UNIT 16 *Inequalities* Activities

Activities

ACTIVITY 16.1 *Archimedes' Inequality for π*

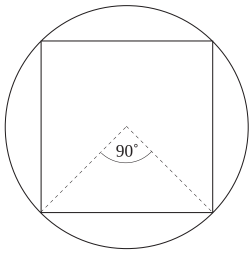
For over 200 years, mathematicians have been attempting to find accurate values for π and whether this number has any pattern to its decimal form. An early approxiation was given by *Archimedes* (in about 250BC) who used 96-sided regular polygons to show that

$$3\frac{10}{71} < \pi < 3\frac{1}{7}.$$

Here we will follow a quick method.

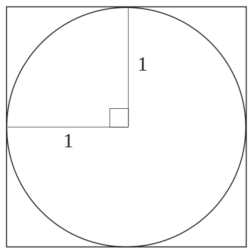
1. First start with a *square*.

(a)



For a circle of radius 1, what is the area of the *inscribed* square?

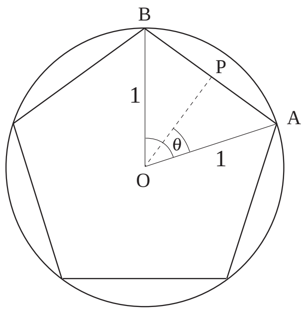
(b)



What is the area of the *circumscribed* square?

2. Deduce the upper and lower bounds for π based on your answers to Question 1. [Hint: the area of the circle is $\pi \times 1^2 = \pi$.)

3. Now consider a regular *5-sided polygon*.



(a) What is the value of θ?

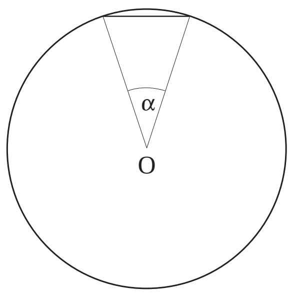
(b) What are the height, OP, and the base length, AB, of the triangle?

(c) What is the total area of the inscribed triangle?

(d) Repeat (a) to (c) but with a circumscribed regular 5-sided polygon.

(e) Deduce improved upper and lower bounds for π .

Extension



Generalise your method to a regular *n*-sided polygon.

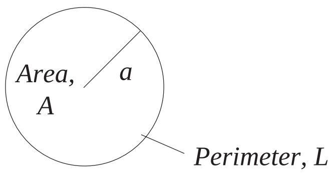
1. For the inscribed polygon, what is the angle α?
2. Find the total area of the inscribed polygon.
3. What is the area of: (a) the inscribed polygon
   (b) the circumscribed polygon?
4. (a) Deduce bounds for π when: (i) $n = 96$ (ii) $n = 1000$ (iii) $n = 1000\,000$.
   (b) In (i) how accurate is your estimate of the value of π?

ACTIVITY 16.2 *Isoperimetric Inequalities*

According to legend, *Princess Dido*, a clever Greek princess fleeing from the tyranny of her brother, landed at Constantinople in Turkey and gained a concession from the local people, who said that she could have *all the land which could be encompassed by the skin of an ox*.

She took the biggest ox skin she could find, cut it into very thin, long strips which she joined together and then placed on the ground, claiming all the land within the skin.

The shape she used to make the enclosed area as large as possible was a *circle*.



A circle is symmetrical about any diameter and for a given perimeter, L, the circle gives the maximum area, A, enclosed.

For a circle of radius a, $A = \pi a^2$ and $L = 2\pi a$, or $a = \dfrac{L}{2\pi}$.

Substituting for a: $A = \pi\left(\dfrac{L}{2\pi}\right)^2 = \dfrac{L^2}{4\pi}$

Hence for a circle, $\dfrac{4\pi A}{L^2} = 1$.

This result is used to measure how close any plane shape is to a circle and the quantity $\dfrac{4\pi A}{L^2}$ is called the *Isoperimetric Quotient* (IQ) of the shape.

It is conjectured that IQ ≤ 1, with equality occuring only for the circle.

Questions

1. Find the IQ for the following plane shapes and show that they support the conjecture above:

(a)	square, side a	(b)	rectangle, sides a and $2a$
(c)	rectangle, sides a and $5a$	(d)	equilateral triangle
(e)	'3, 4, 5' triangle	(f)	semicircle
(g)	'5, 12, 13' triangle	(h)	regular pentagon.

2. (a) Draw up a table of IQ numbers in ascending order.

   (b) What can you conjecture about IQ numbers?

Extension

1. For a regular n-sided polygon, find an expression for its area and perimeter.
2. Use these results to find a formula for the IQ of an n-sided polygon.
3. Evaluate for $n = 10$, $n = 100$, $n = 1000$.
4. What conjecture can you make?

ACTIVITY 16.3 *Inequalities for Mean Values*

You are probably familiar with the *Arithmetic Mean*, A, of a set of positive numbers. It is defined by:

$$A = \frac{1}{n}\left(a_1 + a_2 + a_3 + \ldots + a_n\right).$$

Two other types of mean values are:

(1) the *Geometric Mean*, G, defined by: $G = \left(a_1 \, a_2 \, a_3 \ldots a_n\right)^{\frac{1}{n}}$

(2) the *Harmonic Mean*, H, defined by: $H = n\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}\right)^{-1}$

Questions

1. (a) For the five values, 1, 2, 3, 4 and 5, evaluate A, G and H.

   (b) For the five values, 3, 3, 3, 3 and 3, evaluate A, G and H.

   [*Note*: When all the values are equal, then $A = G = H$ (= common value).
   This is a property which any mean value must have.]

2. Our conjecture is that for any set of values, $A > G > H$, and the equality occurs only when all the values are equal.

   Show that this inequality holds for your own choice of values.

3. For $n = 2$, i.e. two values, say a and b,

$$A = \frac{a+b}{2}, \quad G = \sqrt{ab}, \quad H = \frac{2}{\frac{1}{a} + \frac{1}{b}}$$

   (a) Using the fact that $(a-b)^2 \geq 0$, show that $a^2 + 2ab + b^2 \geq 4ab$.

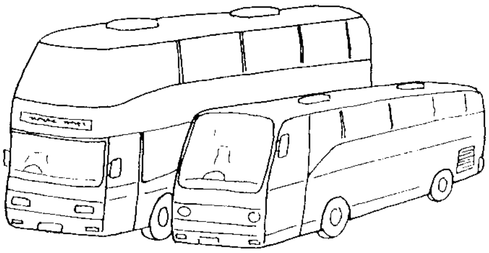
   (b) Hence deduce that $(a+b)^2 \geq 4ab$, and that $A \geq G$.

Extension

Use a similar method to prove that $G \geq H$.

ACTIVITY 16.4 *Linear Programming*

Linear Programming was developed during the *Second World War* to solve complicated optimisation problems.



Sample Problem

To find the best (cheapest) way to organise coaches for a school trip for **560** people (pupils and staff).

The coach firm contacted has two types of coaches:

Coach type	*Capacity*	*Cost per hour*	*No. available*
Double decker	60	£50	6
Single decker	40	£40	15

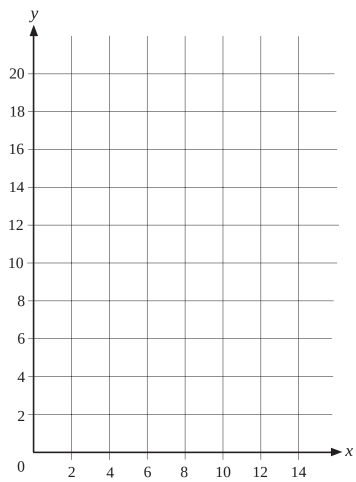
Method

Let x = no. of double deckers and y = number of single deckers.

1. The firm has only 6 double deckers, so $0 \le x \le 6$.
   Write down a similar constraint for y.

2. (a) What is the total number of passengers who can be carried in x double deckers and y single deckers?

   (b) Write down the appropriate inequality to be satisfied.



3. On an appropriate set of axes, similar to those opposite, illustrate all three inequalities on a graph by first drawing $x = 0$, $x = 6$, etc.

4. The boundaries of these lines define the *feasible region*, in which all the inequalities are satisfied.
   Show this region by shading.

5. The cost, C is given by the formula $C = 50x + 40y$.
   Lines given by C = constant are straight, parallel lines.

   (a) Draw $C = 1000$, $C = 900$, $C = 800$, etc. on your graph.

   (b) At what point will C reach its minimum value inside, or on, the boundary of the feasible region?

   (c) What is the optimum (best) solution to the problem?