Mathematical Proof

1.1 True or False

In this section we look at whether statements are true or false. The first point to appreciate is that some statements are *always true*, some are *always false* and others are *sometimes true and sometimes false.*

Worked Example 1

Decide whether the following statements are *always false*, *sometimes true* or *always true*.

(a) Tomorrow is Thursday.

(b) There are 35 days in this month.

(c) If you double a whole number you get an even number.

(d) If you double a number you get an even number.

(e) If you add 1 to an odd number you get an odd number.

(f) If you multiply two odd numbers you get an odd number.

Solution

(a) This statement is *sometimes true.* It is correct on a Wednesday but not correct on other days of the week.

(b) This statements is *always false*. The maximum number of days in any month in our calendar is 31.

(c) This statement is *always true*. For example, if you double 13 you get the number 26.

(d) This statement is *sometimes true*. It is *true* when you double a whole number, but *not true* when you double $2\frac{1}{2}$, as you get the odd number 5, and if you double $\frac{1}{4}$ you get the fraction $\frac{1}{2}$.

(e) This statement is *always false.* For example, if you add 1 to 3 you get the even number 4.

(f) This statement is *always true.* For example, if you multiply 5 and 9 you get the odd number 45; 13×67 is the odd number 871; 6543×9147827 is the odd number 59854232061.

Note

It is fairly easy to see that statement (e) in Worked Example 1 is always false, because adding 1 to an odd number gives the even number that follows it. However, it is not so obvious that statement (f) is always true. We could try a number of different cases and check that it works each time, as shown in the solution given, but this does not *prove* that it works all the time. We will prove this fact in section 1.2.

In Mathematics, we try to discover things that are always true. We might describe important statements of this type as *fundamental truths*. One illustration of this is Pythagoras' Theorem, which states that the sides of every right-angled triangle satisfy the formula $a^2 + b^2 = c^2$. In Mathematics, we try to apply logical processes to verify the certainty of fundamental truths beyond any doubt.

A mathematical statement is *true* if it *always holds.* A statement is *false* if there is *at least one example where it breaks down.*

Worked Example 2

Decide whether the following mathematical statements are *true* or *false*:

(a) Angles at a point add up to $360°$.

(b) The angles of a triangle add up to $270°$.

(c) If a triangle has two equal sides, then it will have two equal angles.

(d) If the sides of a quadrilateral are equal in length, then it is a square.

Solution

(a) This statement is *true* because there are $360°$ in a full turn.

(b) This statement is *false*. The angles of a triangle always add up to $180°$.

(c) This is *true* as the statement relates only to isosceles triangles.

(d) This statement is *false* because a rhombus also has 4 equal sides.

Worked Example 3

Decide whether the following mathematical statements are *true* or *false.*

(a) $x(x+1) = x^2 + 1$.

(b) $x(x+1) = x^2 + x$.

(c) If *x, y* and *z* are even, then $x + y + z$ is even.

(d) For any number *x*, $x^2 > 0$.

(e) For any number *x*, $x^2 \geq 0$.

Solution

(a) If $x = 1$, then

$$x(x+1) = 1(1+1) = 1 \times 2 = 2$$

$$x^2 + 1 = 1^2 + 1 = 1 + 1 = 2$$

so the formula does hold for $x = 1$.

However, if $x = 2$, then

$$x(x+1) = 2(2+1) = 2 \times 3 = 6$$

$$x^2 + 1 = 2^2 + 1 = 4 + 1 = 5$$

so the formula does not hold for $x = 2$.

The fact that we have found a case where the statement does not hold means that the statement is *false*.

(b) This statement is *true*. We can verify this by multiplying out the bracket on the left hand side of the equation.

(c) This statement is *true*. Adding any number of even numbers always gives an even total.

(d) This statement is *false* because $0^2 = 0$ is not positive. Again, we have found a case where the statement does not hold.

(e) This statement is *true*.

1.1 True or False

Exercises

1. Decide whether the following statements are *false, sometimes true* or *always true.*

   (a) Christmas Day is on a Wednesday.

   (b) A year will have 400 days.

   (c) There is a total of 61 days in April and May.

   (d) April has more days than September.

   (e) If 2 May is a Monday, then 9 May will be a Monday.

   (f) There is a gap of 4 years between Olympic Games.

   (g) If you can divide the last two digits of a year exactly by 4, it will be a leap year.

2. Decide whether the following mathematical statements are *true* or *false.*

   (a) Angles on a line add up to $90°$.

   (b) The angles of a quadrilateral add up to $360°$.

   (c) A regular hexagon has interior angles of $120°$.

   (d) A square is a rectangle.

   (e) The circumference of a circle is 3 times the diameter.

   (f) The area of a circle is approximately 3 times the square of the radius.

3. Decide whether the following mathematical statements are *true* or *false.*

   (a) $x + y = y + x$

   (b) $x - y = y - x$

   (c) $xy = yx$

   (d) $\frac{x}{y} = \frac{y}{x}$

   (e) $(x + y)^2 = x^2 + y^2$

   (f) $(x - y)^2 = x^2 - y^2$

   (g) $(xy)^2 = x^2 y^2$

   (h) $\left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}$

1.2 Proof

Proof is the heart of mathematics. If distinguishes mathematics from the sciences and other disciplines. Courts of law deal with the burden of proof, juries having to decide whether the case against a defendant has been proven *beyond a reasonable doubt.* That remaining element of doubt is not acceptable in mathematics. A mathematical proof must be watertight, establishing the truth of the statement *beyond any doubt.*

Scientists advance theories based on, and supported by, all the available evidence. A given theory will survive only so long as no evidence is found to cast doubt on it. When there is doubt, scientists will conjecture new theories that take into account the latest information. It is in this way that science advances. A classic illustration of this occurred in 1543 when Copernicus proposed that the earth and the planets orbited the sun, contrary to the prevailing Ptolemaic view that the sun, moon and stars revolved around the earth.

So what do we mean by a proof?

Put simply, a proof is a chain of reasoning that establishes the truth of a particular statement or *proposition.*

The Greek mathematician Euclid, in his book 'The Elements', written about 300 BC, provided a framework used ever since in many mathematical proofs, especially those in the field of geometry. Such proofs begin with a clear statement of any initial *assumptions.* They then show, using the rules of logic, that if those assumptions are true then the desired conclusion must also be true. During the course of the proof, the argument may quote self-evident facts. These facts are called *mathematical axioms.* Proofs may also make use of other results that have already been established. Important proven results are called *theorems,* for example, Pythagoras' Theorem.

Worked Example 1

Take any three consecutive even numbers and add them together. What do you notice about the totals?

Solution

$4+6+8=18$	$6+8+10=24$	$8+10+12=30$
$10+12+14=36$	$12+14+16=42$	$14+16+18=48$
$16+18+20=54$	$18+20+22=60$	$20+22+24=66$

From these first few cases, we can see that every total is a multiple of 6.

We could go on checking this. For example,

$$342+344+346=1032, \quad \text{which is } 6\times 172$$

However many times we check, we will never cover every possible case because there is an infinite number of possible sets of three consecutive even numbers.

We could use the alternative argument that our first total, 18, was a multiple of 6, and every time the total goes up by 6, so every other total must be a multiple of 6. However, whilst this argument appears convincing, it is not a watertight proof because it relies on the assumption that the observed pattern continues for ever, and we have only a few cases to support that claim.

So how can we prove our theory?

Worked Example 2

The sum of any three consecutive even numbers is always a multiple of 6.

Proof

Being even, the first of the three consecutive numbers must be a multiple of 2. We can therefore write it as $2N$, where N is a whole number.
Being consecutive, the other two even numbers must be $2N+2$ and $2N+4$.

Adding the three numbers gives a sum equal to $2N + (2N+2) + (2N+4) = 6N + 6$ which is 6 times $(N+1)$ This proves that the total must *always* be a multiple of 6.

Worked Example 3

The product of any three consecutive even numbers is always a multiple of 8.

Proof

As in Worked Example 2, we can write the numbers as $2N$, $2N+2$ and $2N+4$.

Multiplying the three numbers gives a product equal to

$$\begin{aligned} 2N \times (2N+2) \times (2N+4) &= 2N \times 2(N+1) \times 2(N+2) \\ &= 2 \times 2 \times 2 \times N \times (N+1) \times (N+2) \\ &= 8N(N+1)(N+2) \end{aligned}$$

which is 8 times $N(N+1)(N+2)$. This proves that the product must *always* be a multiple of 8.

Worked Example 4

In Section 1.1, Worked Example 1, we said that statement (f),

'If you multiply two odd numbers you get an odd number'

was always true. Here we look at the proof for this.

Proof

If the two odd numbers are x and y, then $x-1$ and $y-1$ will both be even. This means that $x-1 = 2p$ and $y-1 = 2q$, where p and q are whole numbers, so $x = 2p+1$ and $y = 2q+1$.

The product $$\begin{aligned} xy &= (2p+1)(2q+1) \\ &= 4pq + 2p + 2q + 1 \end{aligned}$$

$\times$	$2p$	$+1$
$2q$	$4pq$	$+2q$
$+1$	$+2p$	$+1$

The expressions $4pq$, $2p$ and $2q$ are all even, so adding them will give an even number. Adding on the final 1 will give an odd total, which shows that xy must be an odd number.

1.2 Proof

Exercises

1. Prove that the sum of two consecutive numbers must be odd.

2. Prove that the sum of two consecutive odd numbers must be even.

3. Prove that the product of any two consecutive even numbers is always a multiple of 4.

4. (a) Prove that if you add two *consecutive odd* numbers the result will be a multiple of 4.

   (b) Prove that if you multiply two *consecutive odd* numbers, and then add 1, the result will be a multiple of 4.

5. Take any three consecutive numbers and add them together.

   (a) What do you notice about the totals?

   (b) Prove that it always happens.

6. Take any five consecutive numbers and add them together.

   (a) What do you notice about the totals?

   (b) Prove that it always happens.

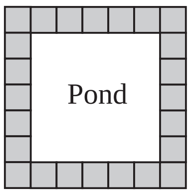
7. Take any four consecutive numbers and add them together.

   (a) What do you notice about the totals this time?

   (b) Prove that it always happens.

8. Prove that if you add the squares of three consecutive numbers and then subtract two, you always get a multiple of 3.

9. The diagram shows a square pond 5 metres long and 5 metres wide. It is surrounded by 24 square paving slabs, each 1 metre long and 1 metre wide.



   (a) Complete a copy of the following table showing the number of paving slabs needed to surround other square ponds.

Size of Pond (metres)	*Number of Paving Slabs*
1×1	8
2×2	
3×3	
4×4	
5×5	24
6×6	
7×7	
8×8	
9×9	
10×10	

(b) Write down a formula for the number of tiles, N, needed to surround a pond of size $L \times L$.

(c) Prove your formula.

10. Roger is an amateur magician. In one of his tricks he invites people in the audience to think of a number. He then asks them to carry out the following simple instructions:

	double your number
then	*add 5*
then	*multiply the number you now have by itself*
then	*subtract 25*
then	*divide by 4*
then	*divide by your original number.*

Roger then predicts that their answer is 5 more than their original number.

For example,

$$3 \rightarrow 6 \rightarrow 11 \rightarrow 121 \rightarrow 96 \rightarrow 24 \rightarrow 8$$
$$4 \rightarrow 8 \rightarrow 13 \rightarrow 169 \rightarrow 144 \rightarrow 36 \rightarrow 9$$
$$11 \rightarrow 22 \rightarrow 27 \rightarrow 729 \rightarrow 704 \rightarrow 176 \rightarrow 16$$

Roger's trick is not magic. Prove that it always happens.

1.3 Algebraic Identities

Some algebraic formulae only work for a limited number of values. For example, in Worked Example 3 of section 1.1, we saw that the equation

$$x(x+1) = x^2 + 1$$

is true for $x = 1$ but not for $x = 2$. In fact, $x = 1$ is the only value for which it is true.

In an *algebraic identity*, both sides of the equation are identical, whatever values are substituted for the variables. We use the symbol $\equiv$ where we have an identity rather than an equation.

For example, the square of $x + 1$ is always equal to $x^2 + 2x + 1$ whatever the value of x, so we write

$$(x+1)^2 \equiv x^2 + 2x + 1$$

So,

> an *equation* (using $=$) may be true for only a limited number of values, or for no values at all;
>
> an *identity* must be true for all substituted values.

Worked Example 1

Decide if the following statement might be an identity.

(A) $\quad 5x - 3(x-2) = 2(x+3)$

Solution

Trying a random selection of values for x:

If $x = 0$, LHS of (A) $= 5 \times 0 - 3(0-2) = 0 - \{3 \times (-2)\} = 0 - (-6) = 6$

RHS of (A) $= 2(0+3) = 2 \times 3 = 6$

$\therefore$ statement A is true for $x = 0$.

If $x = 1$, LHS of (A) $= 5 \times 1 - 3(1-2) = 5 - \{3 \times (-1)\} = 5 - (-3) = 8$

RHS of (A) $= 2(1+3) = 2 \times 4 = 8$

$\therefore$ statement A is true for $x = 1$.

If $x = 2$, LHS of (A) $= 5 \times 2 - 3(2-2) = 10 - \{3 \times 0\} = 10 - 0 = 10$

RHS of (A) $= 2(2+3) = 2 \times 5 = 10$

$\therefore$ statement A is true for $x = 2$.

If $x = 7$, LHS of (A) $= 5 \times 7 - 3(7 - 2) = 35 - \{3 \times 5\} = 35 - 15 = 20$

RHS of (A) $= 2(7 + 3) = 2 \times 10 = 20$

$\therefore$ statement A is true for $x = 7$.

If $x = 39$, LHS of (A) $= 5 \times 39 - 3(39 - 2) = 195 - \{3 \times 37\} = 195 - 111 = 84$

RHS of (A) $= 2(39 + 3) = 2 \times 42 = 84$

$\therefore$ statement A is true for $x = 39$.

From the examples above, we have a reasonable amount of evidence to suggest that statement (A) may well be an identity. However, we cannot be certain until we prove it.

Worked Example 2

Prove the identity

(A) $\quad 5x - 3(x - 2) \equiv 2(x + 3)$

Proof

LHS of (A) $\equiv 5x - 3(x - 2) \equiv 5x - 3x + 6$ *(multiplying out the brackets)*

$\equiv 2x + 6$ *(collecting like terms)*

$\equiv 2(x + 3)$ *(factorising)*

$\equiv$ RHS of (A)

$\therefore$ LHS of (A) $\equiv$ RHS of (A) for any value of x.

$\therefore \; 5x - 3(x - 2) \equiv 2(x + 3)$

Worked Example 3

Decide if the following statement might be an identity.

(B) $\quad 4(x - y) + 5(x + y) = 7(x + y) + 2(x - 3y)$

Solution

Trying a random selection of values for x and y:

If $x = 9$ and $y = 2$, LHS of (B) $= 4(9 - 2) + 5(9 + 2) = (4 \times 7) + (5 \times 11)$

$= 28 + 55 = 83$

RHS of (B) $= 7(9 + 2) + 2(9 - [3 \times 2]) = (7 \times 11) + (2 \times 3)$

$= 77 + 6 = 83$

$\therefore$ statement B is true for $x = 9$ and $y = 2$.

If $x = 8$ and $y = 4$, LHS of (B) $= 4(8 - 4) + 5(8 + 4) = (4 \times 4) + (5 \times 12)$

$= 16 + 60 = 76$

RHS of (B) $= 7(8 + 4) + 2(8 - [3 \times 4]) = (7 \times 12) + (2 \times [-4])$

$= 84 - 8 = 76$

$\therefore$ statement B is true for $x = 8$ and $y = 4$.

If $x = 6$ and $y = -2$,

LHS of (B) $= 4(6 - [-2]) + 5(6 + [-2]) = (4 \times 8) + (5 \times 4)$

$= 32 + 20 = 52$

RHS of (B) $= 7(6 + [-2]) + 2\{6 - (3 \times [-2])\} = (7 \times 4) + (2 \times 12)$

$= 28 + 24 = 52$

$\therefore$ statement B is true for $x = 6$ and $y = -2$.

From the examples above, we have evidence to suggest that statement (B) may also be an identity. Once again, we cannot be certain until we have proved it.

Worked Example 4

Prove the identity

(B) $\quad 4(x - y) + 5(x + y) \equiv 7(x + y) + 2(x - 3y)$

Proof

LHS of (B) $\equiv 4(x - y) + 5(x + y)$

$\equiv 4x - 4y + 5x + 5y$ *(multiplying out the brackets)*

$\equiv 9x + y$ *(collecting like terms)*

RHS of (B) $\equiv 7(x + y) + 2(x - 3y)$

$\equiv 7x + 7y + 2x - 6y$ *(multiplying out the brackets)*

$\equiv 9x + y$ *(collecting like terms)*

Since both sides of (B) simplify to $9x + y$, it follows that

LHS of (B) $\equiv$ RHS of (B) for any values of x and y.

$\therefore \quad 4(x - y) + 5(x + y) \equiv 7(x + y) + 2(x - 3y)$

Worked Example 5

Prove the identity

(C) $\quad (x-8)(x-5)-(x+2)(x+7) \equiv 26-22x$

Proof

We begin by multiplying out the first pair of brackets.

$$(x-8)(x-5) \equiv x^2 - 13x + 40$$

$\times$	x	-8
x	x^2	$-8x$
-5	$-5x$	$+40$

Multiplying out the second pair of brackets gives

$$(x+2)(x+7) \equiv x^2 + 9x + 14$$

$\times$	x	$+2$
x	x^2	$+2x$
$+7$	$+7x$	$+14$

We can now establish the identity

LHS of (C) $\equiv (x-8)(x-5)-(x+2)(x+7)$

$\equiv \left[x^2 - 13x + 40\right] - \left[x^2 + 9x + 14\right]$

$\equiv x^2 - 13x + 40 - x^2 - 9x - 14$

$\equiv 26 - 22x$

$\equiv$ RHS of (C) for any value of x.

$$\therefore\ (x-8)(x-5)-(x+2)(x+7) \equiv 26-22x$$

1.3 Algebraic Identities

Exercises

1. (a) Show that the values $x = 1$, $x = 3$ and $x = 5$ all satisfy the equation

$$7(x - 8) - 3(x - 20) = 4(x + 1)$$

   (b) Prove the identity

$$7(x - 8) - 3(x - 20) \equiv 4(x + 1)$$

2. (a) Show that the values $x = 1$, $x = 3$ and $x = 5$ all satisfy the equation

$$x^3 - 9x^2 + 23x = 15$$

   (b) Show, by substituting the value $x = 4$, that $x^3 - 9x^2 + 23x = 15$ is not an identity.

3. (a) Show that the values $p = 10$ and $q = 5$ satisfy the equation

$$8(p - q) + 3(p + q) = 2(p + 2q) + 9(p - q)$$

   (b) Show that the values of $p = 6$ and $q = 4$ also satisfy the equation in (a).

   (c) Prove the identity

$$8(p - q) + 3(p + q) \equiv 2(p + 2q) + 9(p - q)$$

4. Prove the identity

$$x(m + n) + y(n - m) \equiv m(x - y) + n(x + y)$$

5. (a) Use the multiplication grid to multiply out and simplify $(x + 2)(x + 10)$.

$\times$	x	$+2$
x		
$+10$		

   (b) Use the multiplication grid to multiply out and simplify $(x - 5)(x - 4)$.

$\times$	x	-5
x		
-4		

   (c) Use your answers to parts (a) and (b) to prove the identity

$$(x + 2)(x + 10) - (x - 5)(x - 4) \equiv 21x$$

6. (a) Use the multiplication grid to multiply out and simplify $(x+6)(x+8)$.

$\times$	x	$+6$
x		
$+8$		

(b) Explain how your answer to part (a) proves the identity

$$\frac{x^2+14x+48}{x+6} \equiv x+8 \quad \text{provided } x \neq -6.$$

7. (a) Prove the identity

$$a^2-b^2 \equiv (a+b)(a-b)$$

(b) Use the identity established in part (a) to calculate the following *without the aid of a calculator.*

(i) 81^2-80^2 (ii) 101^2-99^2

(iii) 2731^2-269^2 (iv) $11.7^2-8.3^2$

(v) 999991^2-9^2 (vi) $75.41^2-24.59^2$

8. (a) Prove the identity $m^2-1 \equiv (m+1)(m-1)$.

(b) Prove the identity $m^4-1 \equiv (m^2+1)(m^2-1)$.

(c) Prove the identity $m^4-1 \equiv (m^2+1)(m+1)(m-1)$.

9. Prove the identities

(a) $(x+y)^2+(x-y)^2 \equiv 2(x^2+y^2)$

(b) $(x+y)^2-(x-y)^2 \equiv 4xy$

10. (a) Use the multiplication grid to prove the identity
$a^3-b^3 \equiv (a-b)(a^2+ab+b^2)$

$\times$	a^2	$+ab$	$+b^2$
a			
$-b$			

(b) Prove the identity
$a^3+b^3 \equiv (a+b)(a^2-ab+b^2)$

1.4 Geometrical Proof

In this section we look at the Euclidean approach to the mathematical proof of geometrical facts. We will start from basic facts, called *mathematical axioms*, or from other previously proven facts. Using these we will establish a chain of reasoning that demonstrates the truth of a particular statement or *proposition*. This is the formal, logical method established by the Greek mathematician Euclid, in his book 'The Elements', written about 300 BC. This approach was used to develop the branch of mathematics called Euclidean Geometry, with each newly verified fact being firmly based on a proven body of knowledge. It would take an entire book to cover all that has since been discovered in Euclidean Geometry. We will only look at a few simple illustrations.

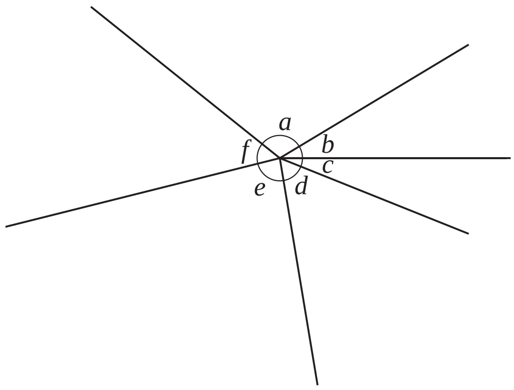
The first basic assumption we will make is that a full turn is $360°$. From this we can establish

Theorem 1 Angles at a point add up to $360°$

Proof

In the diagram opposite, the angles make up a full turn, and a full turn is $360°$, so

$$a + b + c + d + e + f = 360°$$



This argument would hold for any number of angles at a point; we have illustrated it for six angles.

Note

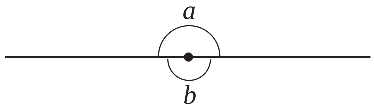
A *corollary* is a fact that results from a significant theorem.

We can use Theorem 1, and the fact that the angles either side of a straight line are equal, to deduce

Corollary 2 The angle on a straight line is $180°$

Proof

In the diagram opposite, a and b are angles at a point so that $a + b = 360°$ by Theorem 1.

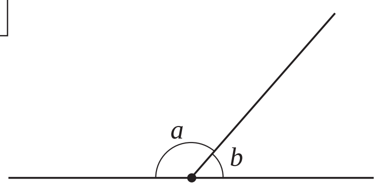


But $a = b$, so $a + a = 360°$, i.e. $2a = 360°$, which gives $a = 180°$.

Corollary 3 Angles on a straight line add up to $180°$

Proof

In the diagram opposite, a and b make up the angle on a straight line, so $a + b = 180°$ by Corollary 2.

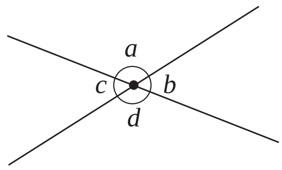


Note

Our first three results are, of course, well known, so it may seem rather unnecessary to have proved them formally. The important point is to see the underlying mathematical development, each fact being derived logically, in sequence, from our single assumption that a full turn is $360°$.

Theorem 4 Vertically opposite angles are equal

Proof



In the diagram opposite, angles a and b make up a straight line, so $a + b = 180°$ by Corollary 3.

Angles a and c also make up a straight line, so $a + c = 180°$, again by Corollary 3,

$$\therefore a + b = a + c$$

from which it is clear that $b = c$, i.e. that the vertically opposite angles b and c are equal.

In the same way, $a + b = 180°$ *(angles on a straight line)*

$d + b = 180°$ *(angles on a straight line)*

$\therefore \quad a + b = d + b$

$\therefore \quad a = d$

Parallel Lines

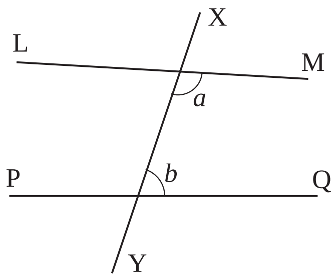
Parallel lines are always the same distance apart. Parallel lines never meet.

Euclid's 5th Axiom

Euclid laid down 5 basic axioms as a foundation for geometry. The fifth of those axioms can be interpreted in a number of different ways but it is normally stated as follows:

If (see diagram opposite) a straight line XY meets two other straight lines, LM and PQ, so that $a + b \neq 180°$, then LM and PQ will meet, i.e. LM and PQ are not parallel.

In the diagram, the fact that $a + b < 180°$ means that LM will cross PQ somewhere to the right of M and Q.



Note

If LM and PQ are parallel in the diagram above, then one immediate consequence of Euclid's 5th axiom is that angles a and b must add up to $180°$, i.e.

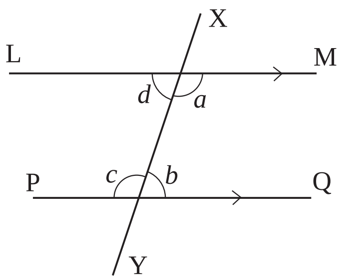
$$\text{LM parallel to PQ} \Rightarrow a + b = 180°$$

Comment

Euclid's 5th axiom may seem rather obvious but it cannot be proved mathematically. Indeed, in the 19th century, mathematicians began to ask what happens when you do not assume the 5th axiom. This led to a whole new branch of mathematics called non-Euclidean Geometry, which was later applied by Einstein, in the 20th century, in his theory of relativity.

Theorem 5 If LM and PQ are parallel lines crossed by a third line XY, then alternate angles are equal

i.e. $a = c$ and $b = d$



Proof

In the diagram above, the fact that LM and PQ are parallel means that

$$a + b = 180°$$

by Euclid's 5th axiom.

However, c and b are angles on a straight line, so

$$c + b = 180°$$

by Corollary 3.

$$\therefore \quad a + b = c + b$$

which shows that $a = c$.

We also have $\quad a + b = 180° \qquad$ *(Euclid's 5th axiom)*

$\qquad a + d = 180° \qquad$ *(angles on a straight line)*

$\therefore \quad a + b = a + d$

so $\quad b = d$

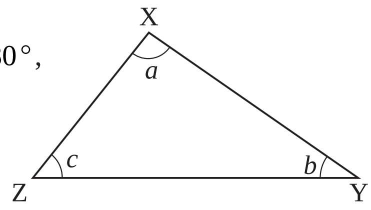
Note

Once Theorem 5 has been proved, it is a simple corollary that corresponding angles are also equal. Establishing this fact is one of the exercises at the end of this section.

Comment

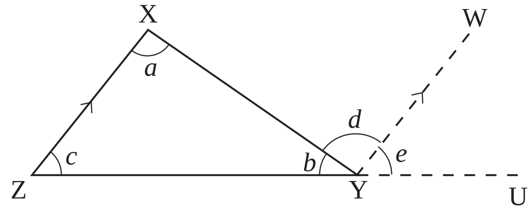
We will now use Theorem 5, and its consequences, to prove that the angles of every triangle add up to $180°$. It is important to appreciate that this basic rule would break down in non-Euclidean Geometry where the 5th axiom does not hold.

Theorem 6 The angles of every triangle add up to $180°$,
i.e. $a + b + c = 180°$



Proof

In the diagram above, extend the line ZY to U and draw the line YW parallel to ZX. This is shown in the diagram below.



Using the notation in the diagram, we have

	$a = d$	*(alternate angles,* ZX *parallel to* YW*)*
	$c = e$	*(corresponding angles,* ZX *parallel to* YW*)*
But	$d + b + e = 180°$	*(angles on a straight line)*
$\therefore$	$a + b + c = 180°$	

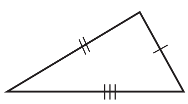
Congruent Triangles

One of the key elements of Euclidean Geometry is the application of *congruency*. This relies on the uniqueness of triangles constructed from three specified pieces of information.

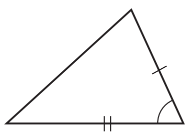
Two shapes are *congruent* if they have exactly the same size and shape. The only difference may be in their position or orientation. In particular, two triangles are congruent if one can be superimposed over the other, so that the three sides and the three angles match identically, one with another.

Geometrical constructions, with a ruler, protractor and a pair of compasses, show that only one triangle can be drawn if we are given measurements for

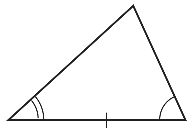
(SSS) the three sides of a triangle



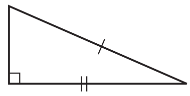
(SAS) two sides and the angle enclosed by those two sides



(AAS) two angles and any side



(RHS) a right angle, the hypotenuse and one other side



Four possible tests for congruency result from the uniqueness of triangles constructed from any one of these sets of information. The four tests for congruent triangles are :

SSS SAS AAS RHS

Any two triangles that share the same three pieces of information must be identical in every respect, i.e. their angles must match and their sides must also match.

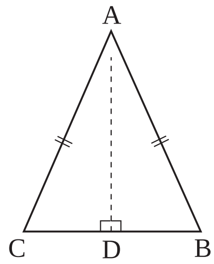
We now use this fact to prove one of the key facts about isosceles triangles.

> **Theorem 7** The angles that face the equal sides in an isosceles triangle are equal, i.e. in Δ ABC, if AB = AC, then $\angle$ABC = $\angle$ACB

Proof

Draw the line AD perpendicular to BC and meeting it at D.

Then, in triangles ABD and ACD



$$AD = AD \quad \textit{(common side)}$$

$$\angle ADB = \angle ADC \quad (90°\textit{, by construction)}$$

$$AB = AC \quad \textit{(equal sides of isosceles triangle)}$$

$\therefore$ Δ ABD is congruent to Δ ACD (RHS).

From this we can conclude that corresponding sides and angles are equal, so that

$$\angle ABD = \angle ACD, \text{ i.e. } \angle ABC = \angle ACB$$

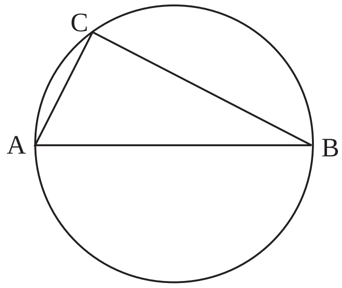
Note

The proof of Theorem 7 also proves that BD = CD, i.e. that D is the midpoint of BC, i.e. that the perpendicular from the point of intersection of the equal sides of an isosceles triangle bisects the third side.

Also the proof of Theorem 7 proves that AD bisects $\angle$CAB , i.e. that the perpendicular from the point of intersection of the equal sdes of an isosceles triangle bisects the angle between the equal sides.

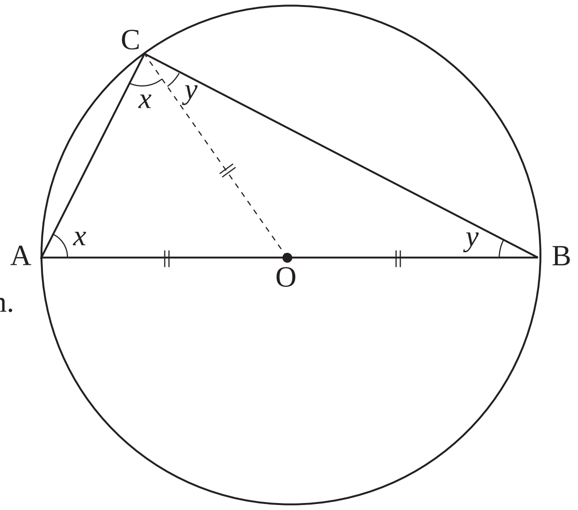
We conclude this section with an algebraic proof of an important result for circles that builds on the facts established in Theorems 6 and 7.

Theorem 8 The angle in a semicircle is a right angle, i.e. if AB is a diameter of a circle and C is any other point on the circumference of the circle, then $\angle ACB = 90°$



Proof

If O is the centre of the circle, join O to C.



In ΔOAC,

$OA = OC$ *(radii)*

$\therefore$ ΔOAC is isosceles

$\therefore$ $\angle OAC = \angle OCA$

and we mark these as angle *x* in the diagram.

In ΔOBC,

$OB = OC$ *(radii)*

$\therefore$ ΔOBC is isosceles

$\therefore$ $\angle OBC = \angle OCB$

and we mark these as angle *y* in the diagram.

Adding the angles of ΔABC now gives

$$\angle BAC + \angle ABC + \angle ACB = 180°$$

i.e. $$x + y + (x + y) = 180°$$

i.e. $$2x + 2y = 180°$$

Dividing by 2 now gives $$x + y = 90°$$

i.e. $$\angle ACB = 90°$$

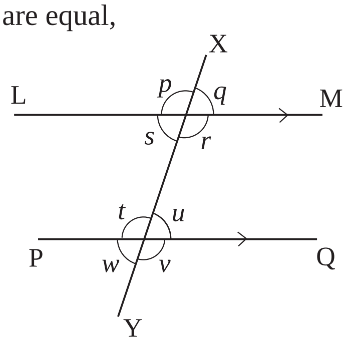
1.4 Geometrical Proof

Exercises

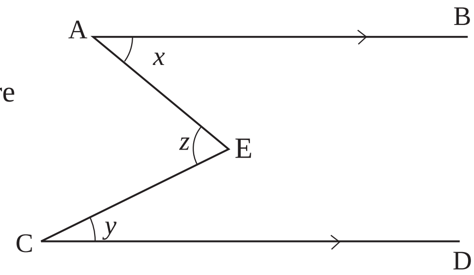
1. Using the facts established in Corollary 3 and Theorem 5, about angles on a straight line and alternate angles, prove that, where a third line XY intersects two parallel lines, LM and PQ, the corresponding angles are equal, i.e.

$p = t \qquad q = u$

$r = v \qquad s = w$

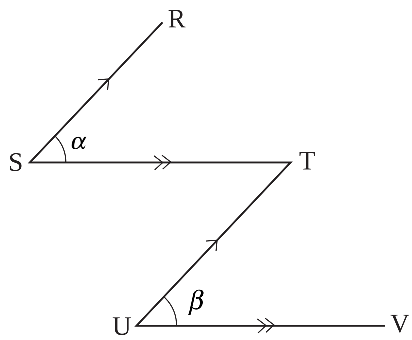


2. Using the fact established in Theorem 6, about the sum of the angles of a triangle, prove that the angles of every quadrilateral add up to $360°$.



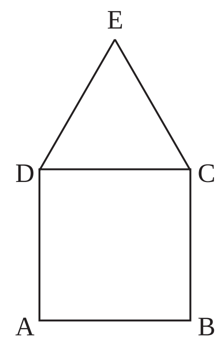
3. In the diagram opposite, lines AB and CD are parallel.

   Prove that $x + y = z$.



4. In the diagram opposite, RS is parallel to TU and ST is parallel to UV.

   Prove that $\alpha = \beta$.



5. In the diagram opposite, ABCD is a square and ΔEDC is equilateral.

   (a) Prove that Δ EDA and Δ ECB are congruent.

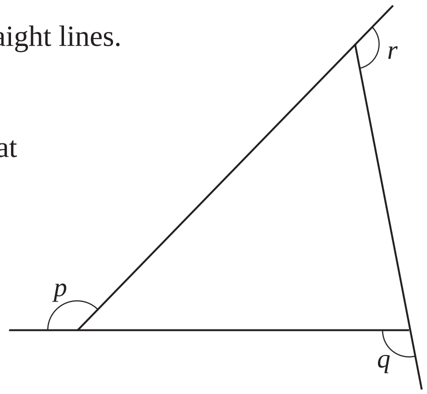
   (b) Deduce that AE $=$ BE.

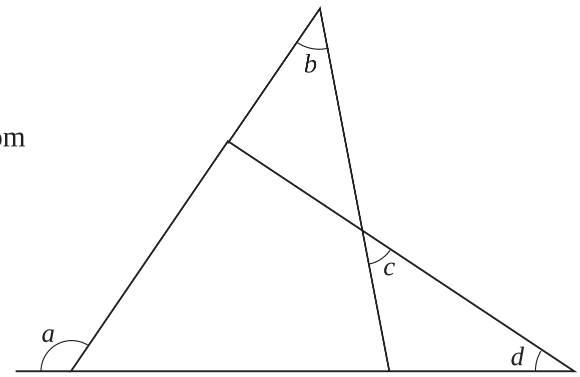
   (c) Calculate the angles of Δ EAB.

6. The diagram opposite is formed from 3 straight lines.

    (a) Prove that $p + q + r = 360°$.

    (b) If, in addition, $p + q = 3r$, prove that the triangle is right-angled.



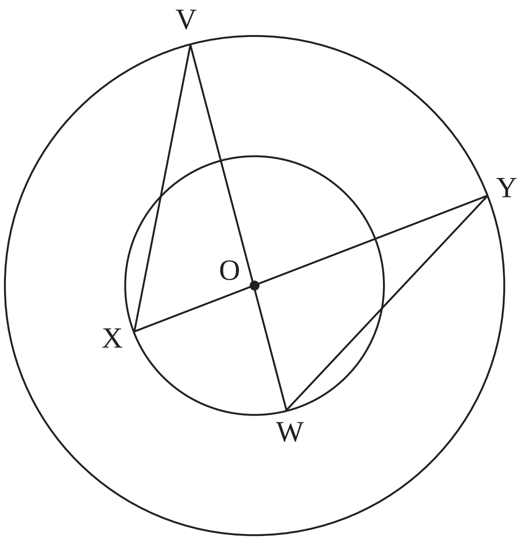


7. The diagram opposite is formed from 4 straight lines.

    Prove that $a = b + c + d$.

8. O is the centre of each of the circles in the diagram below.

    VOW and XOY are both straight lines.



(a) Prove that Δ VOX and Δ WOY are congruent.

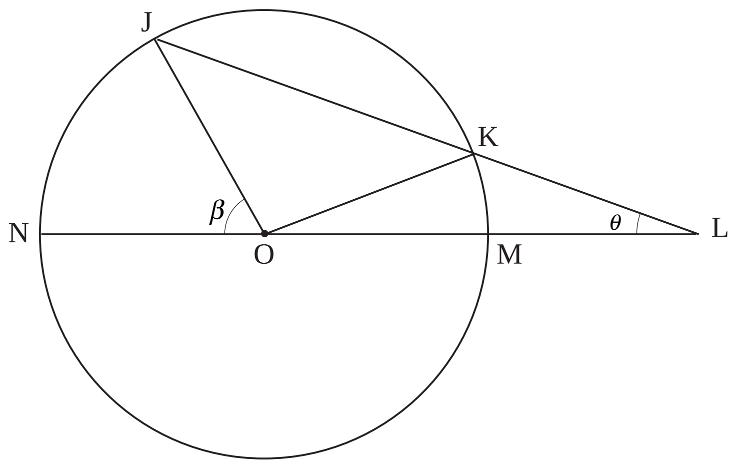
(b) Deduce that

(i) VX = WY,

(ii) $\angle$OVX = $\angle$OYW

(iii) $\angle$OXV = $\angle$OWY

9.



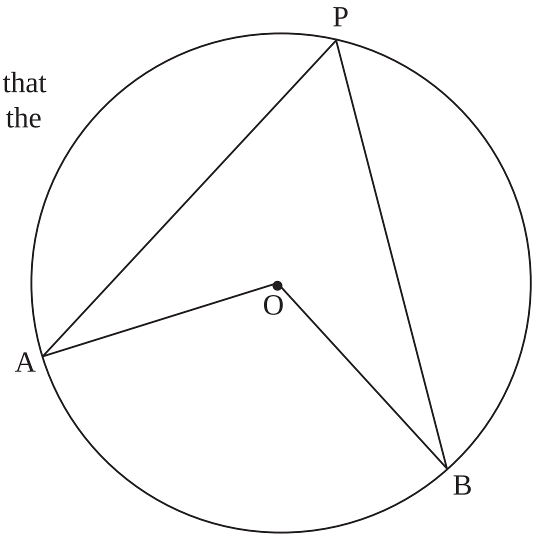
In the diagram above, O is the centre of the circle. LMON and JKL are straight lines. $\angle\text{JON} = \beta$ and $\angle\text{KLM} = \theta$.

The length KL is equal to the radius of the circle.

Prove that $\beta = 3\theta$.

10. Follow the instructions below to prove that the angle which the arc AB subtends at the centre, O, of the circle is double the angle which the arc subtends at the circumference of the circle, i.e.

$$\angle\text{AOB} = 2 \times \angle\text{APB}$$

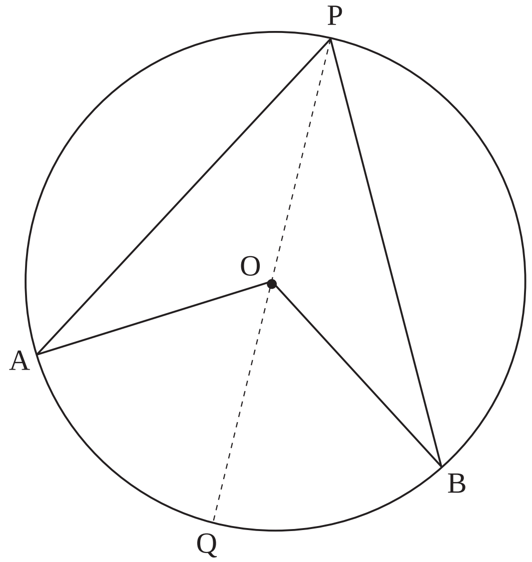


(a) Copy the diagram, join P to the centre of the circle, O, and extend PO until it meets the circle again at Q.

(b) Explain why $\angle\text{OAP} = \angle\text{OPA}$ and label them as angle x on your copy of the diagram.

(c) Explain why $\angle\text{OBP} = \angle\text{OPB}$ and label them as angle y on your copy of the diagram.

(d) Express $\angle\text{APB}$ in terms of x and y.



(e) Use the sum of the angles in a triangle to explain why $\angle AOP = 180° - 2x$ and $\angle BOP = 180° - 2y$.

(f) Use the fact that $\angle AOP$, $\angle BOP$ and $\angle AOB$ are angles at a point to show that $\angle AOB = 2x + 2y$.

(g) Combine parts (d) and (f) to prove that $\angle AOB = 2 \times \angle APB$.

11. O is the centre of the circle in the diagram opposite.

M is the midpoint of the chord GH.

Prove that OM is perpendicular to GH.

(*Hint*: Join O to G, H and M.)