Algebra 4a

What do you know about algebra?

The nth term of a sequence $n^2 + 1$.

Write down the first 4 terms of the sequence.

A sequence is defined by the mapping $n \rightarrow n^2 + n$

Write down the first 4 terms of the sequence.

If $n = 5$

What is $n(n + 1)$?

If $n = 5$

What is $n^2 + 3n + 2$?

Look at this growing pattern



Shape 1 Shape 2 Shape 3

Describe in words what shape 10 would look like.

Generalise your answer

Number sequences

Linear sequences

Number sequences that go up or down by the same amount each time are called **linear sequences**. The difference between successive terms is called the **common difference**. A linear sequence is completely defined if you know the first term and the common difference.

1 The first term of a linear sequence is 4 and the common difference is 5. Write down the first four terms of the sequence and the rule for the n*th* term.



Alice

Alice's explanation

The sequence will be 4, 9, 14, 19, 24, ⋯
The sequence is going up in 5's and so is connected to the 5 times table.
So the rule for the n^{th} term will contain 5n
That's the five times table!
If you look at 4, 9, 14, 19, 24, ⋯
the rule for the nth term is 5n − 1
Because it is the five times table take away one each time.

Fitness check

In each of these questions, write down the first 4 terms and the rule for the n^{th} term.
In each case then work out the 20^{th} term.

1. First term 6, common difference 3
2. First term 2, common difference 8
3. First term -1, common difference 4
4. First term 14, common difference 9



2 The first term of a linear sequence is 21 and the common difference is - 6.
Write down the first four terms of the sequence and the rule for the n^{th} term.



Matthew

Matthew's work

21, 15, 9, 3, -3, ⋯
The terms are going down by 6 each term.
So it's just like the six times table from 27.
So I am thinking (27 − 6), (27 − 12), (27 − 18), ...
n^{th} term is 27 − 6n
Check 4^{th} term 27 − 6 x 4 = 27 − 24 = 3
It works!

Fitness check

In each of these questions, write down the first 4 terms, the rule for the n^{th} term. In each case then work out the 20^{th} term.

1. First term 6, common difference -3
2. First term 12, common difference -5
3. First term 14, common difference -9
4. First term -1, common difference -4





Evie

Evie's work on Question 1

1. The sequence is 6, 3, 0, -3, ⋯
So using Matthew's idea, we are taking away 3 each time.
The "minus 3 times table" would be
-3, -6, -9, -12, ... but we have to get a first term of 6
So the sequence is
(9 − 3), (9 − 6), (9 − 9), (9 − 12), ...
So the rule is 9 − 3n.
And that makes the 20^{th} term
9 − 3 × 20 = 9 − 60 = -51



Jon

Jon's work on Question 2

2. 12, 7, 2, -3, ⋯
Going down in 5's, so
the first term has to be 17 − 5 = 12
The second term is 17 − 10 = 7
So the rule is 17 − 5n
The 20^{th} term is 17 − 100 = -83



Sarah

Sarah's work on Question 3

3. 14, 5, -4, -13, ⋯
We know part of the rule will be -9n because it is going down in 9's.
All we have to is work out the other part of the rule.
That's easy − you just add 9 to the first term.
So the rule is 23 − 9n.
20^{th} term is 23 − 9 × 20 = 23 − 180 = -157

3 Can you use Sarah's idea to write down the rule for Question 4?

Number sequences

Puzzle Page

1. Place 4 numbers between 9 and 44 to form a linear sequence of 6 numbers.
2. Place 6 numbers between 2 and 50 to form a linear sequence of 8 numbers.
3. The 3rd term in a linear sequence is 12. The 8th term is 77. What is the 10th term?
4. The 5th term in a linear sequence is 58. The 9th term is 26. What is the 12th term?
5. A book contains 20 chapters each with 12 pages. Chapter 4 starts on page 39. On what page does chapter 14 end?
6. The Five Villages Drama Festival is held annually. Its venue rota in a five year cycle amongst Pagham, Aldwick, Bersted, Felpham and Middleton in that order. In 2016, it will be held in Felpham. Where was it held in 2000?
7. The first 12 terms in a non linear repeating sequence are

   $$1, 3, 6, 8, 9, 1, 3, 6, 8, 9, 1, 3, \ldots$$

   What number will the 92nd term?
8. The first number in a sequence of positive integers is 6. The other terms in the sequence follow these rules:

   if the term is even, then divide by 2 to get the next term

   if the term is odd, then multiply by 5 and subtract 1 to obtain the next term.

   Find when the n^{th} term is equal to n.
9. How many terms are in this linear sequence?

   $$6, 13, 20, 27, \ldots, 510$$

10. The nth term of a linear sequence is $57 - 6n$. What is the term number of the first negative term?
11. What is the 10^{th} triangle number?
12. What is the 10^{th} number in Fibonacci's sequence?
13. What is the n^{th} term of this sequence?

    $$2, 8, 18, 32, 50, 72, \ldots$$

14. The n^{th} term of sequence 1 is $4n - 3$.

    The n^{th} term of sequence 2 is $60 - 3n$

    After how many terms in does sequence 1 first exceed the corresponding term in series 2?
15. Using the distributive law show that

    $$2(3n - 1) + 3(2n + 1) = 12n + 1$$

Number sequences

1, 3, 7, 13, 21, ...

Look carefully at this number sequence for 30 seconds.

Now tell your partner exactly what you see.

What do you think happens next?

What does this make you wonder?



Matthew

Well it starts with a 1 and then goes up by 2, then by 4, then by 6, then by 8.
So it's not a linear sequence.
So the next term would be 21 + 10 = 31
I wonder what the general rule would look like?

Matthew is right. This certainly isn't a linear sequence.

In the next section we will be looking at some **non-linear number sequences**.

4 The each of these sequences below write down the next 3 terms. Explain what is happening in words.

1. 2, 5, 10, 17, 26, ...
2. 5, 9, 17, 29, 41, ...
3. 4, 7, 13, 22, 34, ...
4. -1, 1, 5, 11, 19, ...



Sarah

Sarah's work

2, 5, 10, 17, 26, ...

+3 +5 +7 +9 +11 +13 +15

+2 +2 +2 +2 +2 +2

Look
Going up in 2's

The next 3 terms will be 37, 50 and 65.
The difference gets bigger by 2 each time.



Evie

Evie's work

5, 9, 17, 29, 41, ...

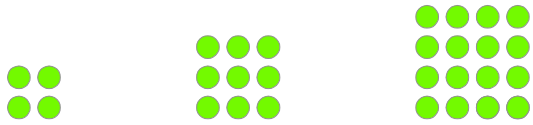
+4 +8 +12 +16 +20 +24 +28

+4 +4 +4 +4 +4 +4

Look
Going up in 4's

The next 3 terms will be 61, 85 and 113.
The difference gets bigger by 4 each time.

How many dots do you need?



Shape 1 Shape 2 Shape 3



Evie

Evie's idea



Shape 1 Shape 2 Shape 3

2 x 2 3 x 3 4 x 4

Each shape is a square with the number of dots along each side one more than the shape number.
So the n^{th} shape will be a square with $n + 1$ dots along each side.
So my general rule is $(n + 1) \times (n + 1)$



Matthew

Matthew's idea



Shape 1 Shape 2 Shape 3

I coloured in a square in each corner with sides equal to the shape number (in red). Then I coloured in a column and a row each equal to the shape number (in blue). I then had one dot left in the top right hand corner (in black).
So the n^{th} shape will be a square with n dots along each side. A row and a column with n dots each and 1 dot left over.
So my general rule is $n^2 + 2n + 1$)



Sarah

So there are two different rules but they don't look the same!

Evie's rule for the number of dots is $(n + 1) \times (n + 1)$.

Matthew's rule is $n^2 + 2n + 1$

And Sarah says that they don't look the same.

5 Can you explain why these two rules are actually the same?



Amber

Amber's work

Evie's rule is $(n + 1) \times (n + 1)$

$= n(n + 1) + 1(n + 1)$

$= n \times n + n \times 1 + n + 1$

$= n^2 + n + n + 1$

$= n^2 + 2n + 1$

Which is the same as Matthew's rule

This process of **expanding two brackets** and collecting up like terms is a very important idea that you will meet again later in the year.

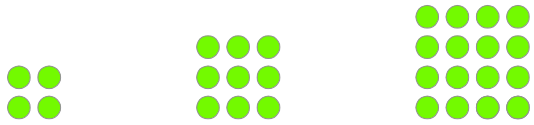
Fitness check

In each of these questions, expand the two brackets and collect up like terms.

1. $(n + 2) \times (n + 3)$
2. $(m + 1) \times (m + 4)$
3. $(p + 4) \times (p + 3)$
4. $(r + 1) \times (m + 4)$



How many dots do you need?



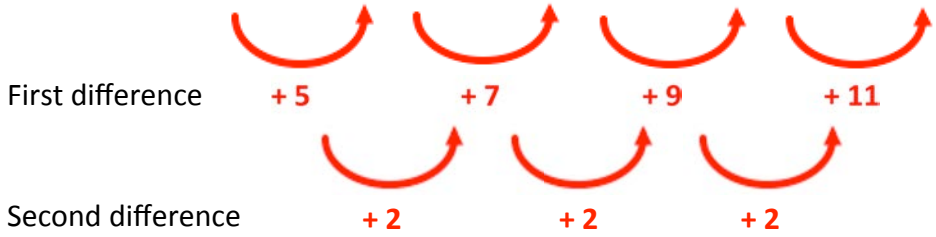
Shape 1 Shape 2 Shape 3



Alice

Sometimes it helps me to put my results in a table and see if I can spot any patterns.

Shape number	1	2	3	4	5
Number of dots	4	9	16	25	36



Quadratic sequences

A quadratic sequence is a sequence of numbers in which the second difference between any two consecutive terms is constant. The rule for the nth term will contain n^2.

Quite often we want to generate quadratic number sequences without drawing pictures.

Sometimes you will be given the rule as a **mapping**. For example, the rule above could be written as

$$n \rightarrow n^2 + 2n + 1$$

But sometimes you will be given a **general rule** for the n^{th} **term**. For example, the rule above could be written as

$$n^{th} \text{ term} = n^2 + 2n + 1$$

6 Write down the first 4 terms of the sequence defined by $n \rightarrow n^2 + 3n - 1$. Check that the second difference is constant.



Jon

Jon's work

$1 \rightarrow 1^2 + 3 - 1 = 3$

$2 \rightarrow 2^2 + 6 - 1 = 9$

$3 \rightarrow 3^2 + 9 - 1 = 17$

$4 \rightarrow 4^2 + 12 - 1 = 27$

$5 \rightarrow 5^2 + 15 - 1 = 39$

3, 9, 17, 27, 39, ...

+6 +8 +10 +12 +14

+2 +2 +2 +2 +2 +2

Look constant difference of 2

The next term will be 53

The difference gets bigger by 2 each time.

Fitness check

In each of these questions, work out the first 5 terms and check the second difference is a constant..

1. $n \rightarrow n^2 - 2n + 1$
2. $n \rightarrow 2n^2 + 4n - 3$
3. n^{th} term = $n^2 - 3n - 2$
4. n^{th} term = $2n^2 + n - 2$





Evie

I think I've noticed something else about the second difference.

7 What might Evie have noticed about the second difference?

Question Creation

Question creation – Test your theories by creating problems for you and your friends to solve. Do they support your theory?

How many dots do you need?



Shape 1 Shape 2 Shape 3

1. Can you draw the next shape in this sequence?
2. What do you notice about the number of dots that are added each time?
3. Can you describe what the tenth shape would look like?
4. How many dots would it have?
5. Can you describe shape 100?
6. How many dots would it have?
7. Can you generalise your answer in words?
8. Can you generalise using algebra?



Evie

Evie's work

1 

Shape 4

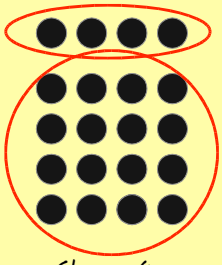
2. It isn't the same each time!
3. The 10th shape will be a 10 x 11 rectangle of dots
4. so it has 110 dots!
5. Shape 100 would be a 100 x 101 rectangle of dots
6. so it has 10100 dots.
7. You always have a rectangle of dots which is the shape number upwards and the shape number + 1 across
8. For shape n, you would have n (n + 1)



Matthew

Matthew's work

I did it a different way.
The tenth shape would be
A 10 x 10 square plus 10
= 100 + 10 = 110



4

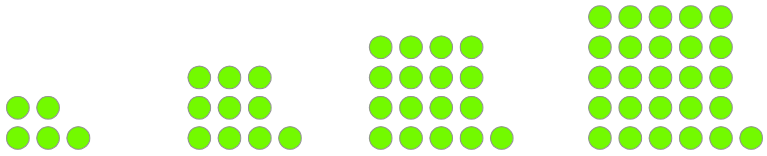
4 x 4

Shape 4

8 How would Matthew have generalised his results?

Number sequences

Look at this growing pattern



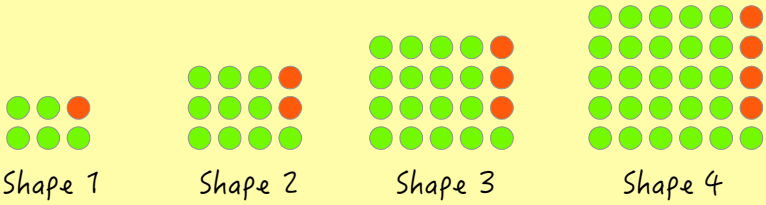
Shape 1 Shape 2 Shape 3 Shape 4

9 By just looking at the shapes, can you find a general rule for the number of dots?



Matthew

Matthew's idea



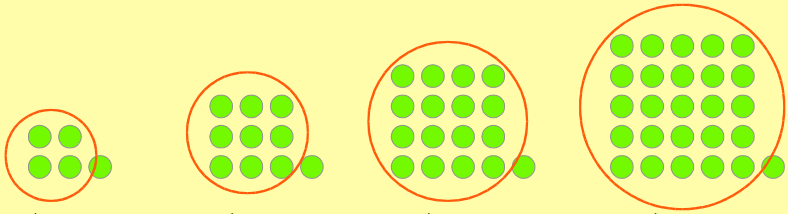
Shape 1 Shape 2 Shape 3 Shape 4

I have made the shapes in rectangles by adding red dots.
My rule is $(n + 2) \times (n + 1) - n$



Alice

Alice's idea



Shape 1 Shape 2 Shape 3 Shape 4

I arranged the dots in squares with one left over each time.
My rule is $(n + 1)^2 + 1$

10 Are they both correct? Check it for shape 4.

How many dots do you need?



Shape 1 Shape 2 Shape 3

1. Can you draw the next shape in this sequence?
2. Can you think of quick way of counting the dots in shape 4?
3. Can you describe shape 10 in words?
4. Can you find a quick way of counting the dots in shape 10?

11 Can you generalise your result?

Matthew's work



Matthew

1 

Shape 4

2. If you add the same number of dots again rotated through 180°, you get a 4 x 5 rectangle. So the number of black dots is ½ x 4 x 5 = 10 dots.



3. The 10th shape will look like a right angled triangle with 10 across the bottom and 10 up on the left hand side.
4. The number of dots in the 10th shape will be ½ x 10 x 11 = 55 dots.

5. In general you will have ½ n (n + 1) black dots

Amber's work

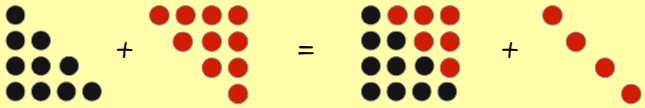


Amber

1 

Shape 4

2. Add on the same shape rotated through 180°. Move the top right corner triangle over to make a square. You then left with the diagonal.



So the number of black dots for shape 4 is ½ (4^2 + 4) = ½(20) = 10

For Shape 10 you will need ½ (10^2 + 10) = ½ (110) = 55

For shape n you will need ½ (n^2 + n)

Triangle numbers



Shape 1 Shape 2 Shape 3

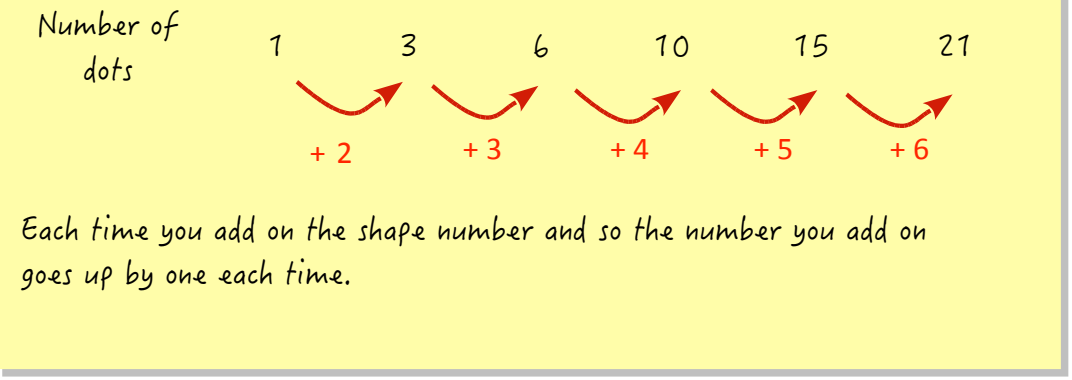
The numbers generated by this pattern are 1, 3, 6, 10, 15, …
These numbers are known as the **triangle numbers**.

12 By just looking at the number sequence can you always generate the next term?



Alice

Alice's explanation



Fitness check



1. Write down the next 3 numbers in the sequence 1, 3, 6, 10, 15, 21, …
2. Using Matthew's formula for triangle numbers, ½ n (n + 1), calculate the
   a) 12^{th} triangle number
   b) 20^{th} triangle number
   c) 100^{th} triangle number
3. Using Amber's formula for triangle numbers, ½ (n^2 + n), calculate the
   a) 15^{th} triangle number
   b) 50^{th} triangle number
   c) 200^{th} triangle number.

13 Explain why it is not practical to use Alice's method to find the 100^{th} term.

Algebra 4a: Mastery questions

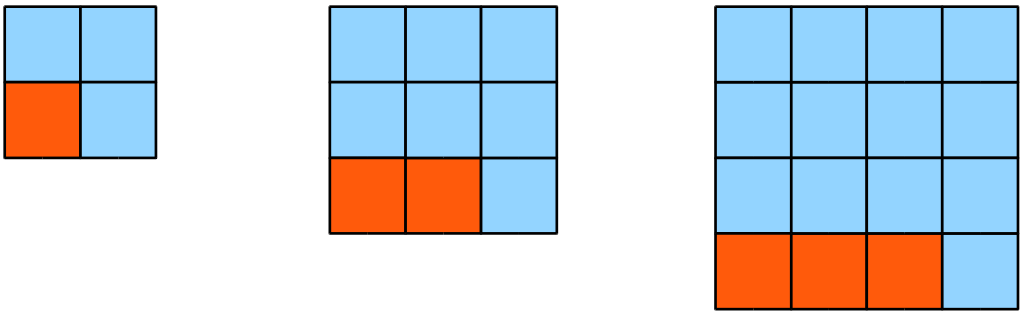
In questions 1 – 10, four possible answers are given. Only one of them is the correct. Record the letter which you think indicates the correct answer. You must be able to give reasons for your choice.

1. Find the nth term for this sequence

$$6, 15, 24, 33, \ldots$$

A	$9n + 3$	B	$9n$	C	$9n + 6$	D	$9n - 3$

2. Here is a pattern made of red and blue tiles.



Shape 1 Shape 2 Shape 3

The n^{th} term for the number of blue tiles is

A	$n^2 + 1$	B	$n^2 - 1$	C	$(n + 1)^2 - n$	D	$n^2 - n$

3. The n^{th} term of this quadratic sequence is

$$0, 3, 8, 15, 24, \ldots$$

A	$n^2 - 1$	B	$2n^2 - 2$	C	$n^2 - n$	D	$2n^2 - 2n$

4. Here are the first four terms of a quadratic sequence

$$86, 78, 68, 56, \ldots$$

The next term is

A	44	B	42	C	48	D	40

5. The nth term of a sequence is $n^2 - 4n + 2$. What is the 10^{th} term of this sequence?

A	142	B	58	C	62	D	38

Chapter Mastery Questions (A)

In questions 1 – 10, four possible answers are given. Only one of them is the correct. Record the letter which you think indicates the correct answer. You must be able to give reasons for your choice.

6. The n^{th} term of this quadratic sequence sequence

-1, 5, 15, 29, 47, ...

A	$n^2 - 2$	B	$2n^2 - 1$	C	$n^2 - 2n$	D	$2n^2 - 3$

7. Which of these sequences will not have 55 as one of its terms?

A	1, 1, 2, 3, 5, …	B	1, 3, 6, 10, ...
C	1, 8, 15, 22, ...	D	3. 7, 13, 21, ...

8. The n^{th} term of a sequence is $(n + 2) \times (n + 3)$. What is the 5^{th} term?

A	15	B	56	C	80	D	88

9. Expand $(n + 2) \times (n + 3)$

A	$n^2 + 6$	B	$n^2 + 6n + 5$
C	$n^2 + 5n + 6$	D	$n^2 + 5n + 5$

10. The first three terms of a sequence are

$$\frac{3}{5}, \frac{8}{10}, \frac{15}{17}, \ldots$$

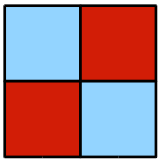
What is the rule for the nth term of this sequence?

A. $\frac{5n-2}{5n}$ B. $\frac{n+2}{n+4}$ C. $\frac{n^2-1}{n^2+1}$ D. $\frac{n^2+2n}{n^2+2n+2}$

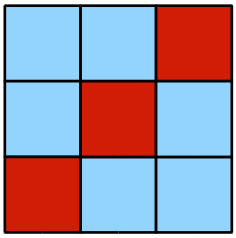
Chapter Mastery Questions (A)

In questions 11 - 12, you must show all of your working.

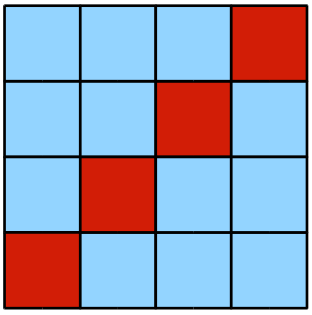
11. Look at this pattern made of red and blue tiles.



Shape 1



Shape 2



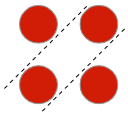
Shape 3

a) How many red tiles would there be in (i) shape 10 (ii) shape n?

b) How many tiles would there be in (i) shape 10 (ii) shape n?

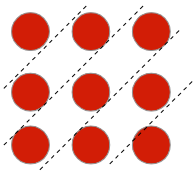
c) How many blue tiles would there be in (i) shape 10 (ii) shape n?

12. Look at this growing pattern



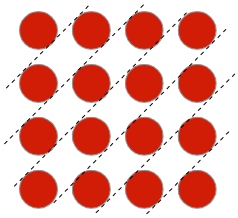
Shape 1

1 + 2 + 1 = 4



Shape 2

1 + 2 + 3 + 2 + 1 = 9



Shape 3

1 + 2 + 3 + 4 + 3 + 2 + 1 = 16

a) What would shape 10 look like?

b) What is the associate sum for shape 10?

c) What is the value of 1 + 2 + 3 + … + 18 + 19 + 20 + 19 + 18 + … + 3 + 2 + 1?

d) Can you generalise this result?