

UNIT 18 *Arithmetic Coding*

Teacher Resource Material

Key Stage: 4 / A-level

Target: Gifted and talented students (of any age!)

This is another example of a method of compression. It is very mathematical, requiring the students to work with complex fractions and binary representations not only for whole numbers but also for decimals. In short, it is for the most able who will understand the innovative techniques that are employed and why they are used.

The Lesson Plans have been written assuming that the students are familiar with binary representations (at least one additional lesson will be needed if they are not).

The notation $[a, b)$, meaning any number x such that $a \leq x < b$, is used throughout the unit.

Solutions and Notes

Exercise 1 a) i) $56 = 32 + 16 + 8 = 1 \times 32 + 1 \times 16 + 1 \times 8 + 0 \times 4 + 0 \times 2 + 0 \times 1$
 $\Rightarrow 111000$ in binary

ii) $147 = 128 + 16 + 2 + 1 = 1 \times 128 + 0 \times 64 + 0 \times 32$
 $+ 1 \times 16 + 0 \times 8 + 0 \times 4 + 1 \times 2 + 1 \times 1$
 $\Rightarrow 10010011$ in binary

iii) $623 = 512 + 64 + 32 + 8 + 4 + 2 + 1 = 1 \times 512 + 0 \times 256 + 0 \times 128$
 $+ 1 \times 64 + 1 \times 32 + 0 \times 16 + 1 \times 8 + 1 \times 4 + 1 \times 2 + 1 \times 1$
 $= 1001101111$ in binary

b) i) $11011 = 1 \times 16 + 1 \times 8 + 0 \times 4 + 1 \times 2 + 1 \times 1$
 $= 16 + 8 + 2 + 1$
 $= 27$ in base 10

ii) $1000011 = 1 \times 64 + 1 \times 2 + 1 \times 1$
 $= 67$ in base 10

iii) $101010101 = 1 \times 256 + 1 \times 64 + 1 \times 16 + 1 \times 4 + 1 \times 1$
 $= 341$ in base 10

Exercise 2 a) i) $\frac{1}{16} = 0.0001$ in binary ii) $\frac{5}{16} = \frac{1}{4} + \frac{1}{16} \Rightarrow 0.0101$ in binary

iii) $\frac{9}{64} = \frac{1}{8} + \frac{1}{64} \Rightarrow 0.001001$ in binary

b) i) $\frac{1}{2} + \frac{1}{4} + \frac{1}{16} = \frac{13}{16}$ ii) $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} = \frac{21}{32}$

iii) $\frac{1}{2} + \frac{1}{32} + \frac{1}{64} = \frac{35}{64}$

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Exercise 3 $\frac{7654320}{16777216} = \frac{2^4 \times 478395}{2^{24}} = \frac{478395}{2^{20}}$

But as the powers of 2 are

1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768,
65536, 131072, 262144, ...

you can deduce that

$$478395 = 262144 + 131072 + 65536 + 16384 + 2048 + 1024 + 128 \\ + 32 + 16 + 8 + 2 + 1$$

$$\Rightarrow 1110100110010111011 \text{ in binary}$$

As we are now dividing this by 2^{20} , we have

$$0.01110100110010111011$$

as the binary representation of $\frac{7654320}{16777216}$.

- Exercise 4* (a) 796 (b) 803 (c) 63 808 (d) 510 208 (e) 4 081 664
(f) 4 081 920 (g) 4 082 432 (h) 4 082 688 (i) 4 083 712

(j) $\left[\frac{4082688}{16777216}, \frac{4083712}{16777216} \right)$

k) First endpoint: $4082688 = 2^{10} \times 3987$

$$16777216 = 2^{24}$$

$$\frac{4082688}{16777216} = \frac{3987}{2^{14}}$$

$$3987 = 2^{11} + 2^{10} + 2^9 + 2^8 + 2^7 + 2^4 + 2^1 + 2^0 = 111110010011$$

$$\frac{4082688}{16777216} = .00111110010011$$

Second endpoint: $4083712 = 2^{12} \times 997$

$$16777216 = 2^{24}$$

$$\frac{4083712}{16777216} = \frac{997}{2^{12}}$$

$$997 = 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^2 + 2^0 = 1111100101$$

$$\frac{4083712}{16777216} = .001111100101$$

So $\left[\frac{4082688}{16777216}, \frac{4083712}{16777216} \right) = [.00111110010011, .001111100101)$

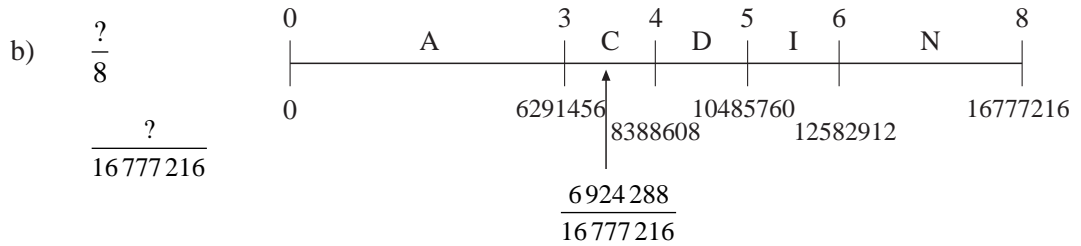
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Teacher Resource Material (continued)

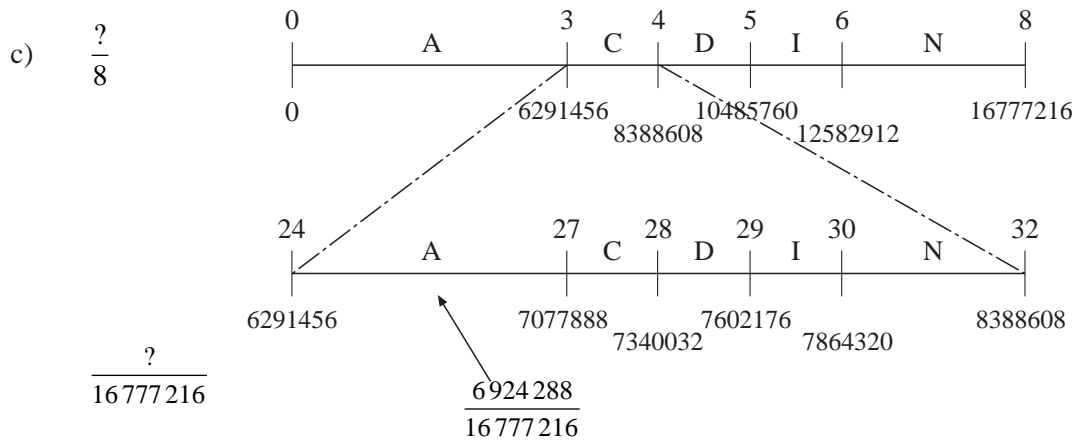
- 1) First endpoint = .00111110010011
 2nd endpoint = .001111100101
 As the interval is open at the far end, we cannot use .001111100101 as a binary representation of LOSSLESS (it actually represents LOSSLEE). So we use .00111110010011 instead – 14 digits. (All the other numbers in the interval need more binary digits.)

Exercise 5 a) 0110100110101 = 3381

$$\text{So } 0.010110100110101 = \frac{3381}{8192} = \frac{6924288}{16777216}$$



Hence the first letter of the message is C.



Hence the second letter of the message is A.

- c) The message is CANADIAN.