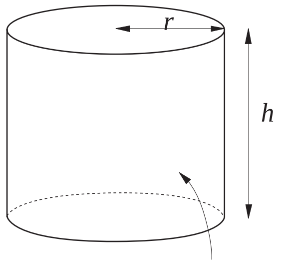
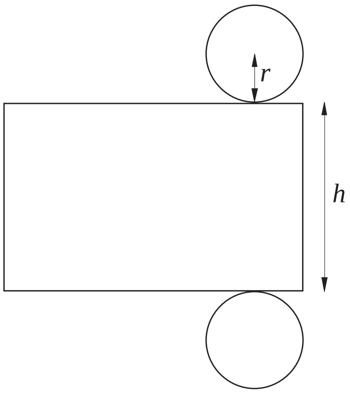
Tin cans come in various shapes and sizes, but what factors influence their design? In particular, is minimizing the area of tin used to make a can an important factor?

Suppose that a manufacturer wishes to enclose a fixed volume V, using a cylindrical can, as shown. The height of the cylinder is denoted by h and the radius of the circular can-section by r.



A net for the cylinder is shown below.



What is the length of the rectangle?

Since the circumference of the circular cross section is equal to the length of the rectangle, it must be $2\pi r$, and the total *surface area*, S, is given by

$$S = \pi r^2 + 2\pi r h + \pi r^2$$
$$= 2\pi r^2 + 2\pi r h$$

S is a function of two variables r and h, but these are related by the volume formula

$$V = \pi r^2 h$$

Problem 1

Write S as a function of r by eliminating h.

Solution

Since $h = \dfrac{V}{\pi r^2}$, then

$$S = 2\pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right)$$

$$\boxed{S = 2\pi r^2 + \frac{2V}{r}}$$

It is helpful to see graphically the relationship between S and r.

Problem 2

(a) What does S tend to as $r \to \infty$?

(b) What does S tend to as $r = 0$?

Sketch the graph of S against r.

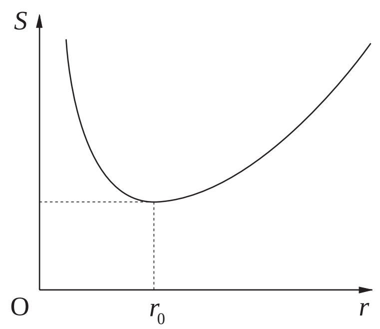
Solution

(a) As $r \to \infty$, S behaves like $2\pi r^2$ i.e. S tends to ∞.

(b) Again, as $r = 0$, S behaves like $\frac{2V}{r}$ and so tends to ∞.

Putting all these facts together results in a graph of the form shown.

There will be a value $r = r_0$ which minimizes the surface area.



Problem 3

Find the values of *r* which minimizes the surface area, *S*.

Solution

One method if finding this minimum is to use *differentiation*, since stationary points of the function *S* occur when

$$\frac{dS}{dr} = 0$$

This gives

$$\frac{dS}{dr} = 4\pi r - \frac{2V}{r^2}$$

$$= 0$$

when

$$4\pi r = \frac{2V}{r^2}$$

$$\Rightarrow \quad V = 2\pi r^3 \Rightarrow r_0 = \left(\frac{V}{2\pi}\right)^{\frac{1}{3}}$$

To find the optimum ratio of *h* to *r*, note that

$$V = \pi r_0^2 h = 2\pi r_0^3$$

$$\Rightarrow \quad h = 2r_0$$

or, alternatively,

height = diameter

But how do we know that this stationary point is, in fact, a minimum?

One way to check that this is a *minimum* is to find the second differential

$$\frac{d^2S}{dr^2} = 4\pi + \frac{4V}{r^3}$$

and, at $r = r_0$, $\frac{d^2S}{dr^2} = > 0$. This shows that there is a minimum of S at $r = r_0$.

Activity 1

For various tins, measure their height and diameter. Do your results support the argument that minimizing surface area is a key factor in the design of tin cans?

Most tins do not seem to support the 'minimizing surface area' result, so can you suggest important factors that were neglected in the analysis above?

It appears that aesthetic appeal is of more importance in the design than the minimum surface area. One case where aesthetic appeal is of little importance though, is that of commercial tins.

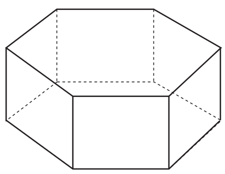
Activity 2

Repeat Activity 1 using commercial tins. Does the mathematical analysis hold for these tins?

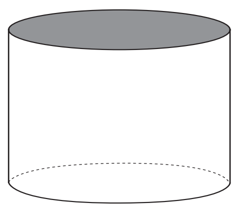
Exercises

1. What is the three-dimensional shape that, for a given volume, minimizes the surface area?

2. Repeat the problem of minimizing surface area for closed hexagonal cylinders (as shown) to enclose a fixed volume.



3. Find the ratio of height to diameter for an open-topped cylindrical container (as shown) which minimizes the surface area to enclose a fixed volume, V.



Answers to Exercises

1. sphere
2. height $= \sqrt{3} \times$ side length
3. 1 : 2