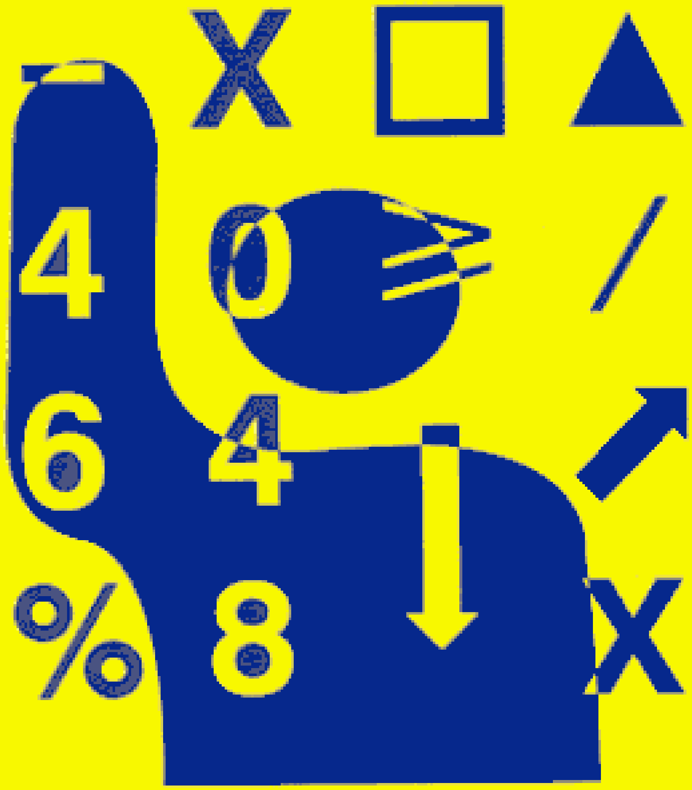
Mathematics Enhancement Programme

Primary Demonstration Project

7B Mensuration

Help Booklet



Support for Primary Teachers in Mathematics

Primary Project funded by Pricewaterhouse Coopers

in association with
British Steel
Garfield Weston Foundation

Sponsored by
ESSO

CIMT
School of Education
University of Exeter

Mathematics Enhancement Programme

Help Module 7

MENSURATION

Part B

Contents of Part B

Contents of Part A

CIMT
University of Exeter

Sponsored by
ESSO

Gatsby Technical
Education Project

PREFACE

This is one of a series of *Help Modules* designed to help you gain confidence in mathematics. It has been developed particularly for primary teachers (or student teachers) but it might also be helpful for non-specialists who teach mathematics in the lower secondary years. It is based on material which is already being used in the *Mathematics Enhancement Programme: Secondary Demonstration Project*.

The complete module list comprises:

1.	ALGEBRA	6.	HANDLING DATA
2.	DECIMALS	7.	MENSURATION
3.	EQUATIONS	8.	NUMBERS IN CONTEXT
4.	FRACTIONS	9.	PERCENTAGES
5.	GEOMETRY	10.	PROBABILITY

Notes for overall guidance:

- Each of the 10 modules listed above is divided into 2 parts. This is simply to help in the downloading and handling of the material.
- Though referred to as 'modules' it may not be necessary to study (or print out) each one in its entirely. As with any self-study material you must be aware of your own needs and assess each section to see whether it is relevant to those needs.
- The difficulty of the material in **Part A** varies quite widely: if you have problems with a particular section do try the one following, and then the next, as the content is not necessarily arranged in order of difficulty. Learning is not a simple linear process, and later studies can often illuminate and make clear something which seemed impenetrable at an earlier attempt.
- In **Part B, Activities** are offered as backup, reinforcement and extension to the work covered in Part A. **Tests** are also provided, and you are strongly urged to take these (at the end of your studies) as a check on your understanding of the topic.
- The marking scheme for the revision test includes B, M and A marks. Note that:

  M *marks* are for method;

  A *marks* are for accuracy (awarded only following a correct M mark);

  B *marks* are independent, stand-alone marks.

We hope that you find this module helpful. Comments should be sent to:

Professor D. N. Burghes
CIMT, School of Education
University of Exeter
EXETER EX1 2LU

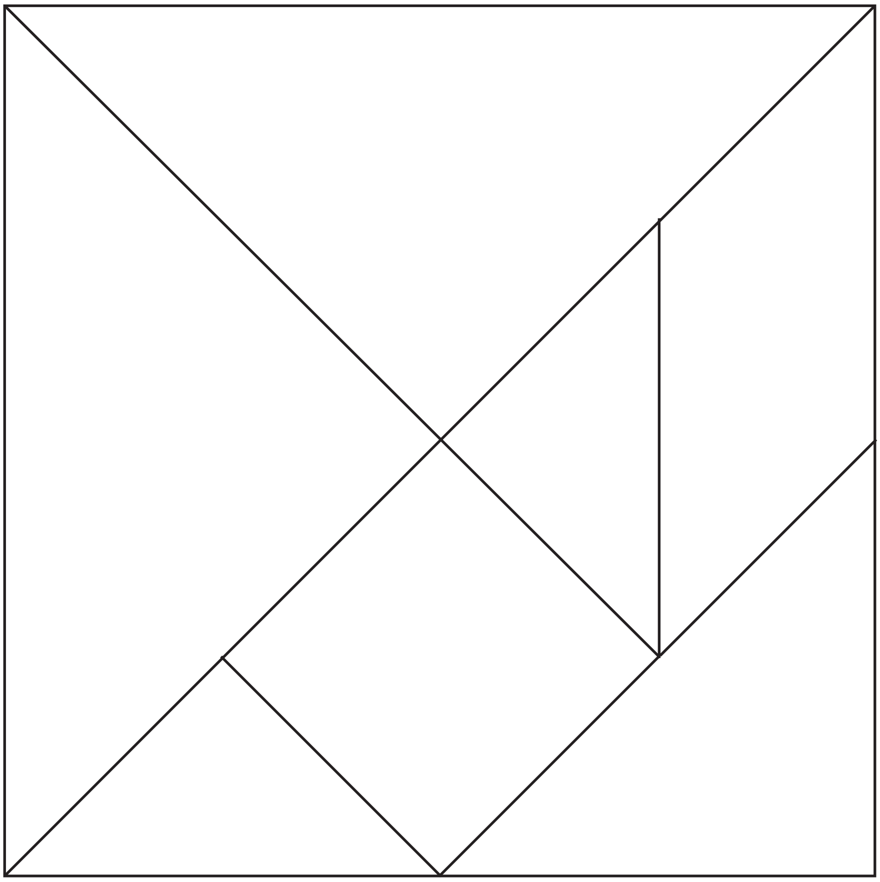
The full range of Help Modules can be found at

www.ex.ac.uk/cimt/help/menu.htm

ACTIVITIES

ACTIVITY 7.1 *Tangram*

Cut out the square below into 7 shapes.



This is a very old Chinese puzzle known as a *tangram*. Cut out the 7 shapes and rearrange them to form:

(a) a square from two triangles, and then change it to a parallelogram;

(b) a rectangle using three pieces, and then change it into a parallelogram;

(c) a trapezium with three pieces;

(d) a parallelogram with four pieces;

(e) a trapezium from the square, parallelogram and the two small triangles;

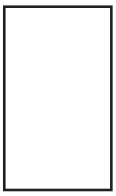
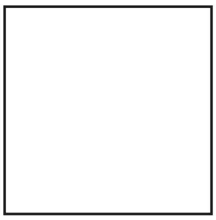
(f) a triangle with three pieces;

(g) a rectangle with all seven pieces.

Finally, put the pieces back together to form the original square.

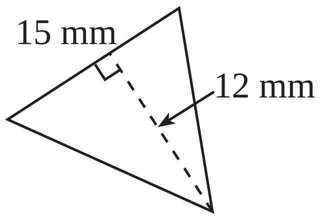
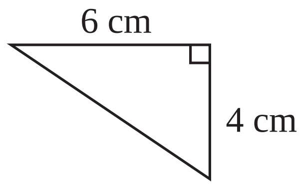
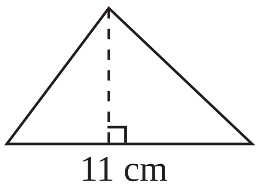
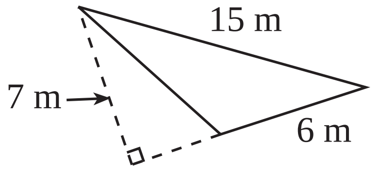
ACTIVITY 7.2 *Areas of Rectangles*

Complete the table. Diagrams are not drawn to scale.

Rectangle	*Length*	*Width*	*Area*
	6 cm		
			15 cm^2
		1 cm	
			36 cm^2

ACTIVITY 7.3 *Areas of Triangles*

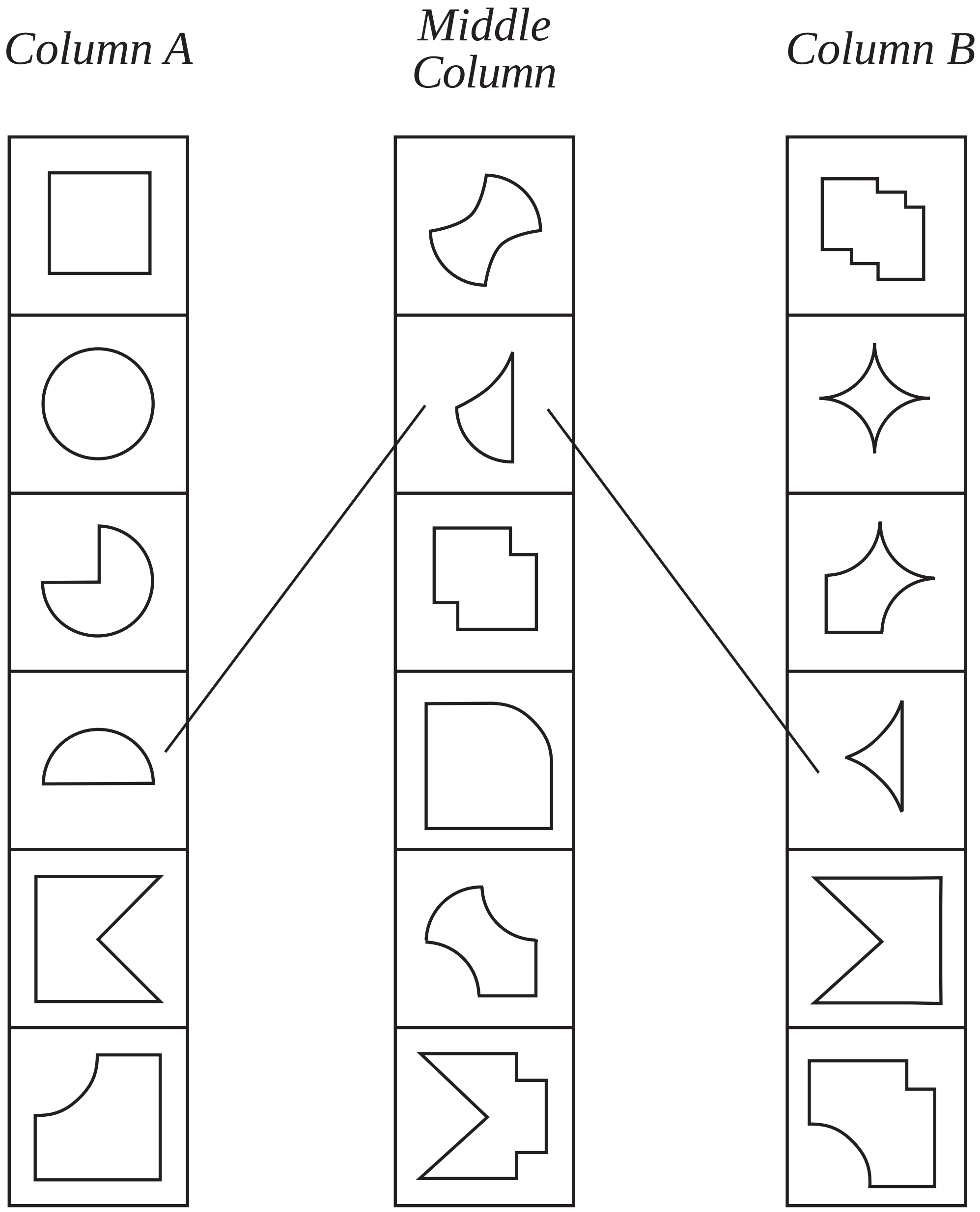
Complete the table. Diagrams are not drawn to scale.

Triangle	*Base*	*Height*	*Area*
	15 mm		
			
			22 cm^2
	6 m		

ACTIVITY 7.4 *Equal Perimeters*

From each figure in the middle column, draw a line to a figure in column A, and one in column B, that have the same perimeter length.

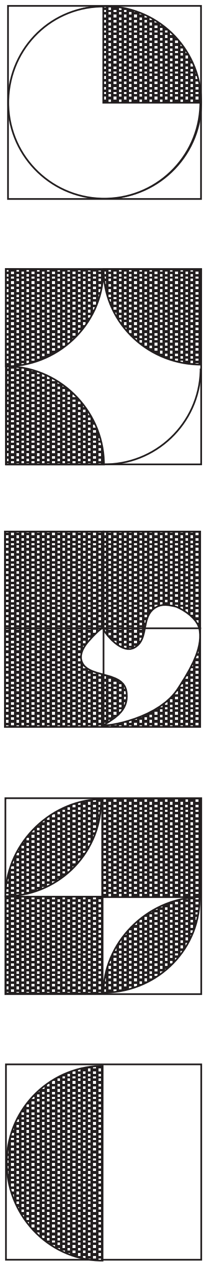
An example is shown for you.



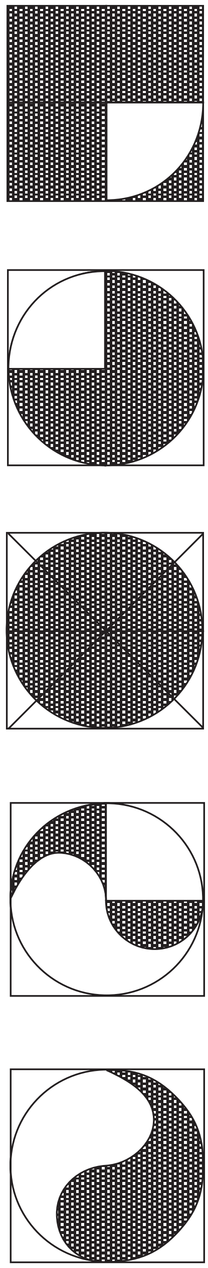
ACTIVITY 7.5 *Equal Areas*

Draw a line connecting a figure in column A to another in column B which has the same area shaded.

Column A

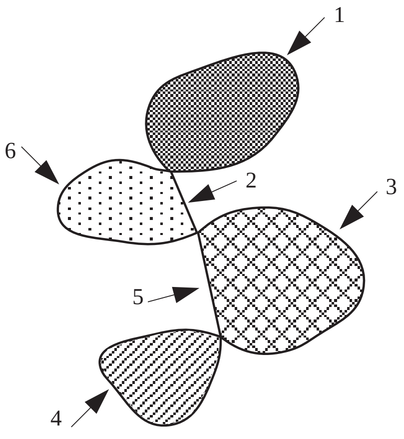


Column B



ACTIVITY 7.6 *Closed Doodles*

The sketch opposite shows an example of a *closed doodle*, that is, it starts and finishes at the same place.



This particular doodle has

3 crossover points

4 internal regions

6 arcs (or branches).

The arcs are labelled 1 to 6 on the sketch.

There is a surprisingly simple relationship between crossovers, regions and arcs.

In drawing a closed doodle, note that the pen or pencil must never leave the paper, and must not go over any part of the doodle more than once.

1. Draw 6 closed doodles of your own.

   Copy and complete the table.

Crossovers	*Regions*	*Arcs*
3	4	6
. . .	. . .	. . .
. . .	. . .	. . .

2. From your regions, and writing

   c = no. of crossovers, r = no. of inside regions, a = no. of arcs

   deduce a simple formula which connects these 3 numbers.

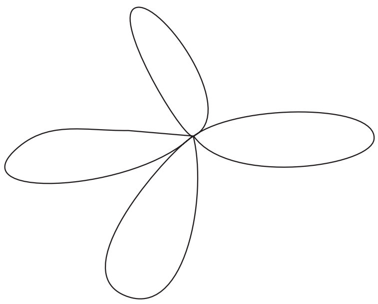
3. Draw 2 more complicated closed doodles and show that your formula still holds.

4. Does the *circle* obey your rule?

5. Does your rule hold for 'cloverleaf' type doodles, as shown opposite?

   Design a more complicated doodle of this type.

   Does it still obey the rules?



6. Can you construct closed doodles with

   (a) $c = 6$, $r = 7$, $a = 12$

   (b) $c = 8$, $r = 3$, $a = 9$

   (c) $c = 5$, $r = 9$, $a = 13$?

   If so, construct an example to illustrate it, but if not, explain why not.

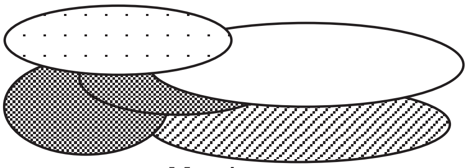
ACTIVITY 7.7 *Map Colouring*

After the introduction of colour printing presses there was much interest in *minimising* the number of colours used to distinguish different countries, in order to reduce costs.

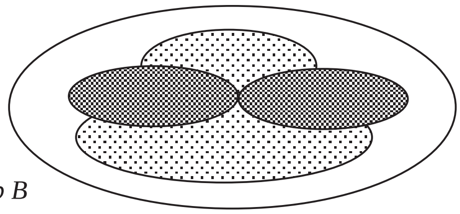
The printers had to be careful that no two countries with shared borders were coloured the same!

Map A below would not have been allowed. Two adjacent countries have the same colour.

Map B is allowed. Countries with the same colour can meet at a point.



Map A

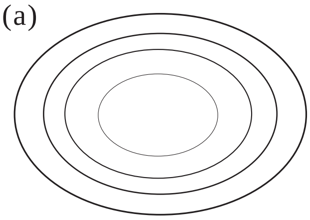


Map B

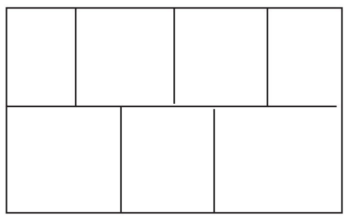
1. How many colours would be needed for *Map A*?

2. Colour the following maps, using a *minimum* number of colours.

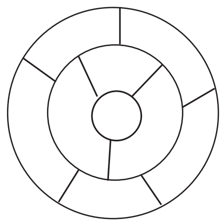
(a)



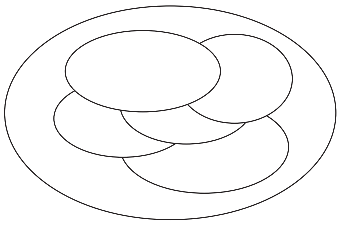
(b)



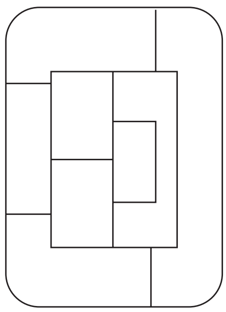
(c)



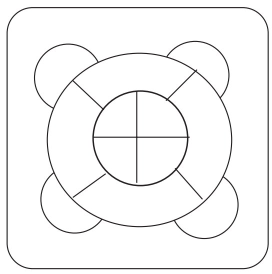
(d)



(e)



(f)



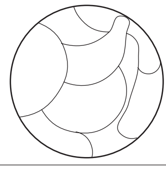
3. Draw some maps to your own design, making them as complicated as you like. Give them to a friend to find out the minimum numbers of colours needed.

Extensions

1. Find a *Map of Europe* which includes the new states which used to make up the USSR. What is the least number of colours needed:

(a) if you do *not* colour the sea (b) if one colour is used for the sea?

2. Map colouring on a sphere is more complicated. Draw patterns on a plain ball and investigate the minimum number of colours needed to colour *any* map.



ACTIVITY 7.8 *Euler's Formula*

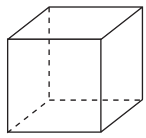
This particular result, named after its founder, the famous Swiss mathematician (1707-1783) is an example of a *topological invariant*; that is, something that remains constant for particular shapes.

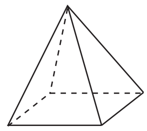
1. For each of the shapes opposite, find
   (a) the number of edges, e
   (b) the number of vertices, v
   (c) the number of faces, f.

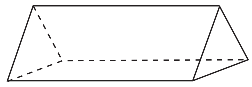
2. Show that

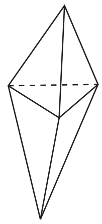
$$e + 2 = v + f$$

for each of these shapes.

(i)


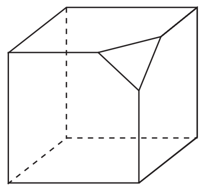
(ii)


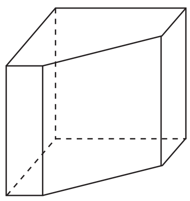
(iii)


(iv)


Of course, verifying a formula for a few examples is no proof that the result is true for all such shapes. We will look at ways of trying to contradict the formula.

3. Suppose we cut off one corner of a cube.
   (a) How many more edges, vertices and faces are there?
   (b) Does the result still hold?



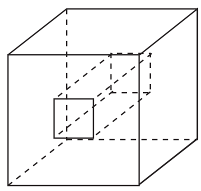


4. A slice is taken off a cube. Again, how many more edges, vertices and faces are there?

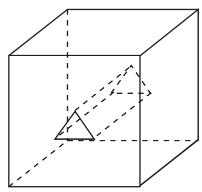
5. Try changing a cube in other ways and in each case check whether Euler's formula still holds.

Extensions

1. Suppose that a square hole is made right through the cube. Does Euler's formula now hold?



2. Try making a triangular hole right through a cube. Does Euler's formula now hold?



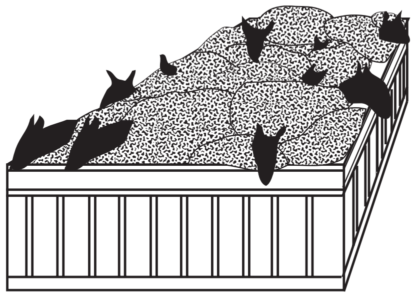
3. Put a similar hole in one of the other shapes used above. Does Euler's formula still hold?

ACTIVITY 7.9 *Fence it off*

We often use mathematics to make the best possible decisions about resource allocation.

In the problems which follow, the farmer has to decide how best to use a limited amount of fencing. Builders, planners and engineers often have similar problems to solve.

1. A farmer has exactly 200 metres of fencing with which to construct a rectangular pen for his sheep. In order to enclose as much grass as possible, the farmer tries out different dimensions and finds the area in each case.

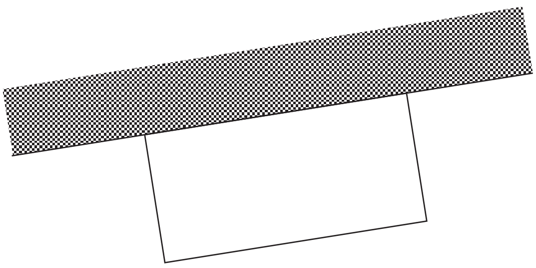


Dimensions (m)	*Area* (m^2)
5 and 95	475
10 and 90	900
15 and 85	1275
....	.....

Complete the table, and find the dimensions which give the *maximum* area.

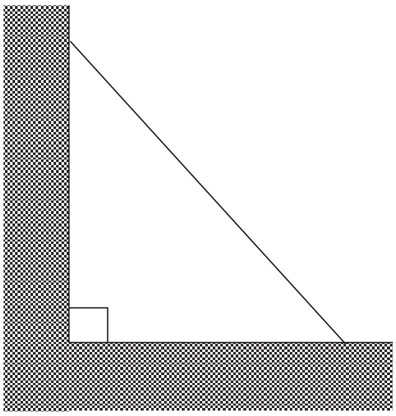
2. The farmer again wants to form a rectangular pen, but this time has a long straight wall which can form one of the sides.

With his 200 m of fencing, what is the largest area of grass that he can enclose?



3. The farmer now wishes to use his fencing to cut off a corner of a field, as in the diagram.

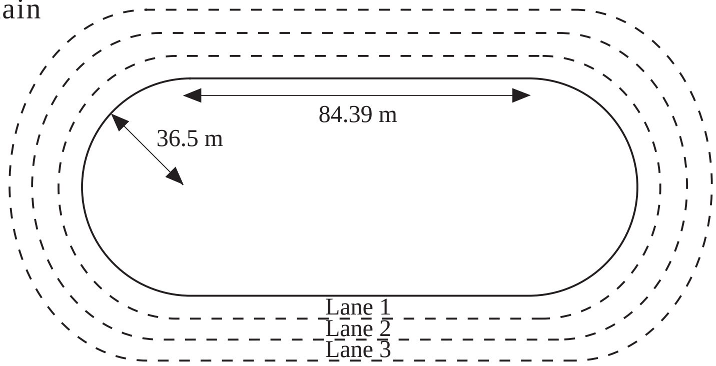
If the length of fencing is again 200 m, what is the maximum area that can be enclosed?



4. Can you generalise the results of the three questions above to a fencing length of x metres?

ACTIVITY 7.10 *Track Layout*

The sketch opposite shows the two main dimensions of a standard 400 metres running track.



1. Find the inside perimeter of this shape.

   Why do you think that it is *not* equal to 400 metres?

The inside runner cannot run at the very edge of his lane (there is normally an inside kerb) but let us assume that the athlete runs at a constant distance of, say, x cm from the inside edge.

2. What is the radius of the two circular parts run by the athlete in the inside lane?

3. Show that the total distance travelled, in centimetres, is

$$2\pi(3650 + x) + 16878$$

   and equate this to 40 000 cm to find a value for x. Is it realistic?

For 200 m and 400 m races, the runners run in specified lanes. Clearly, the further out you are the further you have to run, unless the starting positions are *staggered*.

The width of each lane is 1.22 m, and it is assumed that all runners (except the inside one) run about 20 cm from the inside if their lanes.

4. With these assumptions, what distance does the athlete in *Lane 2* cover when running one complete lap? Hence deduce the required stagger for a 400 m race.

5. What should the stagger be for someone running in *Lane 3*?

6. If there are 8 runners in the 400 m, what is the stagger of the athlete in *Lane 8* compared with that in *Lane 1*? Is there any advantage in being in *Lane 1*?

Extensions

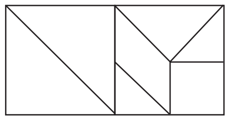
1. The area available for a school running track is $90\text{ m} \times 173\text{ m}$. How many lanes could it be?

2. Design a smaller running track, with lanes, to fit an area $40\text{ m} \times 90\text{ m}$.

ACTIVITIES 7.1 – 7.3 *Notes for Solutions*

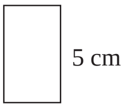
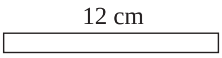
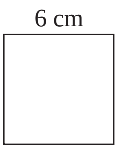
Notes and solutions given only where appropriate.

7.1 Most of these questions should not prove difficult, except possibly the final one, in which you have to form a rectangle. The key to this is to form two squares and join them together as shown (not to scale) here.

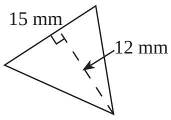
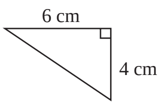
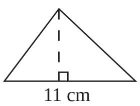
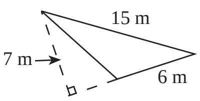


You can also make many other interesting shapes!

7.2

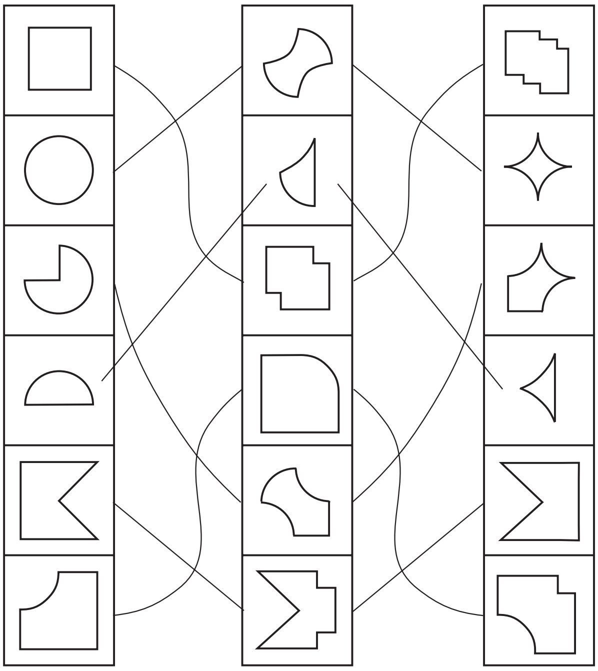
Rectangle	*Length*	*Width*	*Area*
	6 cm	3 cm	18 cm^2
	3 cm	5 cm	15 cm^2
	12 cm	1 cm	12 cm^2
	6 cm	6 cm	36 cm^2

7.3

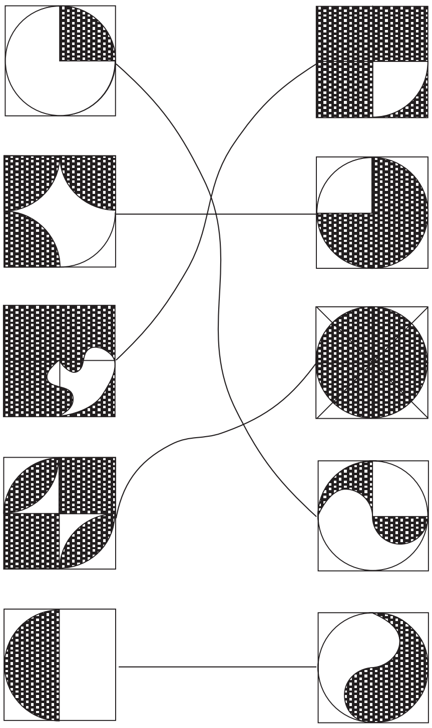
Triangle	*Base*	*Height*	*Area*
	15 mm	12 mm	90 mm^2
	6 cm	4 cm	12 cm^2
	11 cm	4 cm	22 cm^2
	6 m	7 m	21 m^2

ACTIVITIES 7.4 - 7.5 *Notes for Solutions*

7.4 *Column A* *Middle Column* *Column B*



7.5 *Column A* *Column B*



ACTIVITIES 7.6 - 7.10 *Notes for Solutions*

7.6 2. $c + r = a + 1$

4. For a circle, we have $c = 0,\ r = 1,\ a = 0$ (since there are no crossovers)

5. Yes; $c = 1,\ r = 4,\ a = 4$

6. (a) Yes (b) No (since $c + r = 11$, $a + 1 = 10$) (c) Yes

7.7 1. For some time it was *thought* that 4 colours would be sufficient but it was not until 1976 that the American mathematicians, K. Appel and W Haken, gave a valid proof (using a computer search, and equivalent in length to a textbook!).

2. (a) 2 (b) 3 (c) 4 (d) 3 (e) 4 (f) 4

Extension 1 (a) 4 (b) 5

7.8 1. (i) $e = 12,\ v = 8,\ f = 6$ (ii) $e = 8,\ v = 5,\ f = 5$
(iii) $e = 9,\ v = 6,\ f = 5$ (iv) $e = 9,\ v = 5,\ f = 6$

3. 3, 2, 1; Yes 4. 3, 2, 1; Yes

Extensions 1. Yes 2. Yes 3. Yes

7.9 1. $50 \times 50 = 2500 \text{ m}^2$ 2. 50, 100, 50 $\Rightarrow$ 5000 m^2

3. Place symmetrically $\Rightarrow$ area $= 10\,000 \text{ m}^2$ 4. $\frac{x^2}{16}$; $\frac{x^2}{8}$; $\frac{x^2}{4}$

7.10 1. 398.111 m ; the athlete cannot run on the edge

2. $(3650 + x)$ cm 3. about 30 cm 4. 407.0385 m $\Rightarrow$ stagger of 7.04 m

5. 414.7039 m $\Rightarrow$ further stagger of 7.66 m 6. 53.03 m

Extensions 1. 6 lanes 2. A 200 m track with only two lanes.

TESTS

Test 7.1 Mental Practice

Answer these questions as quickly as you can, but without the use of a calculator.

1. What is 823 mm in metres?
2. What is 36 inches in feet?
3. How many pints are there in 4 gallons?
4. Convert 7 inches to cm.
5. Convert 16 km to miles.
6. Convert 40 litres to pints.
7. What is the area of a rectangle, 7 cm by 5 cm?
8. A triangle has area 20 cm^2. If its base is 5 cm, what is its height?
9. What is the volume of a cube of side 4 cm?
10. What is the perimeter of a circle of radius 5 cm? Leave your answer in terms of π.

Test 7.2 Mental Practice

Answer these questions as quickly as you can, but without the use of a calculator.

1. What is 2.37 m in millimetres?
2. What is 4 feet in inches?
3. How many gallons is 40 pints?
4. Convert 10 inches to centimetres.
5. Convert 15 miles to kilometres.
6. Convert 10 kg to lbs.
7. What is the area of a triangle, base 12 cm and height 4 cm?
8. A rectangle has area 35 cm^2. If one side has length 5 cm, what is the length of the other side?
9. What is the volume of a cuboid of sides 2 cm, 3 cm and 5 cm?
10. What is the area of a circle of radius 4 cm? Leave your answer in terms of π.

Test 7.3 Revision

40 minutes are allowed

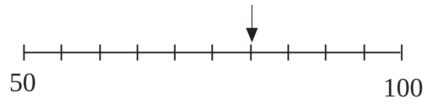
1. Copy and complete the following:

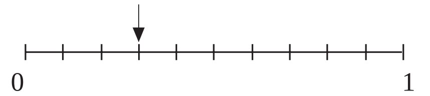
Lengths in

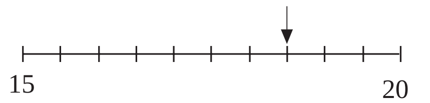
m	*cm*	*mm*
?	?	420
12.3	?	?
?	256	?

(*6 marks*)

2. Read off the values shown by the arrow on each of the following scales:

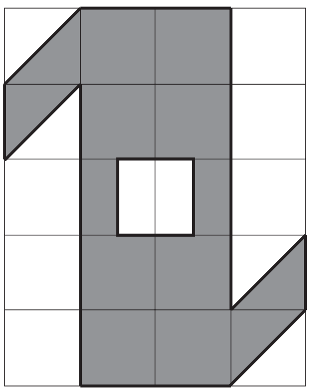
(a)


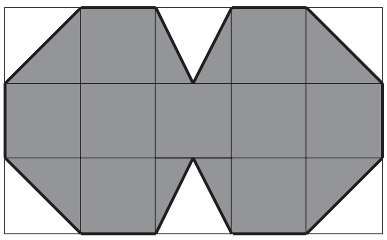
(b)


(c)


(*3 marks*)

3. By counting the number of equivalent squares, find the area of each of the following shaded shapes (the grid lines are each 1 cm apart):

(a)


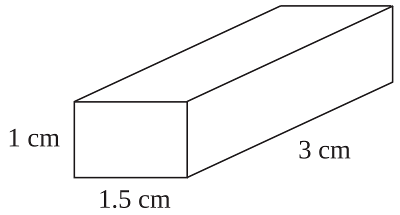
(b)


(*4 marks*)

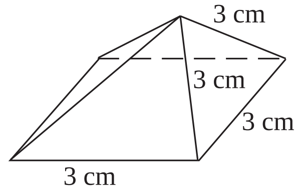
Test 7.3 Revision

4. Draw accurate nets for each of the following two shapes:

(a)

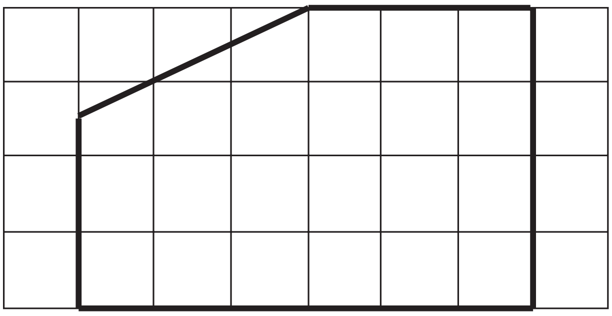


(b)



(*4 marks*)

5.

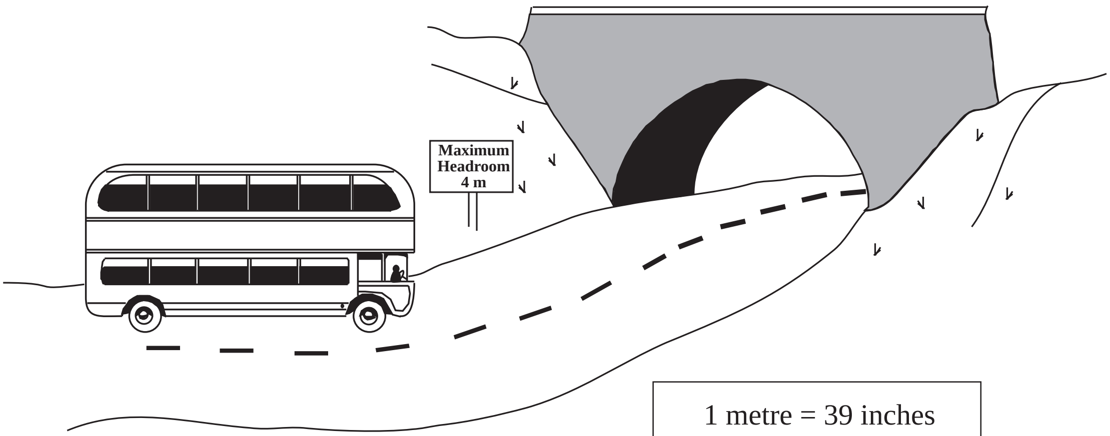


Measure and write down the perimeter of this shape. (*3 marks*)

(*MEG*)

6. On the way to Chester Zoo, the bus driver takes a short cut.

They come to a low bridge with maximum headroom 4 metres.



1 metre = 39 inches

12 inches = 1 foot

The driver knows his bus is 13 feet 1 inch high.

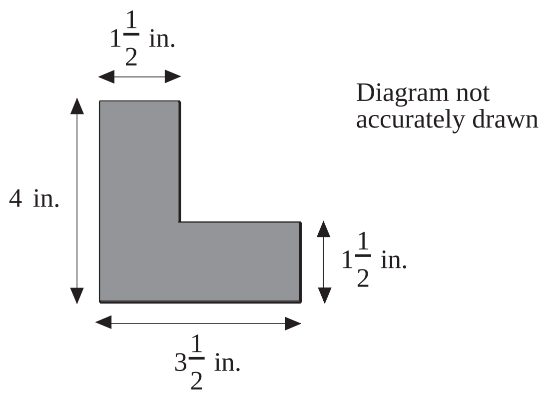
Can the bus go under the bridge?

Show all your working. (*3 marks*)

(*NEAB*)

Test 7.3 Revision

7.

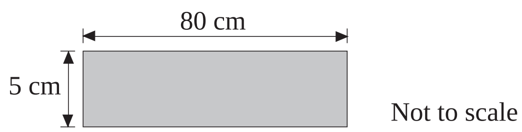


The diagram shows the measurements, in inches, of the 'L' on an 'L' plate.

Work out the area of the 'L'. (*3 marks*)

(*LON*)

8. A cyclist's reflective strip measures 5 cm by 80 cm.



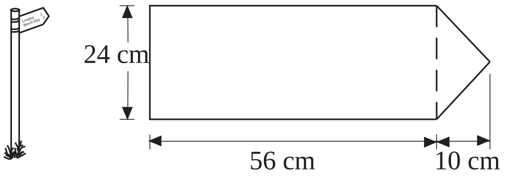
(a) What is the area of the strip? (*2 marks*)

The strip has twelve thousand reflective spots on each square centimetre.

(b) What is the total number of reflective spots on the strip? (*2 marks*)

(*SEG*)

9. A road sign consists of a rectangle and a triangle.

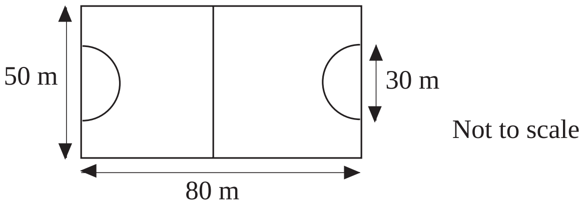


(a) Calculate the area of the rectangle. (*2 marks*)

(b) Calculate the area of the triangle. (*3 marks*)

(*SEG*)

10. A groundsman marks a sports pitch with lines.



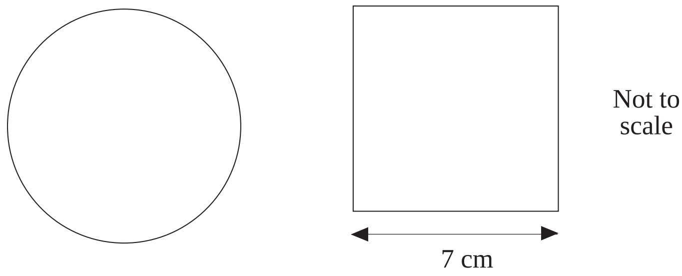
(a) What is the radius of the semi-circles on the pitch? (*1 mark*)

(b) Calculate the total length of the two curved lines on the pitch. (*2 marks*)

(*SEG*)

Test 7.3 Revision

11.

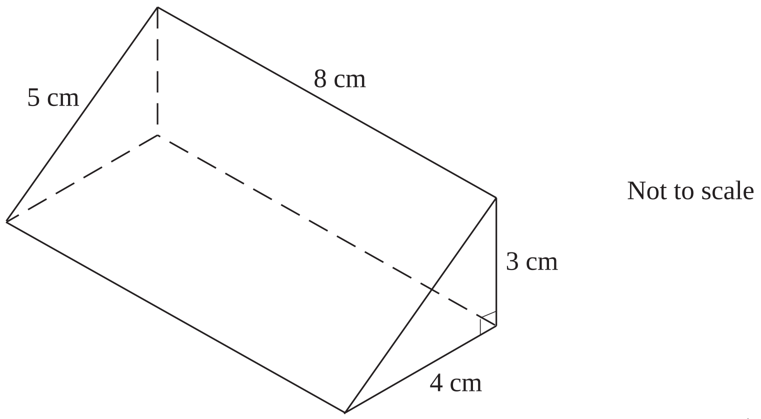


The circumference of the circle and the perimeter of the square are equal.

Calculate the radius of the circle.

(*5 marks*)
(*NEAB*)

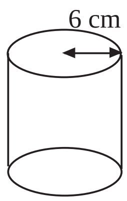
12. The cross section of a prism is a right angled triangle.



Calculate the volume of the prism.

(*4 marks*)
(*SEG*)

13. A cylindrical can has a radius of 6 centimetres.



The capacity of the can is 2000 cm^3.

Calculate the height of the can.

Give your answer correct to 1 decimal place.

(*3 marks*)
(LON)

Tests 7.1 and 7.2 Answers

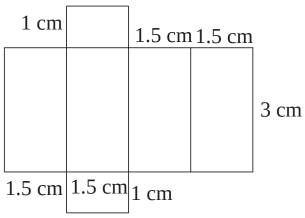
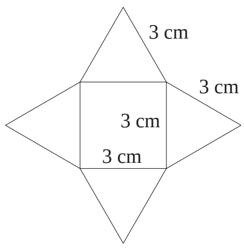
Test 7.1

1. 0.823 m
2. 3 feet
3. 32 pints
4. 17 or 18 cm
5. 10 miles
6. 70 pints
7. 35 cm^2
8. 8 cm
9. 64 cm^3
10. 10π cm

Test 7.2

1. 2370 mm
2. 48 inches
3. 5 gallons
4. 25 cm
5. 24 km
6. 22 lbs
7. 24 cm^2
8. 7 cm
9. 30 cm^2
10. 16π cm^2

Test 7.3 Answers

1.	0.42m, 4.2 cm	B1 B1	
	123 cm, 1230 mm	B1 B1	
	2.56 m, 2560 mm	B1 B1	(*6 marks*)
2.	(a) 80 (b) 0.3 (c) 18.5	B1 B1 B1	(*3 marks*)
3.	(a) 11 cm^2 (b) 12 cm^2	B2 B2	(*4 marks*)
4.	(a) 	B2	
	(b) 	B2	(*4 marks*)
5.	perimeter = 4 + 6 + 2.5 + 3.4 + 3	M1 A1	
	= 18.9 (allow 19)	A1	(*3 marks*)
6.	4 metres $\approx 4 \times 39$ = 156 inches	M1 A1	
	= 13 feet < 13 feet 1 inch	A1	(*3 marks*)
7.	Area = $\left(4 \times 1\frac{1}{2}\right) + \left(2 \times 1\frac{1}{2}\right)$	B1 B1	
	= 9	B1	(*3 marks*)
8.	(a) $5 \times 80 = 400 \text{ cm}^2$	M1 A1	
	(b) $400 \times 12\,000 = 4\,800\,000$ spots	M1 A1	(*4 marks*)
9.	(a) $24 \times 56 = 1344 \text{ cm}^2$	M1 A1	
	(b) Area = $\frac{1}{2} \times 24 \times 10$	M1 A1	
	$= 120 \text{ cm}^2$	A1	(*5 marks*)
10.	(a) 15 m	A1	
	(b) $2\pi \times 15 = 94.25$	M1 A1	(*3 marks*)

Test 7.3 Answers

11.	$2\pi r = 4 \times 7$	B2	
	$r = \dfrac{28}{2\pi}$	M1 A1	
	$= 4.46$	A1	(*5 marks*)
12.	Volume = (area of cross section) $\times$ width	B1	
	$= \left(\dfrac{1}{2} \times 4 \times 3\right) \times 8$	M1 A1	
	$= 48\ \text{cm}^3$	A1	(*4 marks*)
13.	$\pi \times 6^2 \times h = 2000$	B1	
	$h = \dfrac{2000}{36\pi} \approx 17.7$	M1 A1	(*3 marks*)

(TOTAL MARKS 50)