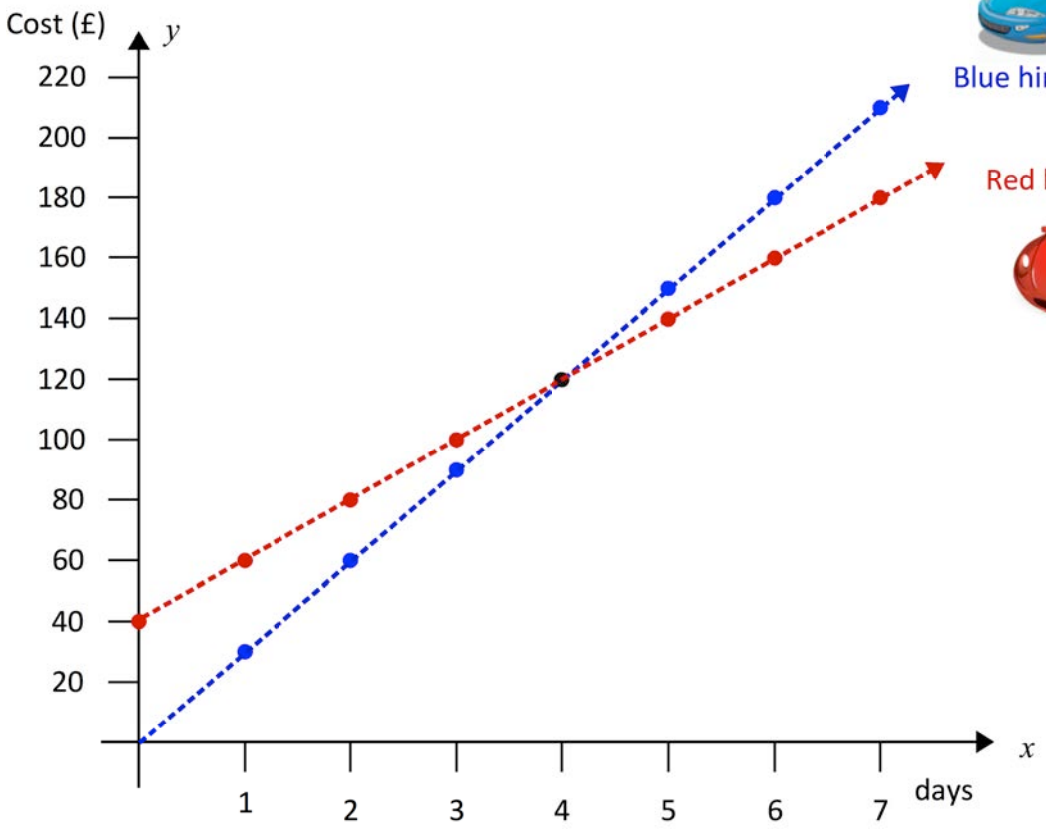
Which car should I hire?



Blue hire company

Red hire company





Two car hire companies rent out cars by the day.
Explain what these graphs tell you about their prices.



Alice

Both cars cost the same for 4 days hire.

The blue car is cheaper for 3 days hire but the red car is cheaper if you hire it for more than 4 days.



Matthew



Sarah

The blue car costs £30 per day.
So the cost is in direct proportion to the number of days hired.

The red car has a booking fee of £40 and then costs £20 for each day's hire.



Evie

1 How much would it cost to hire each car for 12 days?

Which car should I hire?



Jon

The blue car would cost
12 x £30 = £360

The red car would cost
12 x £20 + 40 = £280



Amber

A problem for you to think about

Evie's mum paid £240 to hire a car. Which car might she have hired and for how many days?

Is there more than one possibility?

2

Can you write down a formula for the cost (£y) of hiring the each car for x days?



Alice

For the blue car
it will be $y = 30x$

For the red car it will
be $y = 20x + 40$



Matthew

Variables and linear functions

In this situation we have two quantities that are changing. The cost of the hire (y) and the number of days (x) for which you hired the car.

As both of these quantities can change they are called **variables**.

When we can express y in terms of a linear expression in x, we say ***y* is a linear function of *x***.

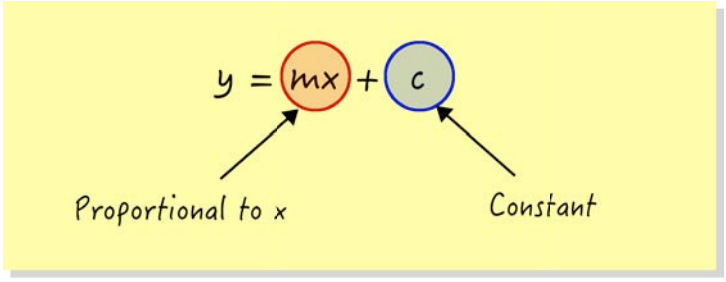
If y is proportional to x, then this can be written $y = mx$ where m is the **constant of proportionality**.This is the case for the blue car.

In general a linear function can be written in the form $\boldsymbol{y = mx + c}$ where m and c are both constants.



Sarah

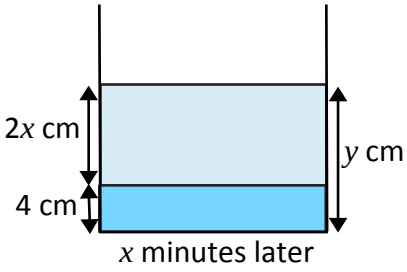
Sarah's notes



Linear functions

Problem 1

In a cylindrical water tank the depth of the water is originally 4cm. Water is poured in so that that the depth increases by 2 cm per minute. If the total depth of the water is y, x minutes after the start of the water being poured in, express y as a linear function in x.



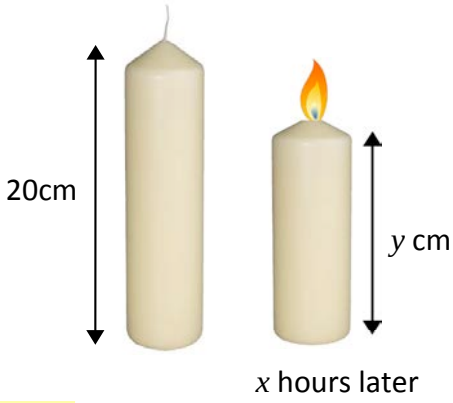


Amber

The water will rise by 2 cm per minute, so in x minutes it will rise $2x$ cm, so the depth will be given by $y = 2x + 4$

Problem 2

A 24cm candle burns at a rate of 2cm per hour. By letting the length of the candle be y cm after it has been burning for x hours, write an expression for y in terms of x.





Evie

Evie's working

The candle goes down by 2cm per hour, so in x hours it goes down by 2x cm.

So the length will be 24 - 2x

So my linear function is y = 24 – 2x

Problem 3

Alice's dad gave her £10 to start off her savings jar. Alice now saves £3 a week. By letting the amount of money in Alice's jar be £y after saving for x weeks, write an expression for y in terms of x.





Jon

Jon's working

Alice's savings go up by £3 a week, so after x weeks her savings will have gone up by £3x.

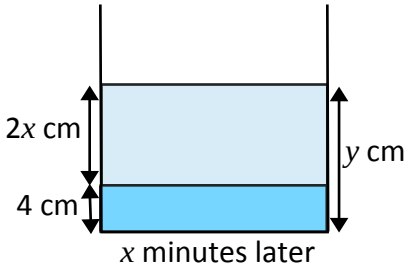
So she will now have £(3x + 10) in her jar.

So my linear function is y = 3x + 10

Changing values in linear functions

Problem 1 revisited

In a cylindrical water tank the depth of the water is originally 4cm. Water is poured in so that that the depth increases by 2 cm per minute. On the previous page, Amber said that the total depth of the water is y, x minutes could be modelled by the function $y = 2x + 4$



3

Create a table that shows how the y values as x increases from 0 to 5.

Investigate what happens as to y as x increases from 2 to 5

Time x (mins)	0	1	2	3	4	5
Depth y (cms)	4	6	8	10	12	14

+ 3

+ 6

As x increases from 2 to 5, the increase in x is $5 - 2 = 3$

And the increase in y is $13 - 7 = 6$

The increase in y is twice the increase in x, in other words, $\frac{\textit{increase in } y}{\textit{increase in } x} = 2$

4

Investigate what happens as to y as x increases from 2 to 6?

What does this make you think happens?

Problem 2 revisited

In the candle problem on the previous page, we modelled the height of the candle y, x hours after it was lit by $y = 24 - 2x$.

Create a table that shows how the y values as x increases from 1 to 4.

Investigate what happens as to y as x increases from 1 to 4.

Do the same thing as x increases from 3 to 8.

What happens to $\frac{\textit{increase in } y}{\textit{increase in } x}$ in both cases?

By comparing this ratio with the function, suggest what the the $\frac{\textit{increase in } y}{\textit{increase in } x}$ will be for the function $y = 3x + 10$?

Rate of change for a linear function



Sarah

For the function $y = 3x + 10$ the $\frac{\textit{increase in } y}{\textit{increase in } x} = 3$

5

Do you agree with Sarah? Can you explain your thinking?

The rate of increase in y as x changes is called the **rate of change**.

An important result

In a linear function, $\boldsymbol{y = mx + c}$, its rate of change is constant and is equal to m.

Rate of change = $\frac{\textit{increase in } y}{\textit{increase in } x} = m$

Fitness check

In each of these linear functions, write down the rate of change. Also calculate the amount of increase in y when x increases by 3.

(1) $y = 4x - 3$ (2) $y = 15 - 2x$ (3) $y = 2.5x + 4$

(4) In a certain port, the temperature at sea level is 22°C. The relationship between the altitude x km, and the temperature y °C, may be expressed as

$y = -5x + 22$ until about 10 km above sea level.

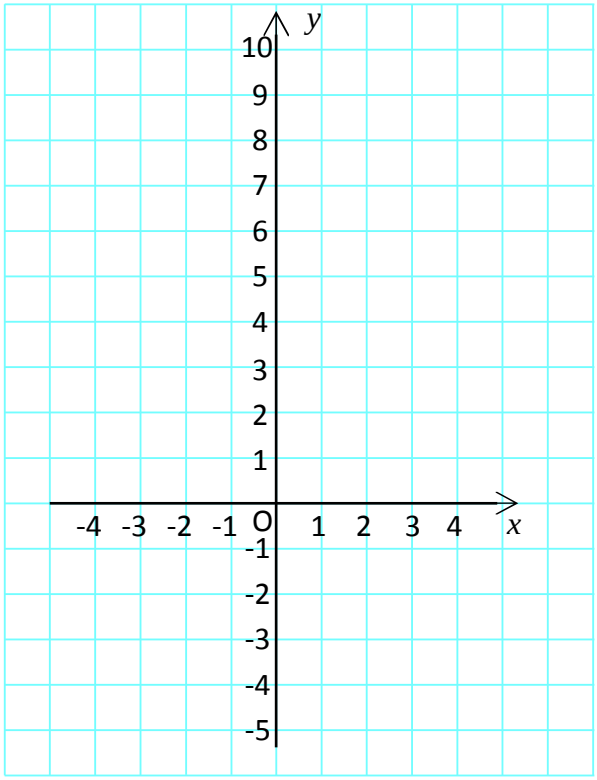
a) What does the rate change, -5 represent?

b) If the altitude increases by 3 km, how much does the temperature decrease?

Graphs of linear functions

Let's draw a graph of the linear function, $y = 2x + 4$, based on the table of its values.

x	...	-4	-3	-2	-1	0	1	2	3	...
y	...	-4	-2	0	2	4	6	8	10	...



6 Plot the pairs of x and y values from the table above on the coordinate grid

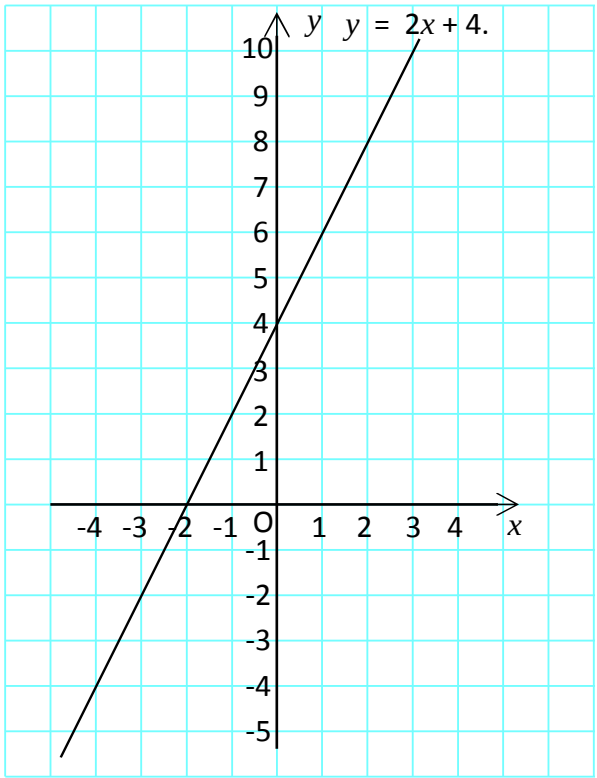


Alice

I am going to add some more points by calculating the value of y as x changes from -4 to 3 as x by 0.5.

7 What happens as we plot more and more points?

As we plot more and more points, the graph will become a line as shown below. This line is the collection of points whose coordinates, (x ,y) are the values of and y that will satisfy $y = 2x + 4$.



Finding more points

You can now use your formula to find other points on the line and to check if points are actually on the line.

For example …

1. the points below are on the graph of the linear function $y = 2x + 4$.

   Write the appropriate number in each box...

   a) (5, ☐)

   b) (-3, ☐)

   c) (☐ , 20)

2. Are these points on the line? Explain your thinking.

   a) (50,104)

   b) (23, 52)

   c) (105, 214)

WHAT'S THE SAME, WHAT'S THE DIFFERENCE?

Let's compare the graphs of

(1) $y = 2x$ and (2) $y = 2x + 4$

For (1) and (2), if we summarize the values of y that correspond to the values of x in a table, it well be like the table below...

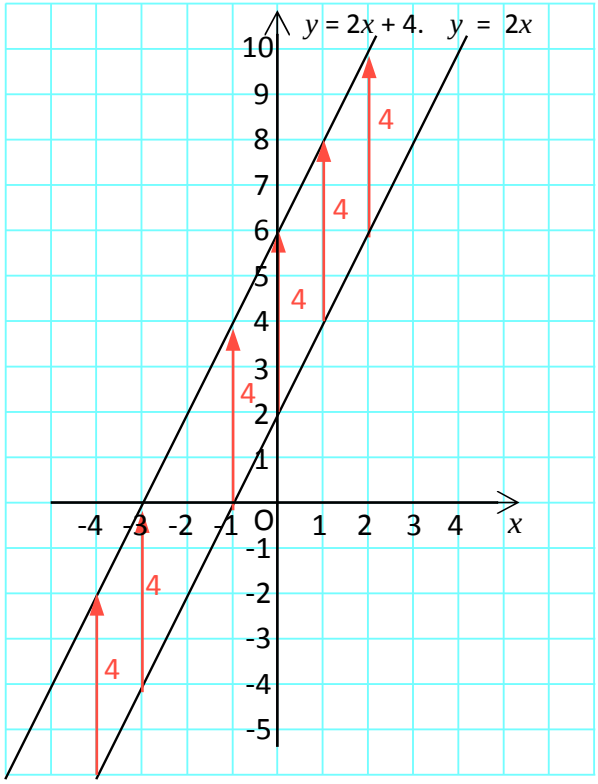
x	...	-4	-3	-2	-1	0	1	2	3	...
$2x$	...	-8	-6	-4	-2	0	2	4	6	...
$2x + 4$	...	-4	-2	0	2	4	6	8	10	

8 Plot the pairs of x and y values from the table for both of these graphs on the same coordinate grid. Describe what you see.



Jon

The two graphs are parallel but $y = 2x + 4$ has been translated three units upwards.



9 What do you think will happen if we try to draw the graph of $y = 2x - 2$?

Create a table of values for $y = 2x - 4$ as shown below.

Add your new points onto your diagram and draw the line $y = 2x - 4$.

What do you see?

Is that what you expected to happen?

x	...	-4	-3	-2	-1	0	1	2	3	...
$2x$	...	-8	-6	-4	-2	0	2	4	6	...
$2x - 1$	...	-12	-11							...



Evie

The line $y = 2x - 4$ is also parallel to $y = 2x$ but this time it has been translated downwards by 4 units.

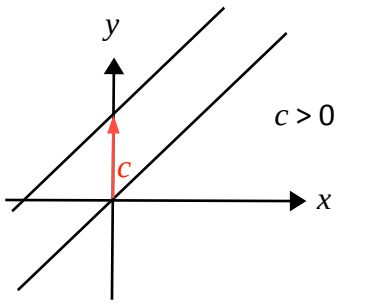
Relationship of graphs of linear functions to graphs of proportional relationships

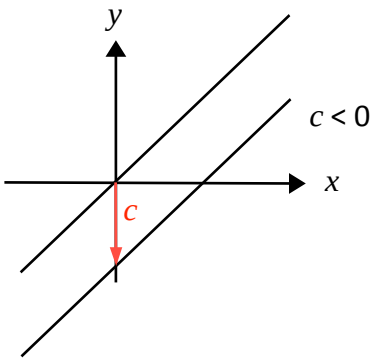
The graphs of the linear function $y = mx + c$ is the line that is obtained by translating the graph of $y = mx$ along the y-axis by c units in the positive direction.

If $c > 0$, the graph is translated upwards, if $c < 0$, the graph is translated downwards.

The constant part of the linear function, $y = mx + c$, is the value of y when $x = 0$ is the y-coordinate of the point where the graph interests the y-axis.

c is called the y-intercept of the graph of the linear function $y = mx + c$.





10 Look at this collection of graphs in Fig 1. One line has equation $y = x + 2$, what are the equations of the other five lines?

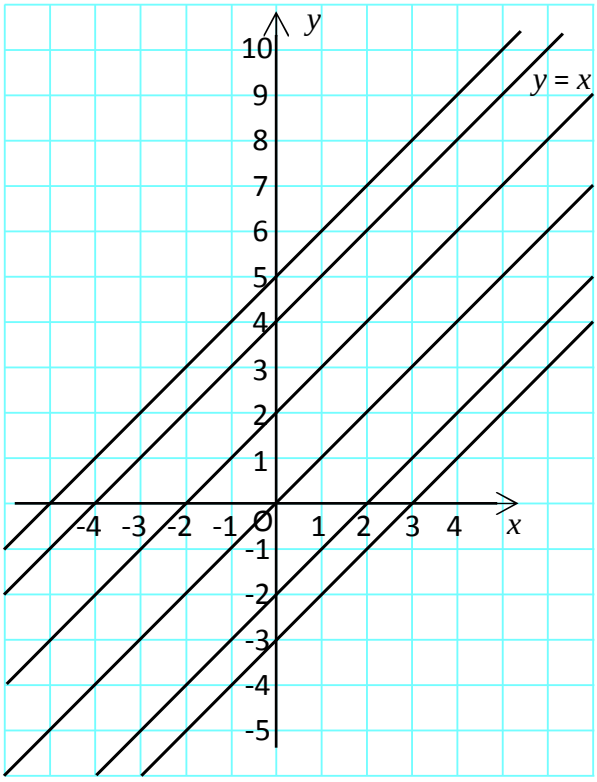


Fig 1

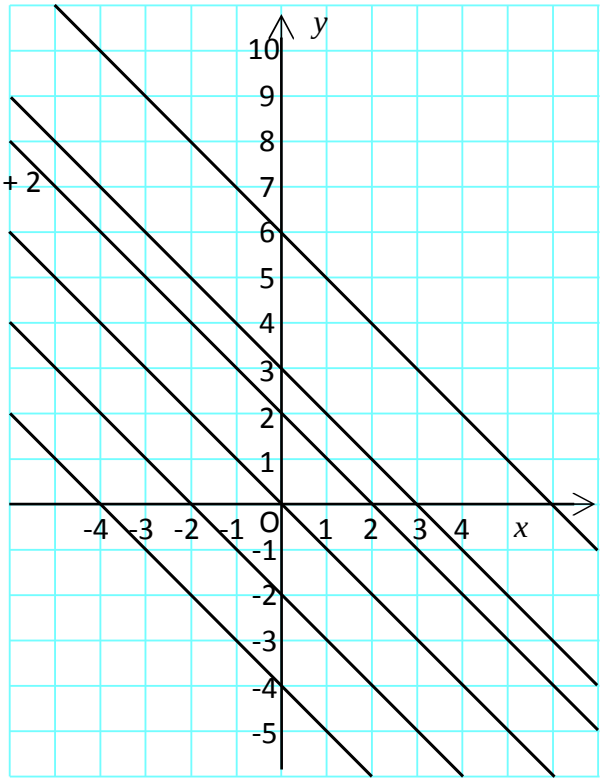


Fig 2

11 Now look at this collection of graphs. One line has equation $y = -x + 2$, what are the equations of the other five lines?

Slopes of graphs

Let's think about the meaning of the rate of change using graphs.

With the linear function $y = 2x + 1$, the rate of change is 2, or

$$\frac{\text{increase in } y}{\text{increase in } x} = 2$$

This means as we move by 1I unit from left to right, the graph goes up 2 units.

1. How far does the graph of $y = 2x + 1$ move up when you move 3 units to the right ?

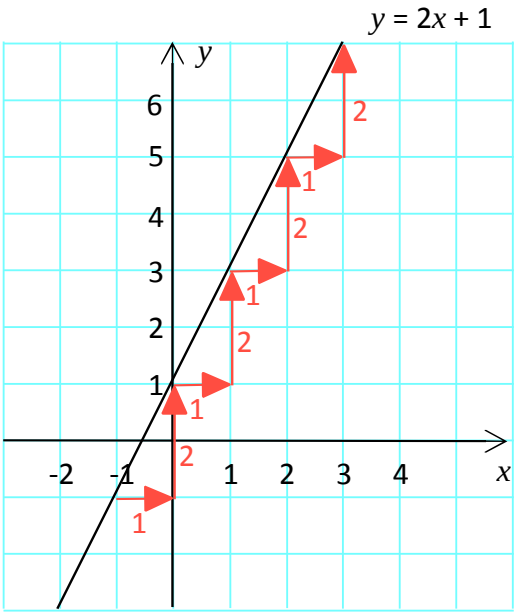
Similarly, with the linear function $y = -2x + 5$, the rate of change is -2. Therefore, on the graph as you move to the right 1 unit, the graph will go down 2 units.

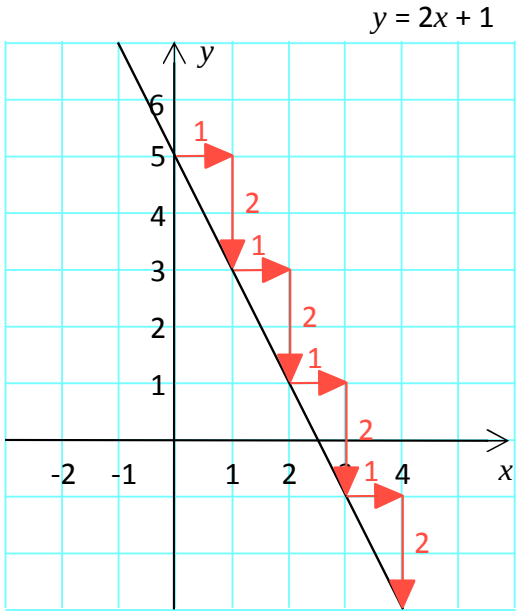
In other words, as you move to the right 1 unit, it will go up -2 units.

2. How far does the graph of $y = -2x + 5$ go down as you move to the right 4 units?

From what we just investigated, the slope of the graph of a linear function, $y = mx + c$, depends on the value of m.

For this reason, we call m the slope or gradient of its graph.



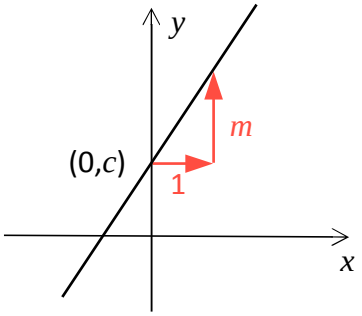


Graph of linear functions

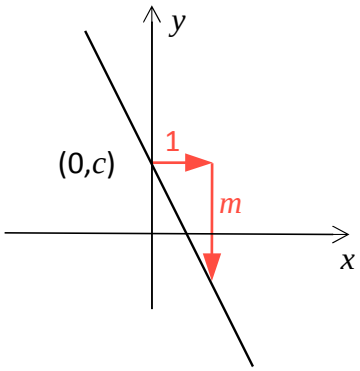
The graph of the linear function, $y = mx + c$ is a line with slope m, and the y-intercept is c.

The slope m is the amount of increase in y when x increases by 1.

When $m > 0$



When $m < 0$



The graph of the linear function, $y = 4x - 5$ is a line with a slope of 4 and a y-intercept of -5.

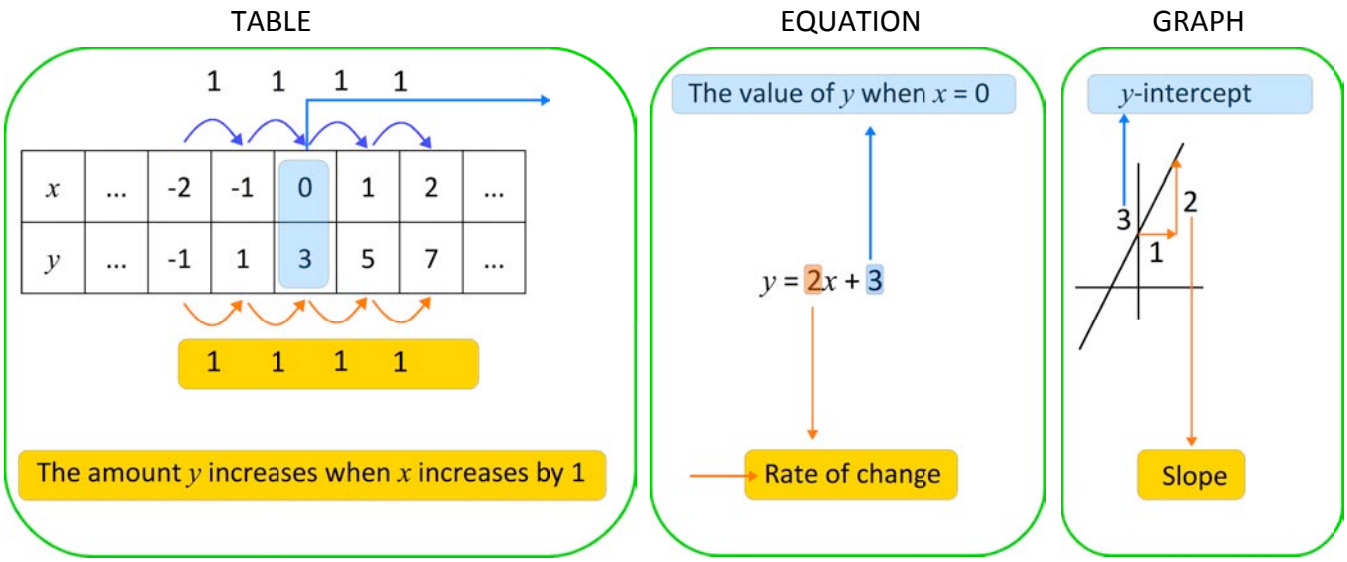
1. Name the slope and the y-intercept of the graph of each of the following linear functions.

   a. $y = 3x + 4$ b. $y = -2x - 1$

2. Name the slope and the y-intercept of the graph of the following linear functions.

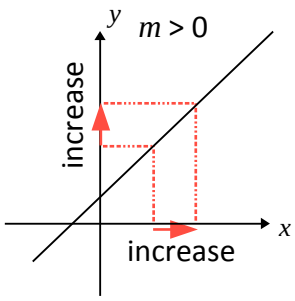
   a. $y = 2x + 3$ b. $y = -4x$ c. $y = 3x - 6$

Using $y = 2x + 3$ as an example, the relationship among tables, equations, and graphs can be summarized as shown below.



Increase and decrease of linear functions and their graphs.

Given the linear function, $y = mx + c$, we can say the following...

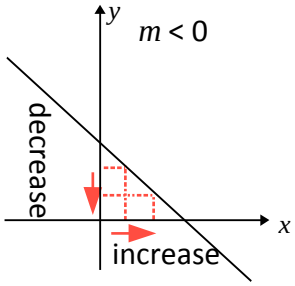


When $m > 0$

as x increases, y will also increase, and the graph will be a line that goes up to the right.

When $m < 0$

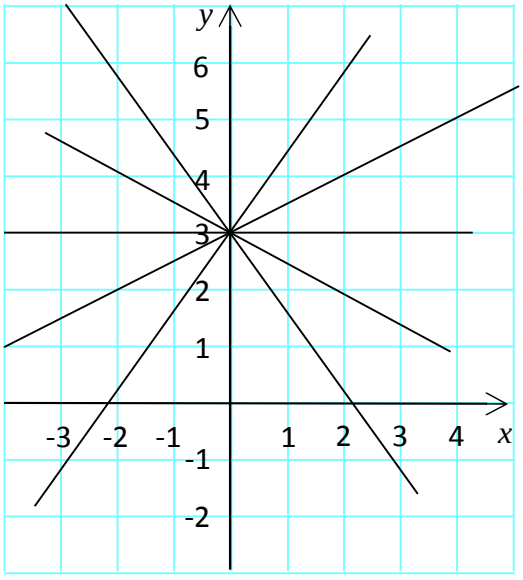
as x increases, y will decrease, and the graph will be a line that goes down as it goes to the right.

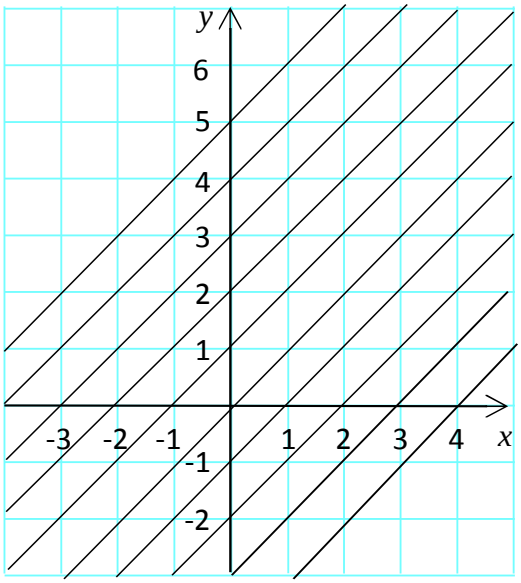


The graph of $y = 2x + 3$ is a line. Sometimes, we say will the line, $y = 2x + 3$, where $y = 2x + 3$ is the equation for the line.

Just say what you see

The figures below show the graphs of $y = mx + c$ as various values were substituted in m and c. Share what you observe.







Evie

When we change the values of m, the slope of the line...

The graphs will go up and to the right when...



Alice



Amber

When the value of m stays the same, the graphs...

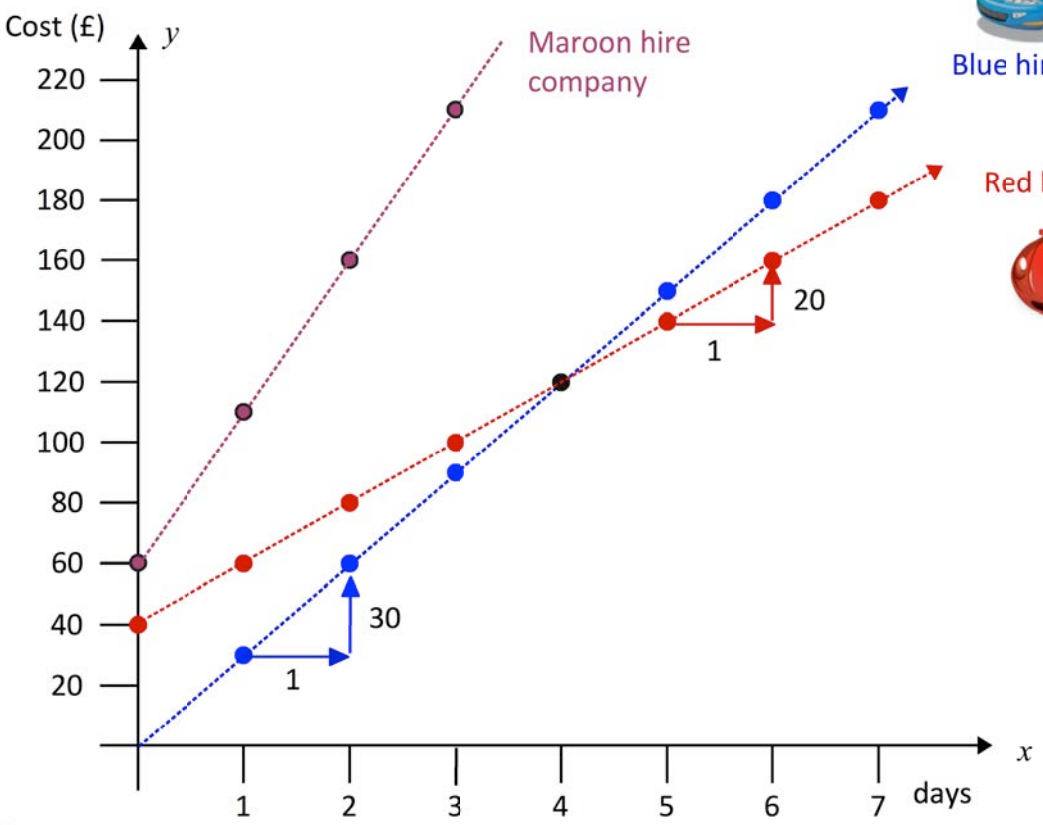
Let's think about the meaning of the slope and the y-intercept in the car hire example from earlier.



Blue hire company



Red hire company



cm



Jon

For the blue car, the gradient is 30 and the y-intercept is zero. That means the the daily rate is £30 and there was no extra initial payment.



Sarah

For the red car, the gradient is 20 and the y-intercept is 40. That means the daily rate is £20 but there is an additional initial payment of £40.

The Maroon hire company

The Maroon hire company hire limousines and their charges have been added to the graph.
Look carefully at the new graph.

1. What is the daily rate for the Maroon hire company?
2. What is their initial charge?
3. What is the linear function for their charges?
4. What would it cost to hire a car from the Maroon hire company for:
   a) 7 days?
   b) 14 days?
   c) 28 days?

Let's look at graphs of linear functions and intervals of values for the variables

12

When heating water in an electric urn, the temperature of water y, x minutes after we started heating it can be expressed as $y = 4x + 20$. What does the 20 represent in this equation?

Write a mathematical expression to show how the temperature of the water increase between 5 minutes and 15 minutes?



Alice

Alice's working

As its a linear function, I am going to draw a graph to help me think.

From my graph, I can see that when $x = 0$, $y = 20$, so that means the water was at 20°C before we switched the boiler on.

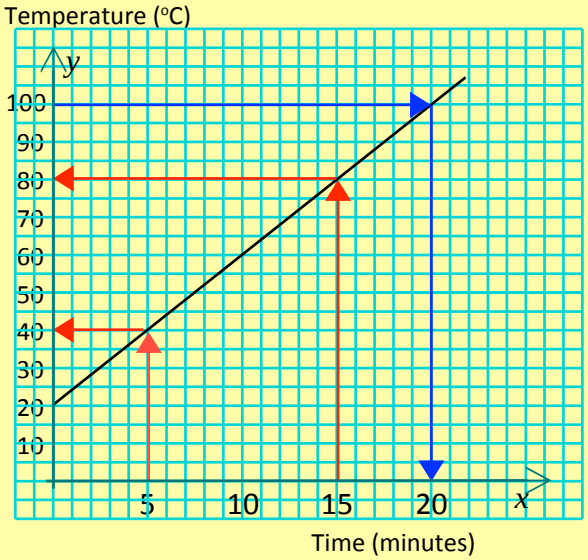
I can see that when $x = 5$, $y = 40$, and when $x = 15$, $y = 80$,

so I could write for x in the interval

$$5 \leq x \leq 15$$

y will be in the interval

$$40 \leq y \leq 80$$



13

How could you use this graph, to estimate the time taken for the water to boil?

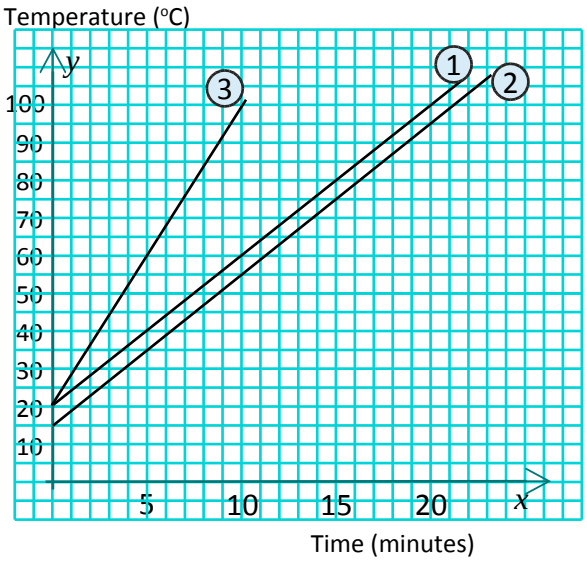


Jon

Easy, water boils at 100°C, so using Alice's graph, this would be after 20 minutes.

Let's think about the meaning of the slope and the y-intercept in the example of boiling water.

Graph (1) below shows the change in water temperature in the urn as discussed previously If the temperature of the water is y^0C, x minutes after the urn is turned on.



14 Can you explain what graphs (2) and (3) might mean?

Let's draw graphs of linear functions based on the slope and y-intercept.

1. Let's draw the graph of the linear function $y = \frac{1}{2}x + 1$


Jon

A line is defined by 2 points, isn't it?

... and we know the y intercept is 1, so (0,1) is one point.

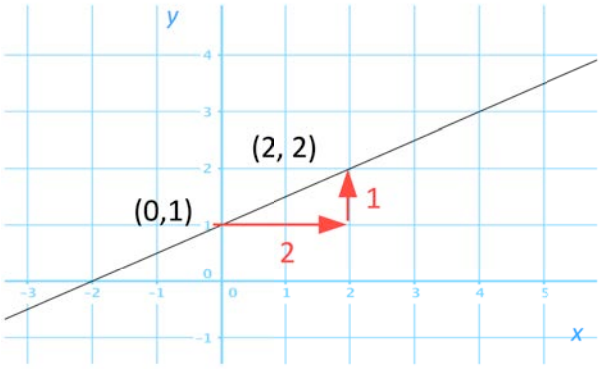

Amber


Evie

... and since the slope is $\frac{1}{2}$, if we move 2 units to the right, we must go up 1 unit.


Sarah

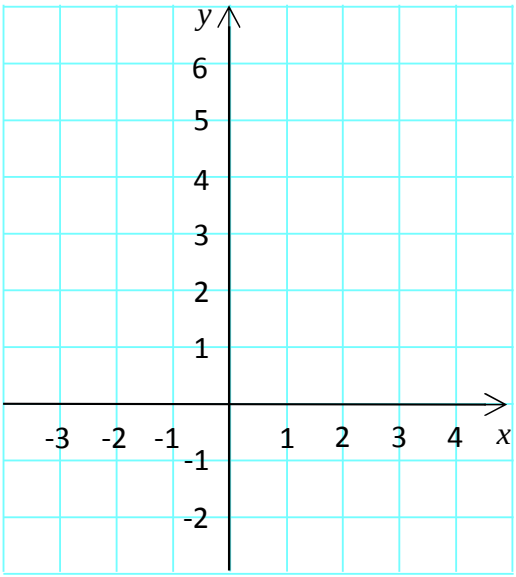
So (2,2) will be on the line.



Fitness Check

Think about the graph the linear function, $y = -2x - 1$

1. Which point on the y-axis does the graph go through? First try and find the coordinates of this point.
2. As we move to the right 1 unit, how far and in which direction does the graph move?
3. Based on the process above, draw the graph on the grid.



4. Draw the graph of the linear function $y = 2x - 1$.

5. Draw the graphs of each of these linear functions.

   a. $y = x + 1$ b. $y = 2x + 2$ c. $y = x - 3$.

   Using the y-intercept and the slope, find two more points that each graph will pass through.

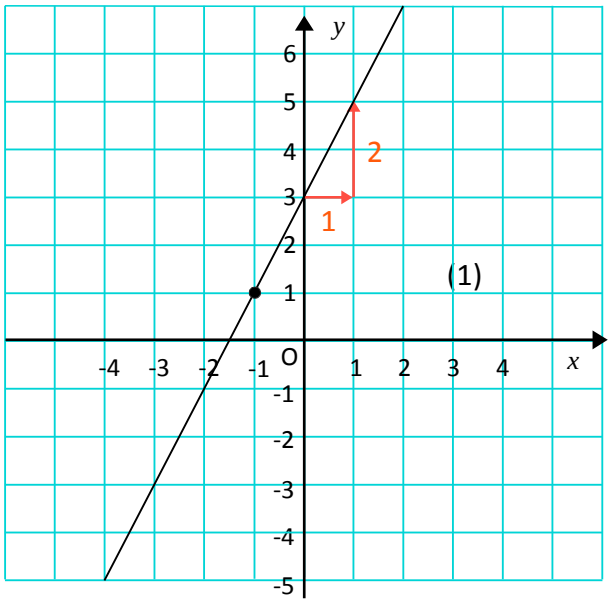
Writing equations for linear functions

Since linear functions can be expressed in the form of $y = mx + c$, if we know the values of m and c, we can write their equations.

Let's write the equation of a graph by investigating its slope and y-intercept.

15

On the graph shown on the right, place a dot where both the x and the y coordinates are integers



Let's write the equation of line (1) shown on the right.
Since this line goes through point (0, 3) on the y-axis, its -intercept is 3.

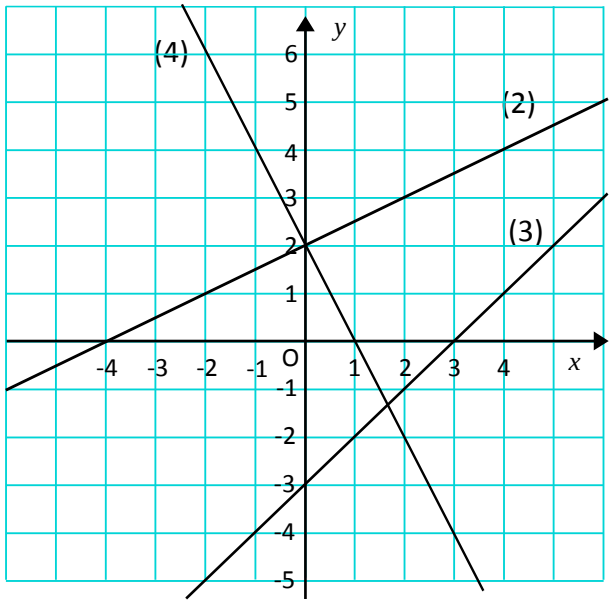
Also, as we move 1 unit to the right, it goes up by 2. So, the slope is 2.

Therefore, the equation for the line is
$y = 2x + 3$

Identify the x and the y coordinates of the dot in the question above. Substitute the values in $y = 2x + 3$ to check if those points are in the line $y = 2x + 3$.

1. Write down the equation of line (2) on the right.

2. Write an equation for lines (3) and (4) shown on the right.



Let's write the equation of a line when given its slope and one point on the line.

16

Find the equation of a line given the slope of the graph is 2 and it goes through point (3,2).



Jon

Jon's working

Since the slope is 2, the equation will be in the form, $y = 2x + c$.

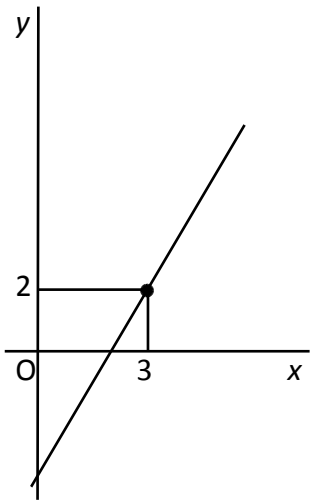
Since the graph goes through (3, 2), we can substitute $x = 3$ and $y = 2$ in the equation above.

$2 = 2 \times 3 + c$

$2 = 6 + c$

$c = -4$

So the equation is $y = 2x - 4$



This question is exactly the same as finding a linear function given its rate of changes is 2 and if $x = 3$, $y = 2$.

Slope is 2 ⟺ **Rate of change is 2**

Goes through (3, 2) ⟺ **if x = 3, y = 2**

1. Write an equation of the linear function that will satisfy each of the following conditions.

   a) The slope of the graph is -3 and it goes through (2, 5).

- The rate of change is 3, and if $x = 2$, $y = 4$.

2. Write an equation of the linear function that will satisfy each of the following conditions.

- The rate of change is -1 and if $x = 2$, $y = -3$.
- The graph is parallel to line $y = 3x + 4$ and goes through point (2, 0).

We can determine the equation when we are given the *y*-intercept and another point on the graph using the same idea

3. Write an equation of the linear function whose graph has the y-intercept of 6 and goes through the point (2, 2).

Let's write the equation of the line when given two points.

17

Given that y is a linear function of x and that the function goes through points (1, 3) and (3, 7), can you find it equation?



Sarah

Sarah's working

Since the slope goes through 2 points,
(1, 7) and (3, 3), the slope of the graph is

$$\frac{3-7}{3-1} = \frac{-4}{2} = -2$$

Therefore, it will be written in the form,

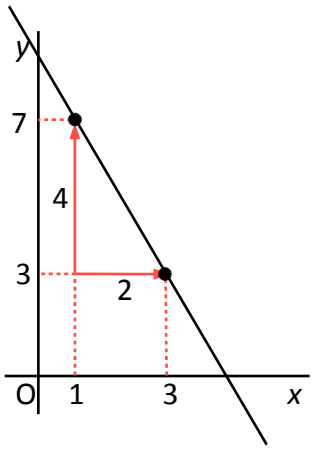
$y = -2x + c$

Since the graph goes through (1, 7), we substitute $x = 1$ and $y = 7$ in the equation above.

$7 = -2 \times 1 + c$

$7 = -2 + c$

$c = 9$ Answer $y = -2x + 9$



Fitness Check

- If y is a linear function of x. and its graph goes through two points (-1, - 2) and (2, 7), what is its equation?
- Find the equation of the linear function given that if $x = 1, y = 3$ and if $x = 4, y = -9$

1. Find the equation of a line that goes through (4, 0) and (8, -2).

Assuming linear function relationships

18

Cooler bags are designed to keep food and drinks cold. On its instruction sheet were the results of an experiment showing how the temperature of a drink in a plastic bottle changed over time. Based on this experiment, predict when the temperature of the drink in the bottle will reach 15°C

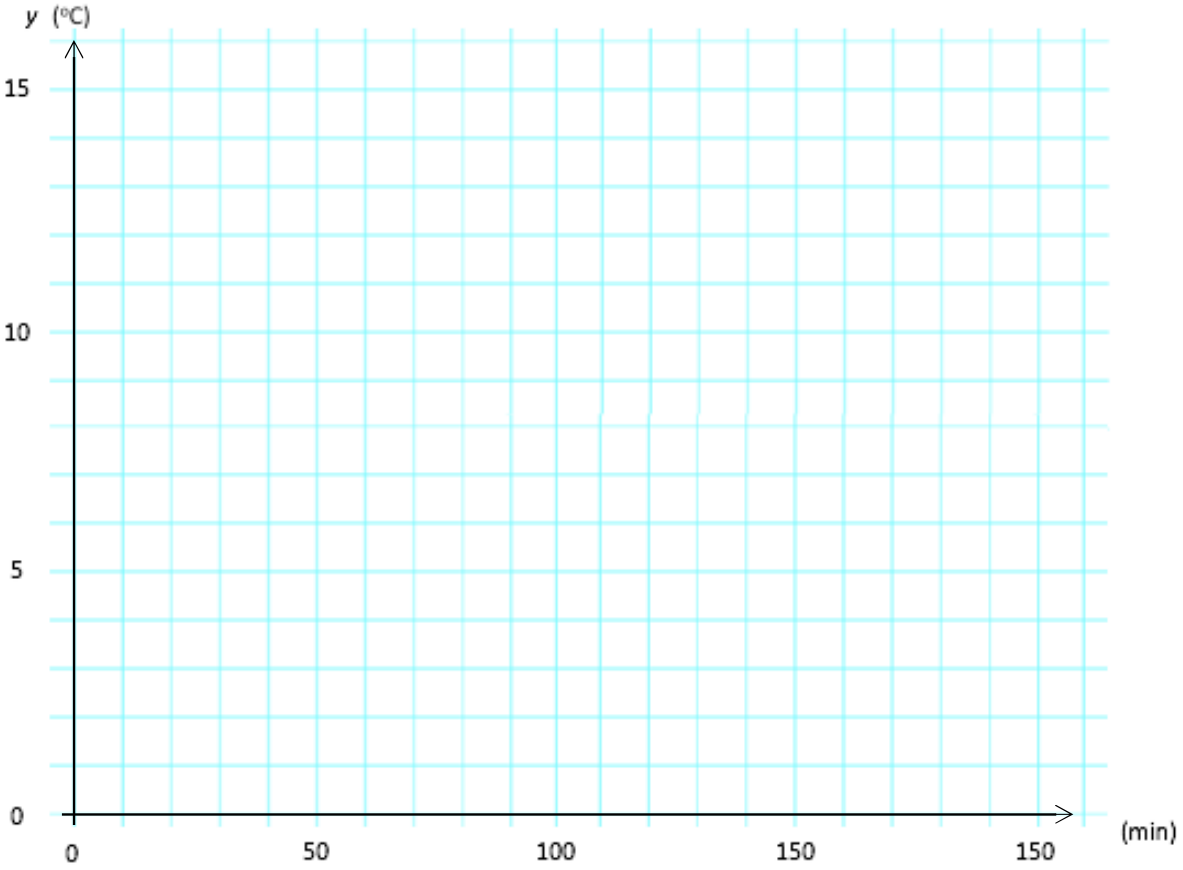
Results of the experiment, when the outside temperature was 28°C.

Time (min)	20	40	60	80	100
Temperature °C	8.2	9.1	10.1	11.1	12.2



Jon

If we compare the changes in temperature every 20 minutes, we could draw a graph...



1. In the question, what can we say about the relationship between time and the temperature? Discuss what you noticed or observed

Since the points that were plotted on the previous page appear to form a line, we can assume y is a linear function of x.

2. Answer the following questions assuming that the points plotted on the previous page will fall on the line that goes through (30, 8.8) and (110, 13.6).

   a) Find the slope of the line. Also, what does the slope of the line represent?

   b) Write the equation of the line for the temperature, y^{o}C, at x minutes.

   c) After about how may minutes will the temperature of the drink reach 20^{o}C?

As shown above, sometime we can grasp the way 2 quantities change by assuming the results obtained from an experiment in a linear functional relationship.

3. Amber wondered how long it would take for a candle to burn down. She measured the length of time it took until the length of the remaining candle was 70mm, 65mm, 60mm and 55mm.

Remaining length (mm)	70	65	60	55
Time (min)	38	63	89	113

a) Predict how much longer the candle will burn

b) About how long was the original length of the candle?



Alice

About how long does it take to burn 5cm?

I wonder if we can think about it by drawing a graph?



Evie



Sarah

I wonder if we can use an equation?

Exercises

1. More water will be poured at a constant rate into a water tank that is filled with 2L. After 3 minutes of pouring water, the total amount of water in the tank was 11L.

a. How much does the water increase per minute?

b. If the total amount of water is yL, x minutes after the pouring began, write the equation of y in terms of x.

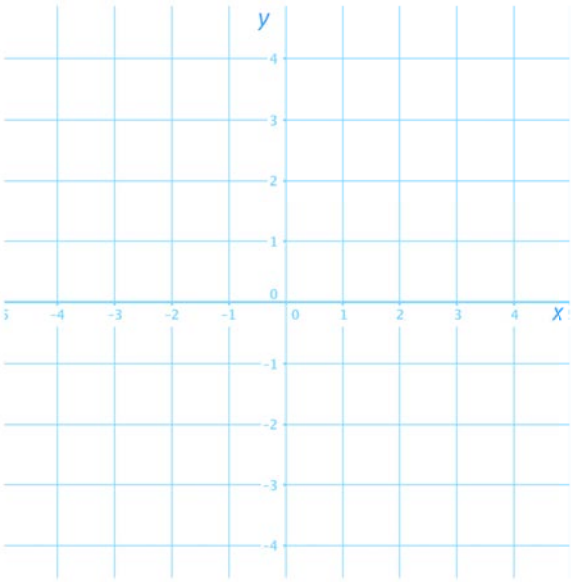
2. Name the rate of change of linear function $y = 4x + 1$.

3. Name the slope and y-intercept of the linear function, $y = 5x - 3$

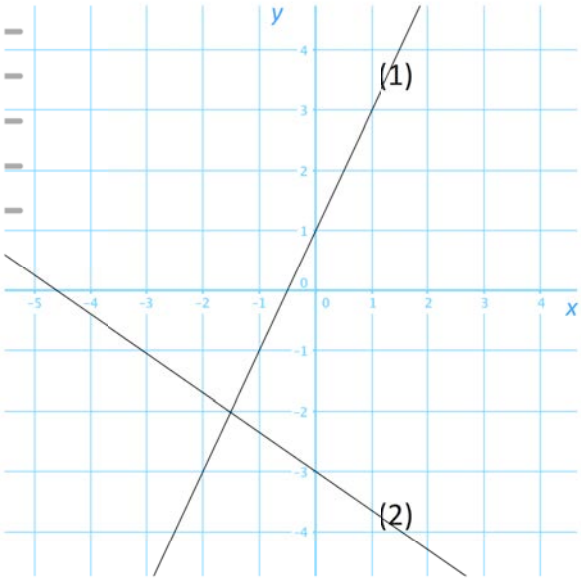
4. Graph each of the following linear functions

a. $y = 3x - 4$

b. $y = \frac{1}{3}x + 2$



5. Write the equations of lines (1) and (2) shown below.



6. Write an equation of the linear function that will satisfy each of the following conditions.

a) The rate of change is 2, and if $x = 1$, $y = -1$

b) The graph will go through (-3,3) and (3, 5)

Graphs of linear equations with two variables

Sometimes we are given a linear equation containing two variables x and y, that is not written as a function.

For example, $x + 2y - 2 = 0$.

Let's think about how would can find pairs of values of x and y that satisfy this equation.

Record your thoughts in a table.

x	...	-4	-3	-2	-1	0	1	2	3	4	...
y	...	3	2.5	2	1.5	1	0.5	0	-0.5	-1	...

If we plot the corresponding points, the graph resembles Fig.1 below. We could, of course, generate an infinite number of points satisfying our equation. The more points we plot, the closer the graph resembles Fig.2 below.

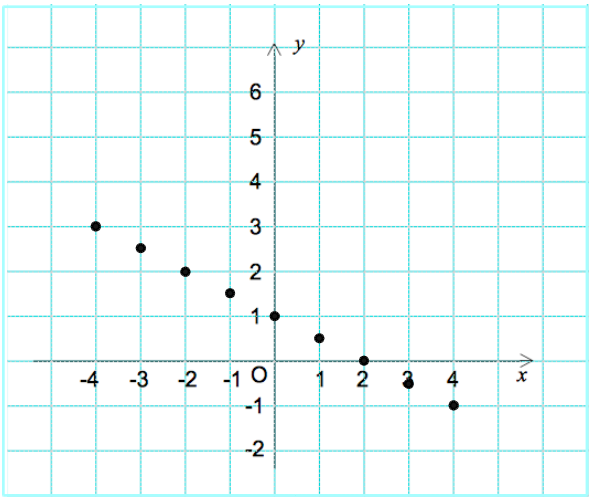


Fig.1

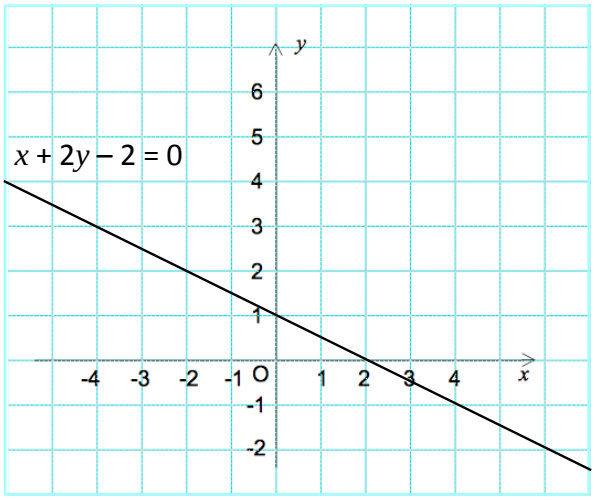


Fig.2

If we fix the value of x, then the corresponding value of y is also fixed. Consequently we can still say y is a function of x.

Functions written in the form $y = mx + c$, where y is the subject of the formula, are called **explicit functions**.

Functions written in the form $ax + by + c = 0$ are called **implicit functions**.



Matthew

But we could rewrite our implicit function and make it explicit!

19

Can you explain Matthew's idea? Why would this help?

Graphs of linear equations with two variables



Sarah

Sarah's working

$x + 2y - 2 = 0$

So $x + 2y = 2$

And $2y = -x + 2$

So $y = -\frac{1}{2}x + 1$

The gradient is $-\frac{1}{2}$ and the y-intercept is 1

Which agrees with the graph.

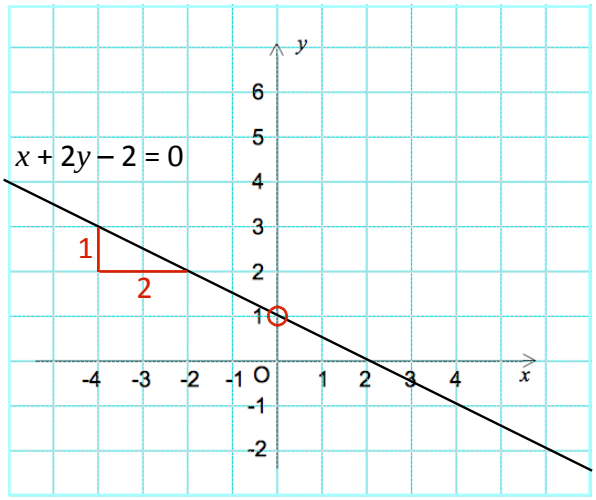


Fig.2

Fitness check

1. Draw the graph of the equation $2x + y - 2 = 0$ for a suitable range of values.
2. From your drawing, write down the gradient and the y-intercept of your graph.
3. Rearrange the implicit function into an explicit function to check your results.
4. Using the same axes, plot the graph of $2x + y - 3 = 0$. What do you notice?
5. Using the same axes, plot the graph of $2x + y = 0$. What do you notice about all three graphs?
6. Can you write down the equations of two more graphs that are parallel to $2x + y - 2 = 0$?
7. Write down the equations of 3 graphs that are parallel to $3x + y = 0$.

20

Can you explain how to write down the equations of parallel graphs if you are given an implicit function? And can you explain why this works?

Graphs of linear equations with two variables



Alice

Alice's working

The equation $2x + y = 0$ can be rewritten as $y = -2x$

This is a line through the origin with gradient -2.

Any graph with gradient -2 will be parallel to $y = -2x$

For example $y = -2x + 3$

I can rewrite this as $2x + y - 3 = 0$

Similarly $2x + y + 2 = 0$ will be parallel.

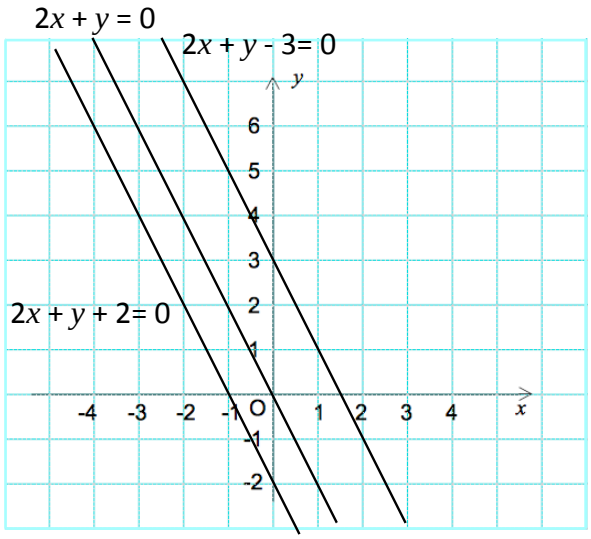


Fig.3



Jon

Jon's work

In general explicit functions can be written as $ax + by + c = 0$.

This can be written as $by = -ax - c$

$$y = -\frac{b}{a}x - \frac{c}{a}$$

So the gradient only depends on the values of $b \div a$.

By changing c we can get a series of parallel lines.

Fitness check

Using Jon's idea, match the parallel lines from this list.

A)	$2x + 3y - 2 = 0$	E)	$x + 1.5y - 2 = 0$
B)	$2x + 5y - 2 = 0$	F)	$4x + 3y + 1 = 0$
C)	$4x + 3y - 2 = 0$	G)	$x + 2.5y + 3 = 0$
D)	$4x + 6y + 2 = 0$	H)	$x + 1.5y + 5 = 0$

What's in the box?



Matthew

I've got an interesting puzzle for you to solve.
I've bought in three boxes that all contain some sweets. You have to find out how many sweets are in each box – without opening them.
But I will give you some clues and you might think about using algebra to help you. I will reveal one clue at a time.



The first clue

I don't know how many sweets are in the first box.



Alice

That's not much of a clue!



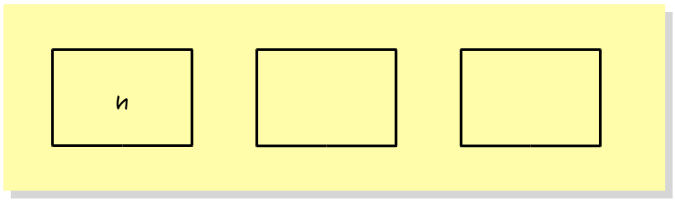
Sarah

But Matthew did say we should use algebra. So I am going to put n in the first box where n could be any number.



Sarah

Sarah's work



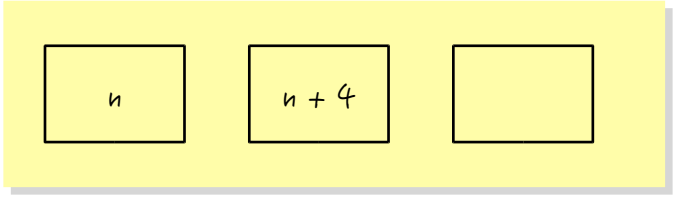
The second clue

There are 4 more sweets in the second box than in the first box.



Sarah

Sarah's work



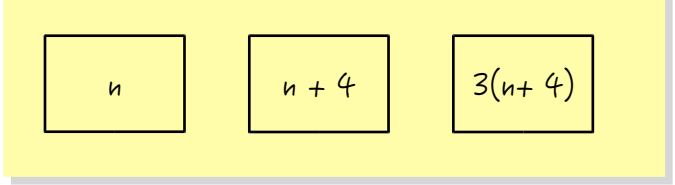
The third clue

There are 3 times as many sweets in the third box than in the second box.



Sarah

Sarah's work



What's in the box?

The fourth clue

Altogether there are 92 sweets.

How many are in each box?



Sarah

Sarah's work

n + $n + 4$ + $3(n+4)$ = 92

$n + n + 4 + 3(n + 4) = 92$

But what do I do with the bracket?

21 Can you help Sarah out? How do you expand brackets?



Amber

Easy

$3(n + 4) = n + 4 + n + 4 + n + 4$

$= 3n + 12$

Even easier. Use the distributive law.

$3(n + 4) = 3n + 12$



Jon



Evie

What's the distributive law?

22 Can you explain how the distributive law works?



Alice

Alice's explanation

$3(n + 4)$ means 3 lots of $n + 4$

Thats 3 lots of n and 3 lots of 4

So $3n + 12$.

You just multiply out term by term

What's in the box?

Solving the equation

Sarah's work



Sarah

n + $n + 4$ + $3(n+4)$ = 92

$n + n + 4 + 3(n + 4) = 92$

$n + n + 3n + 12 = 92$

$5n + 12 = 92$

Now what do I do?

23

Can you solve this two step equation?
Explain your steps.

Amber's working



Amber

$5n + 12 = 92$ — Isolate the "unknown"

$-12 \quad -12$ — So take away 12 from both sides.

$5n = 80$

$\div 5 \quad \div 5$ — Divide both sides by 5

$n = 16$



Jon

So in the first box there are 16 sweets.

… and 20 in the second box



Evie



Alice

… and 3 x 20 = 60 in the third box.

Checking ..,
16 + 20 + 60 = 92
That's correct.



Sarah

24

Now try making up your own "reveal a clue" question.
But be careful. It isn't quite as easy as it looks.

Reveal a Clue



Evie

Evie's question – The Singing Hamsters



There are four groups of singing hamsters
I don't know how many hamsters are in the first group
But there are 2 less in the second group.
There are twice as many hamsters in the third group as the second group.
There are 3 more in the fourth group than in the first group.
Altogether there are 37 hamsters.
How many hamsters are in each group?

25 Can you solve Evie's question?



Jon

Jon's solution

$$\boxed{n} + \boxed{n-2} + \boxed{2(n-2)} + \boxed{n+3} = 37$$

$n + n - 2 + 2(n - 2) + n + 3 = 37$ (Expand the bracket)

$n + n - 2 + 2n - 4 + n + 3 = 37$ (Simplify)

$5n - 3 = 37$ (Isolate the unknown)

$+3 \quad +3$

$5n = 40$

$\div 5 \quad \div 5$

$n = 8$

So 8 in group 1, 6 in group 2, 12 in group 3 and 11 in group 4

Check $8 + 6 + 12 + 11 = 37$

Correct.

Solving equations

Finding the solution to an equation is referred to as solving equations. In order to solve an equation, we need to change the original equation into the form, $n = \square$. Generally we do this one step at a time.

The "equals" sign means that both sides have the same value.

In order to maintain this equality, whatever we do to one side we must also do the same to the other side.

Properties of equalities

1. If we add the same number or expression to both sides of an equality, then it remains an equality.
2. If we subtract the same number or expression to both sides of an equality, then it remains an equality.
3. If we multiply both sides of an equality by the same number, then it remains an equality.
4. If we divide both sides of an equality by the same number, then it remains an equality.
5. If both sides of an equality are switched, it remains an equality.

Let's solve equations using the properties of equalities.

Solve these equations

1. $2n + 3(n + 1) = 78$
2. $5n + 2(n - 1) = 61$
3. $3n + 4(n + 1) + 2(n - 5) = 75$
4. $6n + 5(2n - 5) - (n - 2) = 82$
5. $3n + 5(n + 2) + 2(n - 1) = 118$
6. $2n + 4(n - 2) + 6(n + 2) = 88$
7. $4n + 3(2n + 1) + 3(n - 2) = 114$
8. $8n + 2(n - 5) - 4(n - 3) = 74$



Sarah

Sarah's answer to Q4

$6n + 5(2n - 5) - (n - 2) = 82$ Expand the brackets

$6n + 10n - 25 - n + 2 = 82$ Simplify

$15n - 23 = 82$ isolate the unknown, add 23 to both sides

$15n = 105$ divide both sides by 15

$n = 7$

26

In Sarah's solution, can you explain where the "$-n + 2$" came from?

Solving equations

9. $\frac{3x-7}{5} = 13$

10. $\frac{2(5x+3)}{7} = 28$

11. $\frac{4x+9}{3} = 35$

12. $\frac{3(8x-4)}{11} = 36$

Matthew's answer to Q9



Matthew

$\frac{3x-7}{5} = 13$ Simplify by clearing the fractions

$3x - 7 = 5 \times 13$ Multiply both sides by 5

$3x - 7 = 65$ Isolate the unknown, add 7 to both sides

$3x = 72$ Divide both sides by 3

$x = 24$



Evie

What happens if the "unknown" occurs more than once?

13. Solve $3x + 7 = 72 - 2x$

Jon's solution to Q13



Jon

You just use the rules of equalities.

Firstly you have to get the unknown terms on one side.

$3x + 7 = 72 - 2x$ Add $2x$ to both sides

$5x + 7 = 72$ Isolate the unknown, take away 7 from both sides

$5x = 65$ Divide both sides by 5

$x = 13$

Solve

14. $2(3a - 4) + 13 = 5(2a - 7)$

15. $7(9 - 2b) + 6 = 8 - (5b + 2)$

16. $\frac{3c+1}{2} = 28 - c$

17. $6(2d - 5) + 9 = 5(7 + d)$

18. $3(4 - 3e) + 16 = 7e - 2(e + 14)$

19. $\frac{3f+1}{2} = \frac{18-f}{3}$

Solving equations

Alice's solution to Q16



Alice

$$\frac{3c+1}{2} = 28 - c$$

$3c + 1 = 2(28 - c)$	Multiply both sides by 2
$3c + 1 = 56 - 2c$	Expand the bracket
$5c + 1 = 56$	Add 2c to both sides
$5c = 55$	Take away 1 from both sides
$c = 11$	Divide by 11

Jon's solution to Q19



Jon

$$\frac{3f+1}{2} = \frac{18-f}{3}$$

$\frac{6(3f+1)}{2}$	$= \frac{6(18-f)}{3}$	Multiply both sides by 6
$\frac{\cancel{6}^{3}(3f+1)}{\cancel{2}_{1}}$	$= \frac{\cancel{6}^{2}(18-f)}{\cancel{3}_{1}}$	Cancel both sides
$3(3f+1)$	$= 2(18-f)$	Expand the brackets
$9f + 3$	$= 36 - 2f$	Add 2f to both sides
$11f + 3$	$= 36$	Take away 3 from both sides
$11f = 33$		Divide both sides by 11
$f = 3$		

Just one more ...

20. Solve $\frac{3x-4}{7} = \frac{x+6}{3}$

Inequalities

Earlier we looked at forming equations through clue sharing.
We are now going to use this idea again to expression **inequalities**.

Look at these clues.
Clue 1 - I have 3 boxes each containing some sweets.
Clue 2 - Altogether I have 18 sweets.
How many sweets could be in the first box?




Sarah

Sarah's working

$\square_{n} + \square + \square = 18$

I must have at least 1 sweet in each box, so $n \geq 1$
And the most I can have in the first box is 16, so $n \leq 16$
I could write that as $1 \leq n \leq 16$

Inequalities

An **inequality** restricts the value of our unknown. In the box question, n could be any integer between 1 and 16.

And we write that as $1 \leq n \leq 16$

Another clue

Clue 3 – There are the same number of sweets in the second box as the first box.
How many sweets could be in the first box? Express you answer as an inequality.


Sarah

Sarah's working

$\square_{n} + \square_{n} + \square = 18$

I could still have one sweet in the first box, so $n \geq 1$
But the most I could have is now 8, which would leave 2 in the third box. So $n \leq 8$
I could write this as $1 \leq n \leq 8$

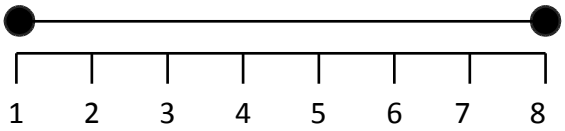
Inequalities

Representing inequalities with a diagram

In our sweet question we know that the number of sweets, n, in the first box satisfies the inequality $1 \leq n \leq 8$.

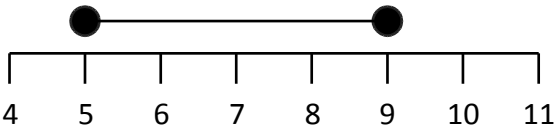
We also known n must be a whole number, so n could be 1, 2, 3, 4, 5, 6, 7 or 8.

Sometimes it is helpful to express this information in a diagram like the one below..



Interpreting inequality diagrams

Look at this diagram. Interpret the diagram in an inequality for n.



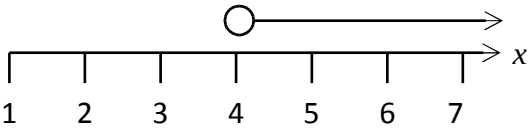
More than … less than ...



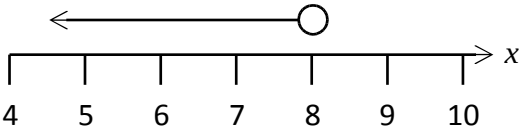
Sarah

If I what to say x is bigger than 4, I know I write that as $x > 4$ but how would I represent that on a diagram?

If we want to show that x is greater than 4, but not equal to 4, we use an empty circle. If there is no upper bound, we put an arrow pointing in the positive direction as shown below.



Now look at this diagram. What do you think it shows?



Inequalities

More than … less than ...

We can use inequalities to clarify rounding.

For example, suppose the length of a caterpillar is 9cm to the nearest cm.



How can we express this as an inequality?

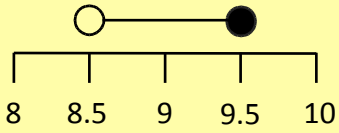
How can we express this diagrammatically?



Matthew

Matthew's working

The caterpillar must be more than 8.5cm long but less than or equal to 9.5cm.
So the length, L, must satisfy this inequality.
$8.5 < L \leq 9.5$
I didn't shade in the circle on the left to show that the length can't actually be 8.5,



Fitness check

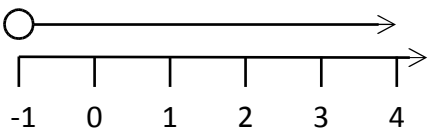
Represent these inequalities on a number line

1. $x < 7$
2. $3 < x < 7$
3. $3 \leq x \leq 7$

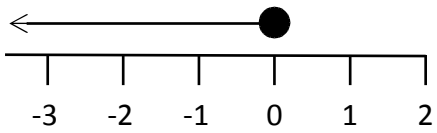
1. $8 < x < 13$
2. $-2 \leq x < 2$
3. $-3 \leq x \leq 5$

Express the information on these number lines as inequalities.

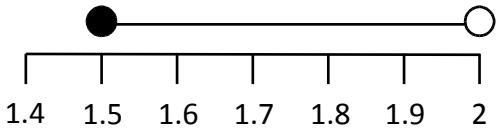
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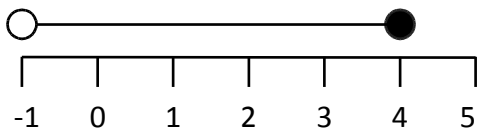
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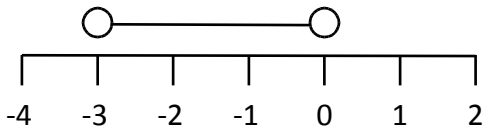
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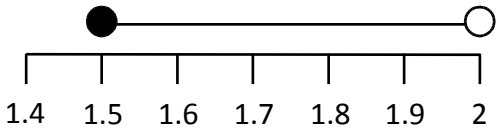
10.



11.



12.



Working with inequalities

Suppose x is a positive integer and $2x - 5 < 13$, what can we say about the value of x?



Alice

Alice's working

I think it will work like an equality. So

$2x - 5 < 13$ (Add 5 to both sides)

$2x < 18$ (Now divide by 2)

$x < 9$

But x is a positive integer, so $1 \leq x \leq 8$

Alice is right, inequalities behave in a similar way to equations – with one or two exceptions!

Properties of inequalities

1. If we add the same number or expression to both sides of an inequality, then it remains an inequality.
2. If we subtract the same number or expression to both sides of an inequality, then it remains an inequality.
3. If we multiply both sides of an inequality by the same **positive** number, then it remains an inequality.
4. If we divide both sides of an equality by the same **positive** number, then it remains an inequality.
5. If both sides of an equality are switched, the **inequality sign is reversed**.

27

Can you explain why rules 3, 4 and 5 are slightly different to the rules of equalities?

Fitness check

Solve these inequalities

1. $4(3x + 2) \geq 20$
2. $2(5x - 4) \leq 4x + 16$
3. $5(4x + 3) \geq 13x + 64$
4. $3(x - 8) \leq 5x - 6$
5. $26 - 2x \geq 9$
6. $2(3 - x) \leq -4$
7. $5(8 - 3x) \geq 10 + 5x$
8. $7(6 - 4x) \leq 15 - 19x$

Expanding brackets

The distributive Law

You may recall from your earlier work that it is often useful to expand brackets.
For expand $5(2x - 3y)$


Evie

Evie's explanation

To work out $5(2x - 3y)$
You just multiply each term in the bracket by 5.

$5(2x - 3y) = 5 \times 2x + 5 \times 3y = 10x - 15y$

The distributive law

Evie is actually using a very important idea in Algebra called the **distributive law**.

That is that $a(b + c) = ab + ac$

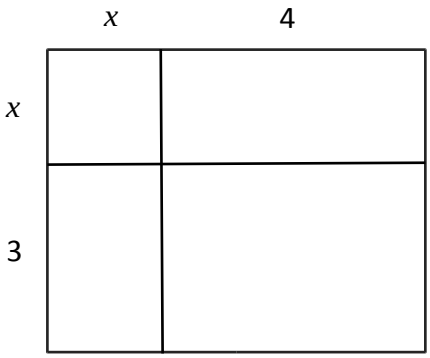
Fitness check

Expand the brackets in these expressions

1. $3(4a - 5b)$
2. $4(7c + 2y)$
3. $-2(9e + 3f)$
4. $-(5g - 2h)$

Expanding double brackets

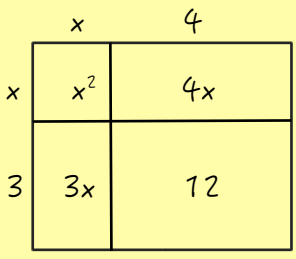
Look at this diagram. The lengths of the sides are $x + 4$ and $x + 3$.
Write down two different expressions for the area of the whole shape .




Sarah

Sarah's working

The area of a rectangle is length x width
So one expression for the area is $(x + 4)(x + 3)$
But if we split the shape into 4 boxes, the area is
$x^2 + 4x + 3x + 12 = x^2 + 7x + 12$
So
$(x + 4)(x + 3) = x^2 + 7x + 12$



Expanding brackets

Use Sarah's method to expand the following brackets.

1. $(x + 5)(x + 2)$
2. $(2x + 3)(x + 4)$
3. $(x - 2)(x - 3)$
5. $(x + 2)(x + 2)$
6. $(x - 1)(x + 1)$
7. $(2x + 1)(2x + 1)$



Alice

You could use the distributive law twice.

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Can you explain what Alice might be thinking?



Alice

Alice's idea

Instead of $(2x + 3)(x + 4)$ *we could write*

$2x(x + 4) + 3(x + 4)$

$= 2x^2 + 8x + 3x + 12$

$= 2x^2 + 11x + 12$

Use Alice's method to expand

1. $(x + 4)(x + 5)$
2. $(2x + 1)(x + 3)$
3. $(x - 2)(x - 1)$
4. $(x + 3)(x + 3)$
5. $(x + 2)(x - 2)$
6. $(3x - 2)(3x - 2)$



Matthew

I've got an even better idea.

29

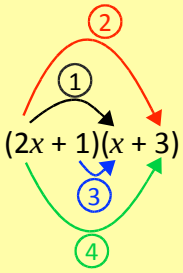
Can you explain what Matthew might be thinking?

Expanding brackets



Matthew

Matthew's idea



Multiply in pairs and then just write down the 4 terms you get.

$2x^2 + 6x + x + 3 = 2x^2 + 7x + 3$

So $(2x + 1)(x + 3) = 2x^2 + 7x + 3$

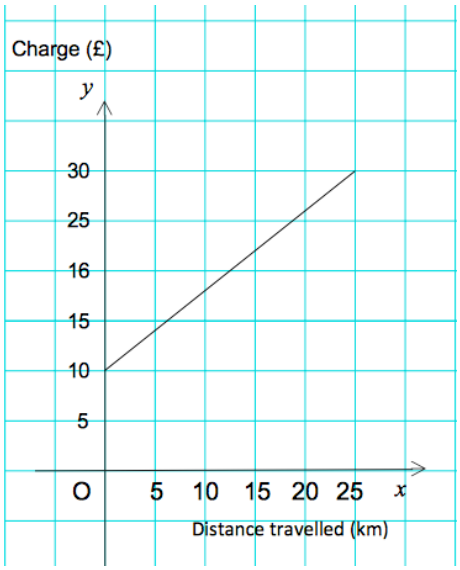
Use Matthew's method to expand

1. $(x + 1)(x + 7)$
2. $(2x + 3)(x + 2)$
3. $(x - 3)(x - 5)$
4. $(x + 5)(x + 5)$
5. $(x + 5)(x - 5)$
6. $(4x - 1)(4x - 1)$

Algebra 4b Mastery

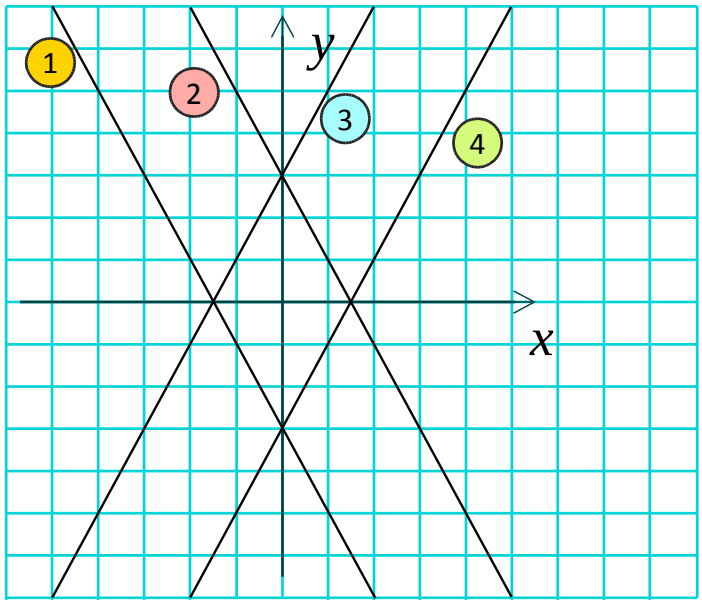
1. This graph shows how a taxi company works out how much to charge a customer.

   What is the additional cost for every km travelled?



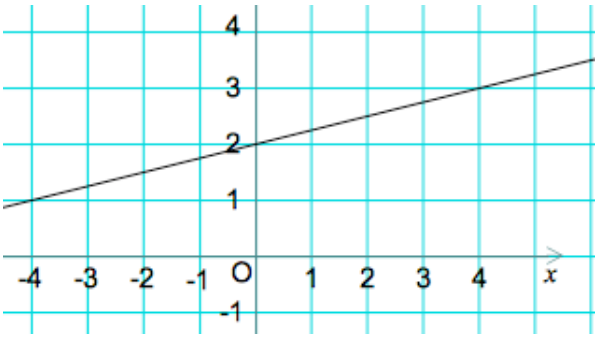
A	B	C	D
£1.25	£1	80p	£1.20

2. Which of these could be the graph of $y = -2x + 3$



3. If you were to plot this line in the form $y = mx + c$

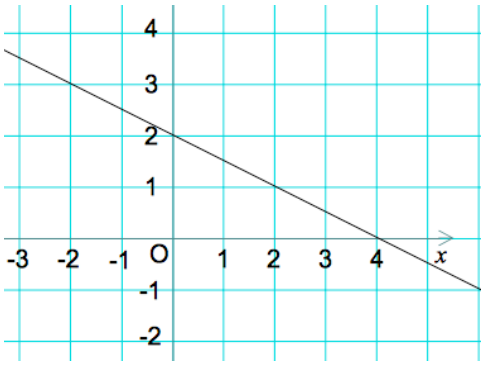
   What is the value of m?



A	B	C	D
2	4	$\frac{1}{4}$	$-\frac{1}{4}$

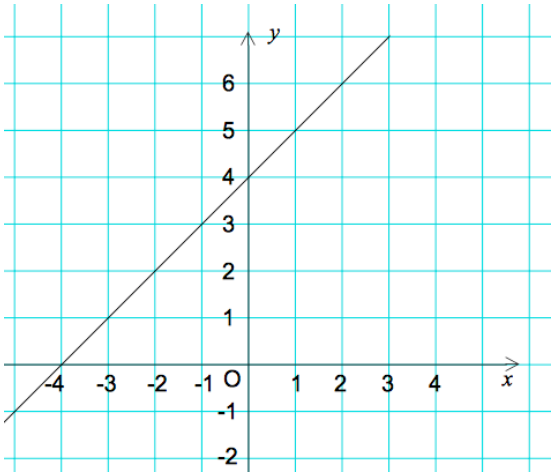
4. If you were to plot this line in the form $y = mx + c$
   What is the value of c?

A	B	C	D
2	4	$\frac{1}{2}$	$-\frac{1}{2}$



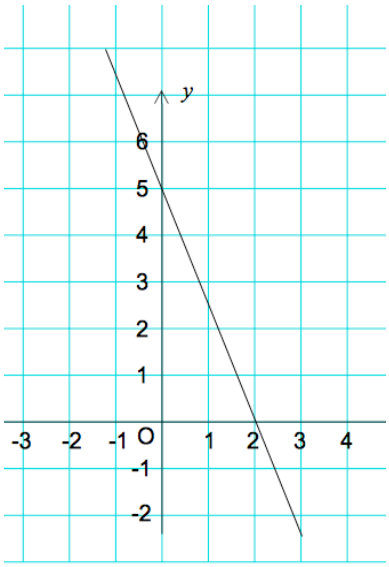
5. What is the equation of this line?

A	$y = 4x + 4$
B	$y = 4x - 4$
C	$y = x + 4$
D	$y = x - 4$



6. What is the gradient of this line?

A	B	C	D
$\frac{5}{2}$	$-\frac{5}{2}$	$\frac{2}{5}$	$-\frac{2}{5}$



7. What is the gradient of the line that passes through (-2, -4) and (5, -1)

A	B	C	D
$\frac{7}{5}$	$\frac{3}{5}$	$\frac{3}{7}$	$-\frac{3}{7}$

8. What is the equation of a line with gradient 3 that passes through the point (5, 17)?

A	$y = 3x + 5$	B	$y = 3x + 17$
C	$y = 3x + 2$	D	$y = 3x - 5$

9. What is the equation of a line that passes through the points (-3, 5) and (-1,1)?

A	$y = 2x + 3$	B	$y = -2x - 1$
C	$y = 2x - 3$	D	$y = -2x + 11$

9. The line passing through the points (-1,3) and (3,k) has gradient $\frac{3}{2}$.
What is the value of k?

A	B	C	D
3	6	7	9

10. Which of these points is not on the line $2x + 3y = 12$?

A	B	C	D
(3,2)	(2,3)	(-3,6)	(9,-2)

11. What is the gradient of the line $2x + 3y = 12$?

A	B	C	D
$\frac{3}{2}$	$-\frac{3}{2}$	$\frac{2}{3}$	$-\frac{2}{3}$

12. Which of these lines is not parallel to the line $2x + 3y = 12$?

A	B	C	D
$2x + 3y = 5$	$3y = -2x + 8$	$2x = -3y + 9$	$3x - 2y = 12$

13. Which of these equations is correct if

$2(3n + 4) – (3n – 2) = 67$

A	B	C	D
$3n – 4 = 67$	$9n – 4 = 67$	$3n + 10 = 67$	$9n + 10 = 67$

14. Which of these equations is correct if

$5(2n - 3) – 3(2n –7) = 94$

A	B	C	D
$4n + 6 = 94$	$9n – 36 = 94$	$4n - 36 = 94$	$16n + 6 = 94$

15. Which is the correct next step in solving this equation

$3x – 4 = x + 8$

A	B	C	D
$4x – 4 = 8$	$2x – 4 = - 8$	$3x – 4 = 8$	$2x – 4 = 8$

16. Which of these equations is correct if

$$\frac{2x - 1}{7} = x - 3$$

A	B	C	D
$2x – 1 = 7x - 3$	$2x – 1 = 7x - 21$	$14x – 7 = 7x - 21$	$2x – 1 = x - 10$

17. Which of these equations are correct if

$$\frac{2x - 1}{3} = \frac{x + 2}{2}$$

A	B	C	D
$2x – 1 = x + 2$	$6x – 3 = 2x + 2$	$4x – 2 = 3x + 6$	$12x – 6 = 6x + 12$

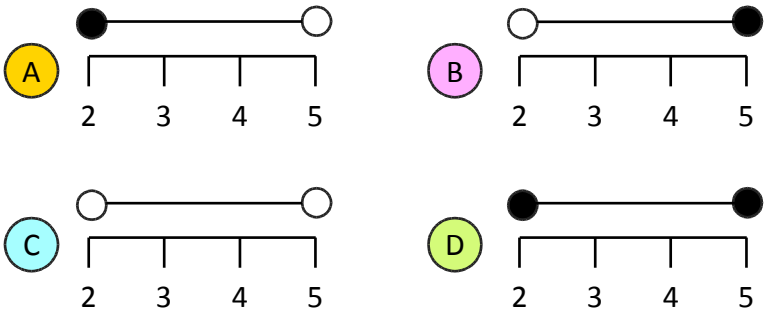
18. Which of these equations are correct if

$$\frac{4x - 2}{3} + \frac{3x - 1}{2} = 13$$

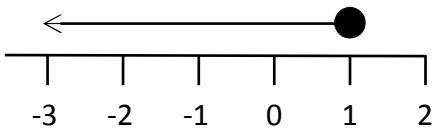
A	$4x - 2 + 7x - 3 = 78$
B	$24x - 12 + 18x - 6 = 78$
C	$8x - 4 + 9x - 1 = 13$
D	$8x - 4 + 9x - 1 = 78$

19. Which of these is the correct representation of the inequality

$$2 < x \leq 5$$



20. Which of these inequalities is represented by this diagram?



21. Which of these is a solution of the inequality

$$3(2x - 4) \leq 24$$

A	B	C	D
$x < 5$	$x \leq 1$	$x \leq 5$	$x \leq 10$

21. Which of these is a solution of the inequality

$$5 - 2x < 17$$

A	B	C	D
$x < -6$	$x > -6$	$x < 11$	$x > 11$

22. Which of these is a solution of the inequality

$$5(x - 4) \leq 3x - 8$$

A	B	C	D
$x \leq -14$	$x \leq -2$	$x \leq -4$	$x \leq 6$

23. Expand ...

$$-(2n - 1)$$

A	B	C	D
$2n - 1$	$-2n - 1$	$2n + 1$	$-2n + 1$

24. Expand ...

$$-2x(3x + 4)$$

A	B	C	D
$6x^2 - 8$	$-6x^2 + 4$	$-6x^2 - 8x$	$3x^2 + 4 - 2x$

25. Factorise fully ...

$$56x - 28$$

A	B	C	D
$7(8x - 4)$	$4(14x - 7)$	$28(2x - 1)$	$8(7x - 2)$

26. Factorise fully ...

$$-8y + 4$$

A	B	C	D
$4(-2y - 1)$	$-4(2y - 1)$	$-4(y + 1)$	$-2(4y - 2)$

27. What goes in the shaded box

$$(2x + 3)(x + 2)$$

	$2x$	3
x		
2		

A	B	C	D
5	6	$7x$	$6x$

28. Expand …

$(x + 4)(x + 2)$

A	B	C	D
$x^2 + 6x + 6$	$x^2 + 8x + 8$	$6x^2 + 6x + 8$	$x^2 + 6x + 8$

29. Expand …

$(x - 4)(x + 1)$

A	B	C	D
$x^2 - 5x - 4$	$x^2 - 3x - 4$	$x^2 + 5x - 4$	$x^2 - 3x - 5$

30. Expand …

$(x - 5)(x + 5)$

A	B	C	D
$x^2 - 10x - 25$	$x^2 + 10x + 25$	$x^2 + 25$	$x^2 - 25$

31. Expand …

$(x - 3)(x - 3)$

A	B	C	D
$x^2 - 6x + 9$	$x^2 + 6x + 9$	$x^2 – 6x + 9$	$x^2 – 9x + 6$

32. Expand …

$(2y + 1)(3y - 2)$

A	B	C	D
$6y^2 - 7y - 2$	$5y^2 - y - 2$	$6y^2 + y - 2$	$6y^2 - y - 2$

33. Expand …

$(2x + 5)(2x + 5)$

A	B	C	D
$4x^2 + 10x + 25$	$4x^2 + 20x + 25$	$4x^2 + 20x + 10$	$4x^2 + 14x + 25$