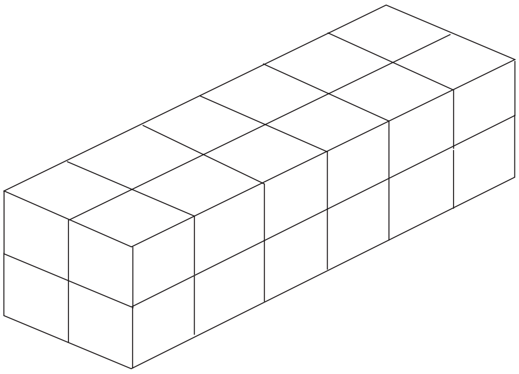


Construct a cuboid with 24 small cubes, as shown opposite.

The surface area of the cuboid is

$$2 \times (\underbrace{2\times 3}_{\text{top}} + \underbrace{4\times 2 + 4\times 3}_{\text{sides}}) = 52 \text{ square units}$$

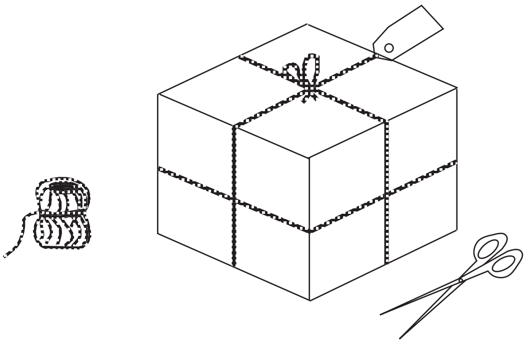
Does the surface area change if the arrangement (configuration) of cubes changes?



1. How many cubes are needed for the cuboid opposite?
2. Find the surface area of this cuboid.
3. Using 24 cubes to make a cuboid, what is the configuration that gives the
   (a) *minimum* surface area,
   (b) *maximum* surface area?

The minimum surface area configuration corresponds to the shape that requires the minimum amount of wrapping paper for a given volume (i.e. 24 cubic units). Of course, overlaps would be needed in practice, but as they would be similar for all shapes, we can disregard them here.

4. Find the *minimum* wrapping for
   (a) 48 cubes (b) 64 cubes.



A related problem is to find the total length of string required to go round the cuboid in each direction (as shown opposite), *not* including the extra needed for the knots.

5. For the cuboids you made earlier from 24 cubes, find the total length of string needed for each configuration.
6. What configuration minimises the amount of string needed to tie up a parcel of volume 24 cubic units?

Extension

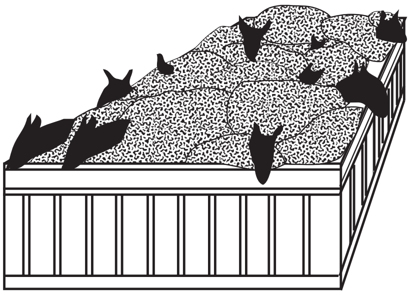
1. Find the cuboid shape that minimises the surface area when it encloses a fixed volume, V .
2. Does this also minimise the length of string needed?

ACTIVITY 7.10 *Fence it off*

We often use mathematics to make the best possible decisions about resource allocation.

In the problems which follow, the farmer has to decide how best to use a limited amount of fencing. Builders, planners and engineers often have similar problems to solve.

1. A farmer has exactly 200 metres of fencing with which to construct a rectangular pen for his sheep. In order to enclose as much grass as possible, the farmer tries out different dimensions and finds the area in each case.

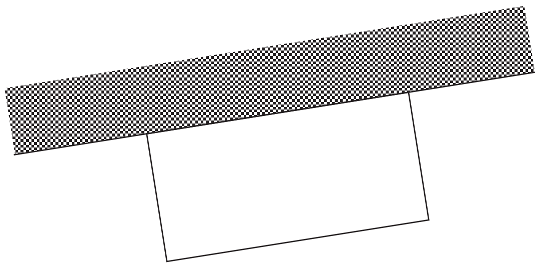


Dimensions (m)	*Area* (m^2)
5 and 95	475
10 and 90	900
15 and 85	1275
....	.....

Complete the table, and find the dimensions which give the *maximum* area.

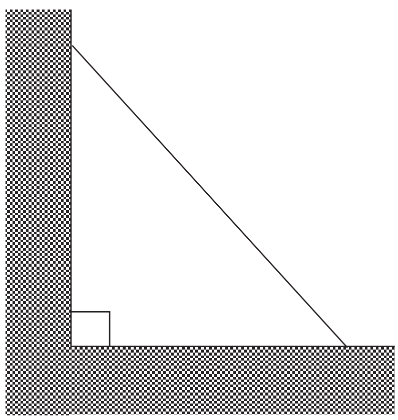
2. The farmer again wants to form a rectangular pen, but this time has a long straight wall which can form one of the sides.

   With his 200 m of fencing, what is the largest area of grass that he can enclose?



3. The farmer now wishes to use his fencing to cut off a corner of a field, as in the diagram.

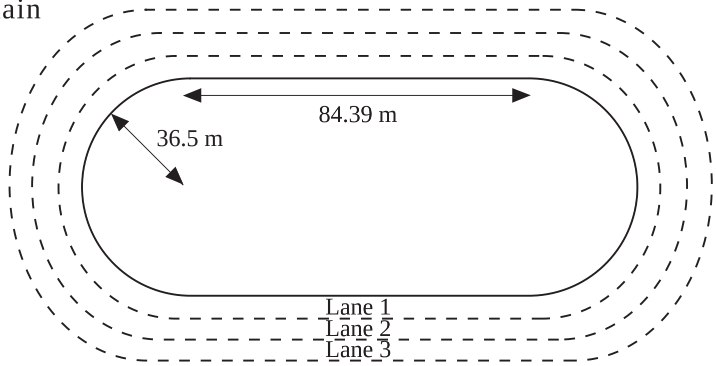
   If the length of fencing is again 200 m, what is the maximum area that can be enclosed?



4. Can you generalise the results of the three questions above to a fencing length of x metres?

ACTIVITY 7.11 *Track Layout*

The sketch opposite shows the two main dimensions of a standard 400 metres running track.



1. Find the inside perimeter of this shape.

   Why do you think that it is *not* equal to 400 metres?

The inside runner cannot run at the very edge of his lane (there is normally an inside kerb) but let us assume that the athlete runs at a constant distance of, say, x cm from the inside edge.

2. What is the radius of the two circular parts run by the athlete in the inside lane?

3. Show that the total distance travelled, in centimetres, is

$$2\pi(3650 + x) + 16878$$

   and equate this to 40 000 cm to find a value for x. Is it realistic?

For 200 m and 400 m races, the runners run in specified lanes. Clearly, the further out you are the further you have to run, unless the starting positions are *staggered*.

The width of each lane is 1.22 m, and it is assumed that all runners (except the inside one) run about 20 cm from the inside if their lanes.

4. With these assumptions, what distance does the athlete in *Lane 2* cover when running one complete lap? Hence deduce the required stagger for a 400 m race.

5. What should the stagger be for someone running in *Lane 3*?

6. If there are 8 runners in the 400 m, what is the stagger of the athlete in *Lane 8* compared with that in *Lane 1*? Is there any advantage in being in *Lane 1*?

Extensions

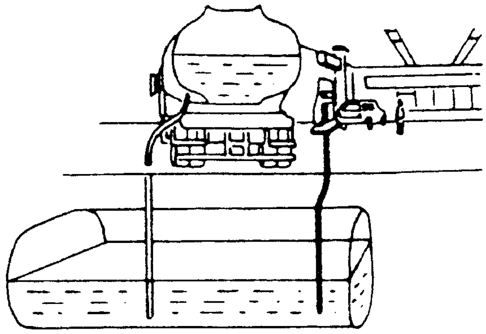
1. The area available for a school running track is $90 \text{ m} \times 173 \text{ m}$. How many lanes could it be?

2. Design a smaller running track, with lanes, to fit an area $40 \text{ m} \times 90 \text{ m}$.

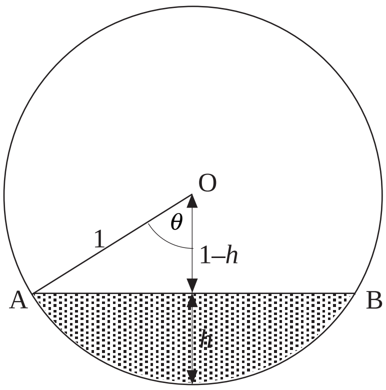
ACTIVITY 7.12 *Dipstick Problem*

Petrol stations very rarely run out of fuel. This is due partly to efficient deliveries but also to precise stock control.

Each type of fuel (4 star, unleaded, diesel) is stored in an underground tank and the amount left in each tank is carefully monitored using some form of dipstick.



It is easy to measure the height, say h, left in the tank. However, the volume will be proportional to the *cross-sectional area* – not the height. Suppose the cross-section is a circle (it is in fact elliptical, but a circle is a good approximation). We will find the relationship between area, A, and height, h, and so provide a ready reckoner to convert height to area.



For simplicity, we will take $r = 1\text{ m}$. For values of h from 0 to 1, we will find the angle θ and the area of fuel.

1. Show that $\cos\theta = 1 - h$.

2. Show that the area of the sector OAB is given by $$\frac{\pi\theta}{180}$$

3. Show that the area of the triangle OAB is $$(1 - h)\sin\theta$$

4 Deduce the area of the cross-section of fuel and express this as a fraction, A', of the complete cross-sectional area of the tank.

5. (a) Using the equation in problem 1, find the value of θ for each value of h in the table opposite.

   (b) Use the formula deduced in problem 4 to find the area fractions.

h	θ	Area fraction
0	0	0
0.1	25.89	
0.2	...	...
...	...	...
...	...	...
1.0	90	0.500

6. Plot a graph of A' (vertical axis) against height h (horizontal axis).

7. Use your graph to estimate the height that corresponds to an area fraction of

   (a) 0.05 (b) 0.10

Extension
Construct a dipstick for this problem, from which you can read off the area fraction against height value.