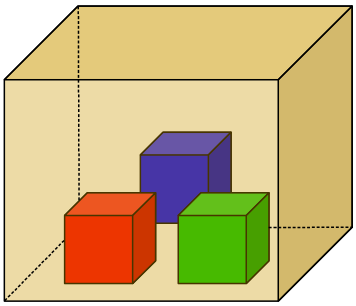
What are the chances of getting two reds?

A box contains a red cube, a blue cube and a green cube.
One cube is randomly drawn out, the colour recorded and then put back into the box.

A second cube is now drawn out and the colour recorded.

What are the chances of getting two reds?.





Matthew

I would think it is about one in six.

I don't know but we could just try it out for ourselves.



Evie

1 Why do you think Matthew said “one in six”?



Mr T

I think we should do as Evie suggested. So I have got a box for each of you. Do the experiment 10 times and record your results.



Alice

I got RR, BB, RB, BB, BG, BB, GG, BR, BB,GG.
So I only got RR once.

I got RG, GB, GB, BR, BB, RR, BB, BR, RR, BB.
So I only got RR twice..



Jon



Sarah

Well I got GB, BG, BB, GG, BR, GB, RG, BB, GR, BR.
So I didn't get RR at all.



Mr T

What we need to do is collect everyone's results.

What are the chances of getting two reds?

The group's results

Outcome	Frequency	%
Two reds	40	11.4%
Two blues	39	11.1%
Two greens	37	10.6%
One red, one blue	86	24.6%
One red, one green	79	22.6%
One blue, one green	69	19.7%
Total	350	

2

Can you now explain why Matthew said “one in six”? But why was he wrong?



Sarah

There does only seem to be six different outcomes which is why Matthew said, “One in six”.

But our results show that these six outcomes are not equally likely to occur.



Taro



Amber

It looks like the chances of getting two reds is about one in nine.

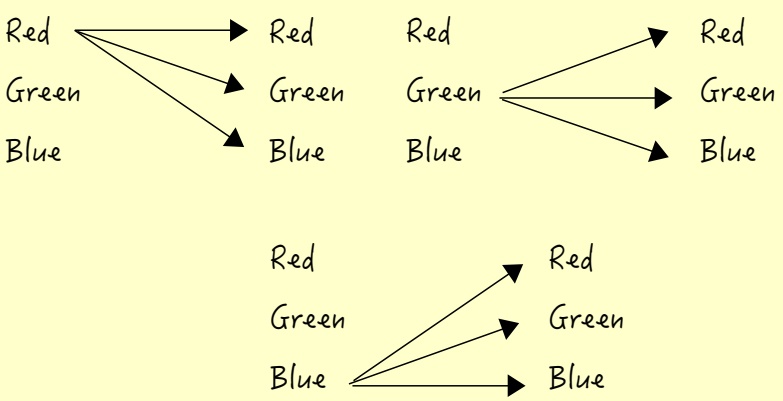
3

Can you make a full list of the outcomes, including the order the cubes were chosen?

Amber's working



Amber



They are nine possible outcomes.



Matthew

Which is why the chances of getting two reds is one in nine.



Evie

And why the chances of getting red and green in any order is two in nine.



Mr T

In probability experiments, it is really important that you list all of the possible outcomes in order!

4 In our next experiment we will be tossing two coins. How many different outcomes are there?

"Horse Racing"

In this race, three horses race along a 6 furlong straight track. The horses are called "Two Heads", "Two Tails", and "One of Each". To decide which horse moves, you toss two coins. The winner is the first to reach the winning post and that takes 6 moves.



	1	2	3	4	5	6
Two Heads						
Two Tails						
One of Each						

5 Pick your favourite. Place your bets, Play the game.



Jon

Jon's idea



Two heads

Two tails

One of each

So 3 outcomes and they all have an equal chance.

Alice's idea



Two heads

Two tails

One of each

One of each

So 4 outcomes and "One of each" occurs twice and should win over 6 furlongs.

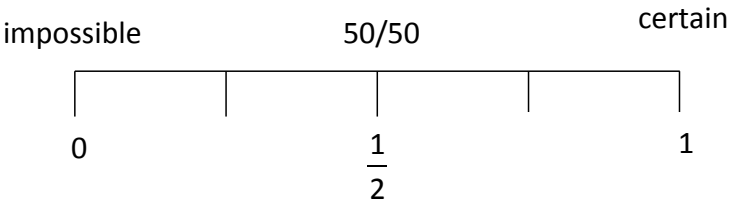


Alice

6 They can't both be right! Who do you think is correct? Explain your reasoning.

Talking about experimental probability

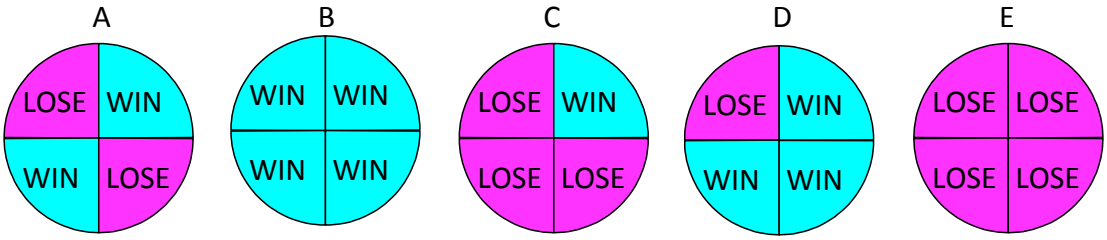
In Probability 2, we introduced the idea of a probability scale. If you are unfamiliar with this scale then look back at Probability 2.





Miss F

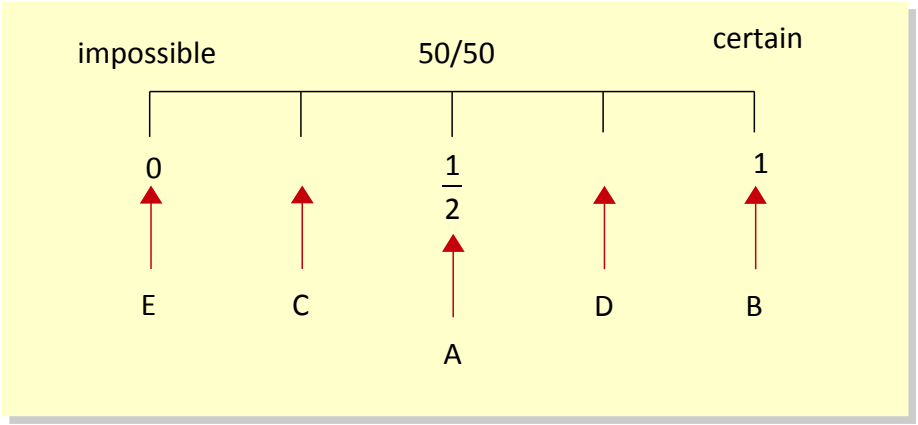
This scale is often referred to as the probability line.
Look at these spinners, mark the chances of you winning with each spinner on the probability line.



Matthew's solution



Matthew





Miss F

In this case the probability line shows the theoretical probability of you winning in each case. This doesn't mean that in an experiment with spinner A, you would actually win half of the time but it does mean in the long run you will win roughly half the time.

7

Construct your own spinner A, and spin it 20 times.
Record how many times you win.

Relative frequency





Amber

In my 20 goes, I won 8 times and lost 12 times.

In my 20 goes, I won 11 times and lost 9 times.



Taro



Alice

In my 20 goes, I won 6 times and lost 14 times.

In my 20 goes, I won 13 times and lost 7 times.



Jon

Putting it altogether



Miss F

We get a much better estimate of the experimental probability when we put the results together. The more results we have, the more reliable our experimental result becomes.

Name	Win	Lose	Totals
Amber	8	12	20
Taro	11	9	20
Alice	6	14	20
Jon	13	7	20
Totals	38	42	80

Relative frequency

When an experiment can be repeated, then the **experimental probability** of an event happening is the proportion of times the event happened in the long run.

The **relative frequency** of an event happening is equal to the number of times the event happens ÷ the total number of trials.

In the example above the experimental probability of winning could be written as

$$P(\text{win}) = \frac{38}{80} = 0.475$$

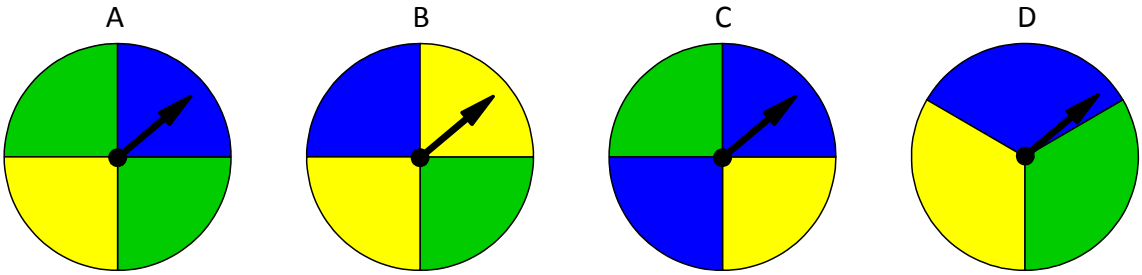
And the experimental probability of losing could be written as

$$P(\text{lose}) = \frac{42}{80} = 0.525$$

The **law of large numbers** says that in the long run the experimental probability or relative frequency will get closer and closer to the **theoretical probability**.

In this case the relative frequency of winning is already getting close to 0.5 which is the theoretical probability.

Let's think about designing spinners



Sarah, Taro and Alice all used the same spinner in a game. Here are their results.


Sarah

Colour	Tally marks	Total
Green	///	3
Blue	𝍸 ///	8
Yellow	𝍸 ////	9


Taro

Colour	Tally marks	Total
Green	𝍸 /	6
Blue	///	3
Yellow	𝍸 𝍸 /	11


Alice

Colour	Tally marks	Total
Green	𝍸 ///	8
Blue	////	4
Yellow	𝍸 ///	8

8 Which spinner do you think they used? Explain your thinking.


Evie

I designed a different spinner. Here are my results.

Colour	Tally marks	Total
Green	𝍸 𝍸	10
Blue	𝍸 𝍸 ////	14
Red	𝍸 𝍸 𝍸 /	16

9 What do you think Evie's spinner might look like? Explain your thinking.



Amber

Amber's working

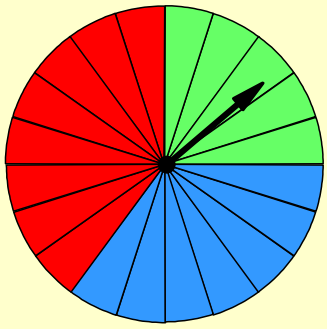
Evie has 40 results

The relative frequency of green is $\frac{10}{40} = \frac{5}{20}$

The relative frequency of blue is $\frac{14}{40} = \frac{7}{20}$

The relative frequency of red is $\frac{16}{40} = \frac{8}{20}$

So if I divide my spinner into 20 equal sectors, 5 should be green, 7 blue and 8 red!



10 Do you agree with Amber?
How could you be more sure?



Evie

This is my second set of results.

Colour	Tally marks	Total
Green	𝍸 ///	8
Blue	𝍸 𝍸 𝍸 ///	18
Red	𝍸 𝍸 ////	14

11 How do these results affect your thinking about Evie's spinner?

A spinner challenge

Taro has designed his own spinner. He says the chances of landing on red is $\frac{1}{3}$.

The chances of landing on blue is $\frac{1}{4}$. The chances of landing on green is $\frac{1}{5}$.

The chances of landing on black are $\frac{1}{6}$ and the chances of landing on gold is $\frac{1}{20}$.

12 Can you design Taro's spinner?

The law of large numbers

Some more results


Evie

Here is my third set of results.

Colour	Tally marks	Total
Green	𝍸 /	6
Blue	𝍸 𝍸 𝍸 //	17
Red	𝍸 𝍸 𝍸 //	17

13 Why doesn't Evie get the same results every time?
How does having more results help you?

14 What do you think Evie's spinner looks like now?

Fitness check

Alice also designed a spinner. Here are her results. What might her spinner look like?

Colour	Tally marks	Total
Green	𝍸 𝍸 𝍸 ////	19
Blue	𝍸 𝍸 𝍸 ////	19
Red	𝍸 𝍸 𝍸 ////	19
Yellow	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 ///	43

The law of large numbers

In probability games and experiments, the law of large numbers (LLN) is a theorem that describes the result of performing the same experiment a large number of times.

According to the law, the cumulative results will get closer to the theoretical results the more trials you have.

For example, if you spin a coin 10 times, theoretically you should get 5 heads and 5 tails but, in reality, you probably won't.

Matthew, Amber and Jon all spun the same coin 10 times. Here are their results.


Matthew

These are my results.
H, H, H, T, H, T, H, T, T, H
That's 6 heads and 4 tails.


Amber

Here are my results.
H, H, T, T, H, H, H, T, H, H
That's 7 heads and 3 tails.


Jon

Here are my results.
T, T, T, H, T, H, T, H, T, T
That's 3 heads and 7 tails.

Are you feeling lucky?

Games are played all over the world by people of all ages. Taking part is fun but it is generally more enjoyable if you win.

Some people appear to win more than they should – they may just be lucky – but they may understand the probabilities involved and that will give them an advantage.

Studying probability will increase your chances of winning many of these games

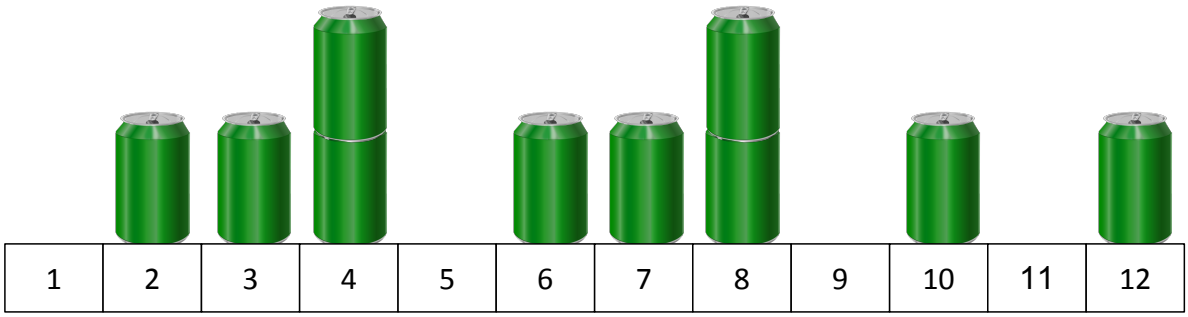
For example let's play, “Knock 'em down”.

“Knock 'em down”

In this case you begin with a wall divided into 12 equal regions as shown below.

1	2	3	4	5	6	7	8	9	10	11	12

You then “stack” your 10 cans on the wall in any of the marked regions until your wall looks something like this ...



To play the game, you have to knock the cans off the wall by rolling two dice.

You then add up the scores and, if you have a can that matches the total, you knock off one can.



Matthew

Does it matter where you put your cans?



Alice

There must be a way that we can work out where the best place is to put the cans.



Amber

I think it's just luck



Jon

Let's play the game and see what happens.

15

Jon's right – so let's play the game a few times before we go any further.

Sample spaces



Sarah

Did you realise that you can't get a total of 1 with two dice.

Well I put a can on 12 and that didn't come up in 3 games!



Taro



Evie

Did you noticed that 6, 7 and 8 seemed to come up a lot?

We need to write down all of the possible outcomes



Matthew

16 Can you think of a way of writing down all of the possibilities?



Amber

Amber's idea

I am going to use a two way table to record the results

Second dice →

First dice ↓

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

17 Look at the different totals. What do you notice?



Sarah

There are 36 outcomes

And 7 comes up the most.



Jon

Frequency tables



Taro

We need to summarise the results in a simple table. For example 12 only comes up once!

And 7 comes up 6 times



Alice



Sarah

So we should expect to get a lot more 7's than 12's.

18 Summarise your results in a suitable table. Perhaps something like this?

Total	Frequency	Total	Frequency
1	0	7	6
2	1	8	5
3	2	9	4
4	3	10	3
5	4	11	2
6	5	12	1

Some useful notation

The probability of an event (X) happening = $\frac{\text{number of successful outcomes}}{\text{total number of outcomes}}$

For example, the probability of getting a 7 could be written as:

$$P(X = 7) = \frac{6}{36}$$

Fitness check



What is …

1. $P(X = 4)$?
2. $P(X = 11)$?
3. $P(X = 6)$?
4. $P(X < 5)$?
5. $P(X \geq 8)$?
6. $P(X \neq 7)$?
7. $P(2 < X < 6)$?
8. $P(4 \leq X \leq 9)$?
9. $P(5 < X \leq 10)$?

Sample spaces

It is often helpful when thinking about probability questions to list all of the possible outcomes in a logical way. This is exactly what Amber did in “Knock 'em down”. A sample space is a logical listing of all the possible outcomes in an experiment. It can be pictures, a table or a list.

Another horse race

What would happen if we have 11 horses and use the sum of 2 6-sided dice again?

Horse	1	2	3	4	5	6
2						
3						
4						
5						
6						
7						
8						
9						
10						
11						
12						

19 Who do you think will win this time Who won't win?



Taro

Horse 7 would probably win

It would probably be 6, 7 or 8.
2 or 12 would never win.
It isn't a fair race!



Sarah



Mr T

Let's play the game a few times and see what happens.

Another game



Jon

As we predicted 2 or 12 will never win.

As it isn't a fair race, maybe we should handicap some of the horses.



Alice

The handicap race

To make our race a little less predictable, we are going to make it a handicap race. We are going to give the "outsiders" a start as shown below.

Horse	1	2	3	4	5	6
2	x	x	x	x	x	
3	x	x	x	x		
4	x	x	x			
5	x	x				
6	x					
7						
8	x					
9	x	x				
10	x	x	x			
11	x	x	x	x		
12	x	x	x	x	x	

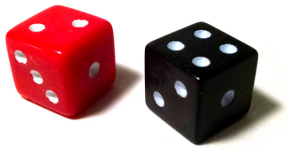
Horse number 2 or number 12 only need to come up once to reach the finishing line. Horse number 7 needs to come up 6 times if he is to win.

20

Who do you think will win this time?
Place your bets and play the game.

Ladders

This is another 2 dice game but this time you multiply the scores together.



So if you throw a two and a four, your score would be 8.

Your “board” has 6 vertical boxes as shown below and a side box that acts as the bin.

After each round, you must enter your score into one of the boxes. Once entered, it cannot be moved.

You are trying to put the numbers you score in ascending order.

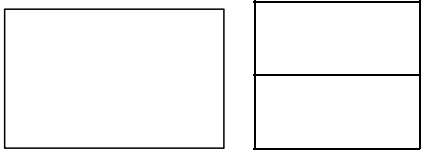
If you can't fit the number into a space, it goes into the “bin”.

The winner is the player who has managed to fit in the most numbers and so has the least in the bin.

21

Play the game a few times and try to develop a winning strategy.

Bin



Evie's working



Evie

3 x 1 = 3
2 x 6 = 12
1 x 3 = 3
5 x 3 = 15
6 x 1 = 6
4 x 3 = 12

Bin: 12

My score is 5

15
12
6
3
3

Jon's working



Jon

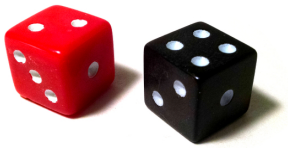
6 x 3 = 18
1 x 5 = 5
3 x 4 = 12
6 x 5 = 30
4 x 5 = 20
2 x 3 = 6

Bin: 20

My score is 5

30
18
12
6
5

22 Create a sample space diagram for this game. Does that help you to win?





Alice

Alice's sample space diagram

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

Fitness check

In this game if X represents your score, what is ...

1. P(X = 4) ?
2. P(X = 12) ?
3. P(X = 25) ?
4. P(1 ≤ X ≤ 6) ?
5. P(7 ≤ X ≤ 12) ?
6. P(13 ≤ X ≤ 18) ?
7. P(19 ≤ X ≤24) ?
8. P(25≤ X ≤ 30) ?
9. P(31 < X ≤ 36) ?

23 Now modify your strategy and play the game again. Did your new strategy help you to do better?

Let's think about sets ...

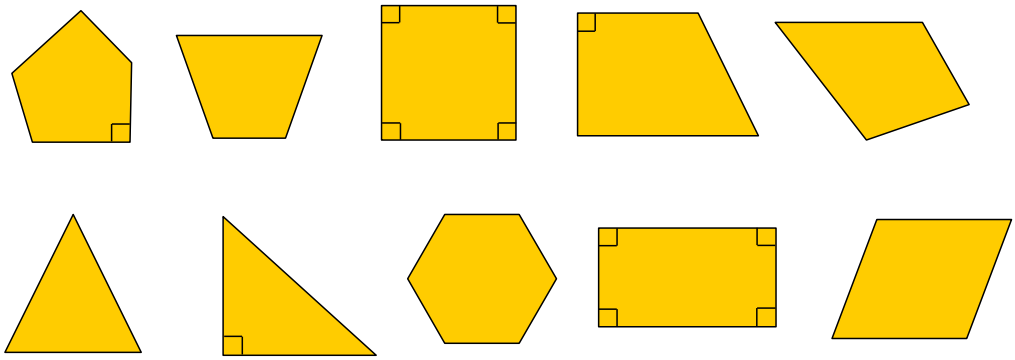
We have already seen that **sample spaces** can be very useful in recording all of the possible outcomes in a probability experiment.

In this section we will be looking at how other diagrams can also be useful.



Miss F

Last night my little sister was playing with her shape sorter. She put the 10 shapes in a bag and then picked one out. Each time, she asked me to guess what the shape might be. Now that got me thinking ...



What is the probability that

a) the shape chosen has four sides?

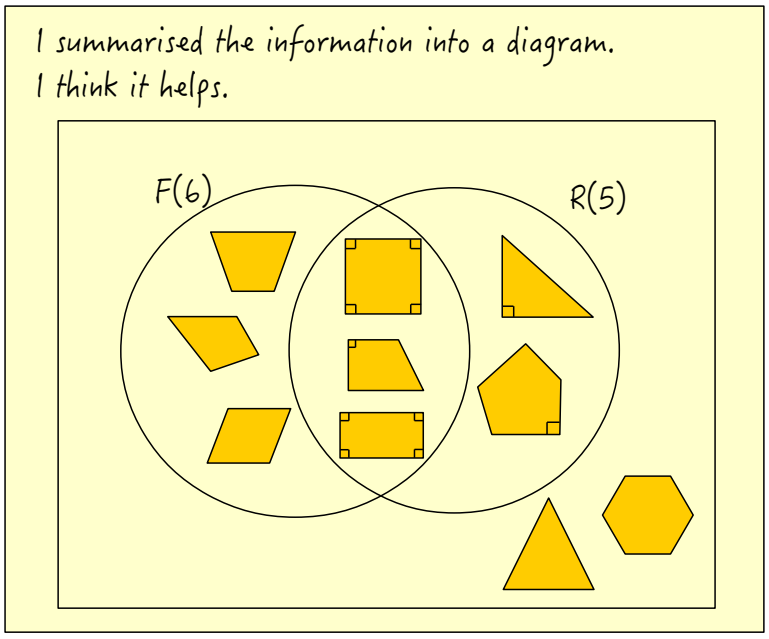
b) the shape contained a right angle?

c) The shape was four sided and contained a right angle?



Amber

Amber's idea



24 Can you explain Amber's diagram?

Some new terminology

A **set** is a list of objects in no particular order; they could be shapes, numbers, letters or even words.

In our example we looked at a set of 10 shapes used in a child's shape sorter. Each shape is called a **member** of the set.

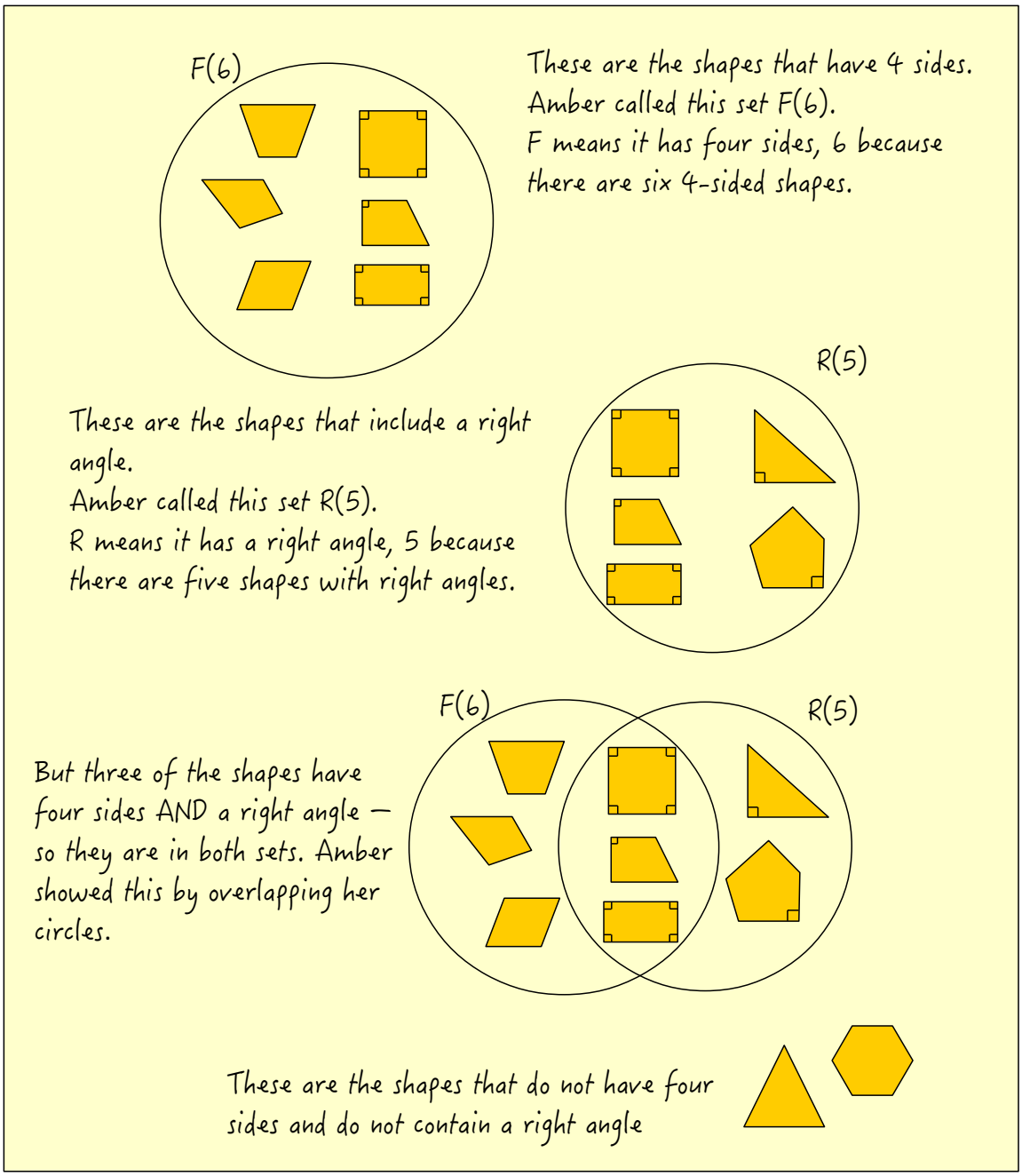
Amber summarised the shapes in a diagram.

This is Alice's explanation of Amber's diagram.

Alice's explanation of Amber's diagram



Alice



Let's think about set notation

Fitness check

1. What are the members of these sets?
   a) The days of the week?
   b) The first five even numbers?
   c) The colours of the rainbow?
   d) The odd numbers between 10 and 20?

Some useful notation

When we talk about sets, the phrase 'the set whose members are' frequently occurs, so we use a shorthand notation. We put the set in curly brackets.

For example, if A is the set whose members are the first five even numbers, we can write this as

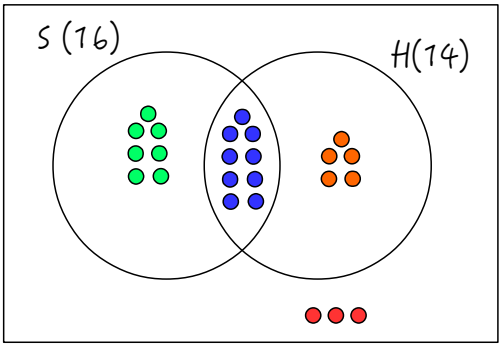
A = {the first five even numbers} or A = {2, 4, 6, 8, 10}.

2. How would you describe these sets in words?
   a) B = {1, 3, 5, 7}
   b) C = {x, y, z}
   c) D = {England, Scotland, Wales}
   d) E = {England, Scotland, Wales, Northern Island}
   e) F = {y, z, x}

25

Are sets C and F the same? Explain your thinking.

3. 8Y and 8Z were asked if they wanted to join the soccer team or the hockey team. Their answers are shown below. How would you interpret this diagram?



4. If one student is chosen at random, what is the probability they
   a) only want to join the soccer team?
   b) want to join both teams?
   c) don't want to join either team?

Venn diagrams

The use of diagrams to represent sets was first used by John Venn (1834 – 1923) and consequently are called **Venn diagrams**.

A Venn diagram begins with a box called the **universal set**, usually represented by ε (epsilon), which contains everything or everybody that we are interested in.

In our last example, the universal set was the students from 8Y and 8Z..

The circles are used to denote **subsets** of the universal set. In our example these were students who wanted to join the soccer team or the hockey team.

The overlapping region means the items belong to both sets.

Anything outside of the circles but within the box represent items in the universal set but not in the subsets. In this case students who don't want to join either team.

All of this sounds very complicated but another example should make it much clearer.

Delivering the news

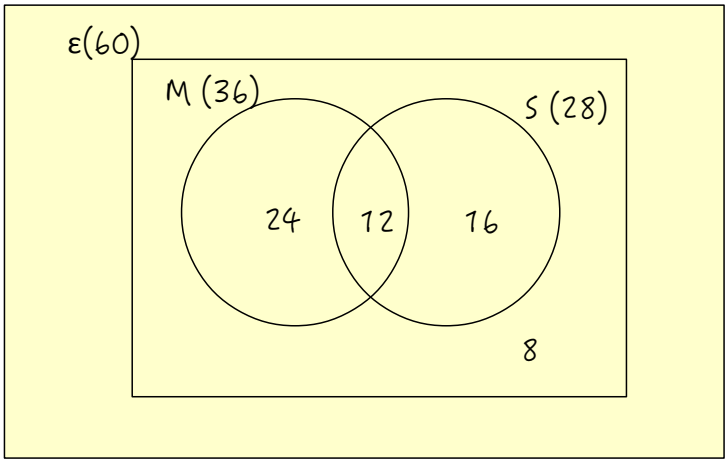
The two most popular newspapers in the UK based on Saturday sales are “The Mail” and “The Sun” (2016 figures).

Evie has a Saturday job delivering newspapers to 60 houses. In her notebook, she drew a Venn diagram to show how many of these houses receive either newspaper.



Evie

Evie's notebook



26

Can you explain the information shown on Evie's Venn diagram?

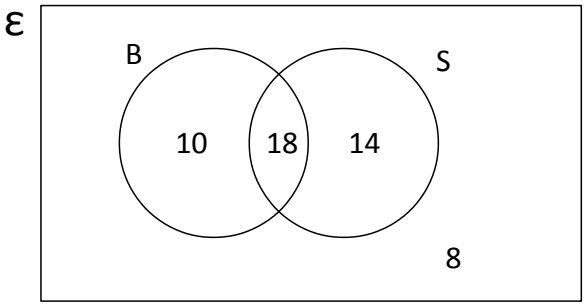
Delivering the news

Of the 60 houses on Alice's paper round, what is the probability that on a Saturday they ...

1. have only the Sun?
2. have only the Mail?
3. have both the Sun and the Mail?
4. do not have the Mail
5. do not have only the Sun?
6. have neither the Sun nor the Mail?
7. have both the Sun and the Mail?
8. have only one of these papers?

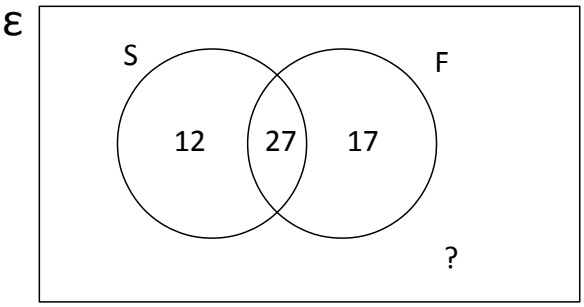
Fitness check

1. 50 people were surveyed and asked “Do you have a brother or a sister?”. Their answers are shown on this Venn diagram.



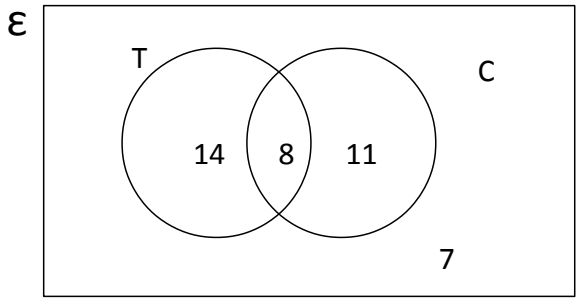
   a) How many people had a brother but not a sister?
   b) How many people had both a brother and a sister?
   c) How many people didn't have a brother or a sister?
   d) What is the probability that a person chosen at random from the group has …
      i. a sister but not a brother?
      ii. either a brother or a sister but not both?

2. 65 people were surveyed and asked “Have you ever been to Spain or France for a holiday?”. Their answers are shown on this Venn diagram.



   a) How many people had not been to Spain or France?
   b) How many people had been to both Spain and France?
   c) What is the probability that a person chosen at random from the group has only been to Spain?

3. All the members of a cricket club were asked if they drank tea (T) or coffee (C) in the break between innings. Their answers are shown in the Venn diagram below



a) How many members are there in the cricket club?

b) How many didn't drink tea or coffee?

c) How many didn't drink tea?

d) How many drank both tea and coffee?

e) What is the probability that a member chosen at random ...

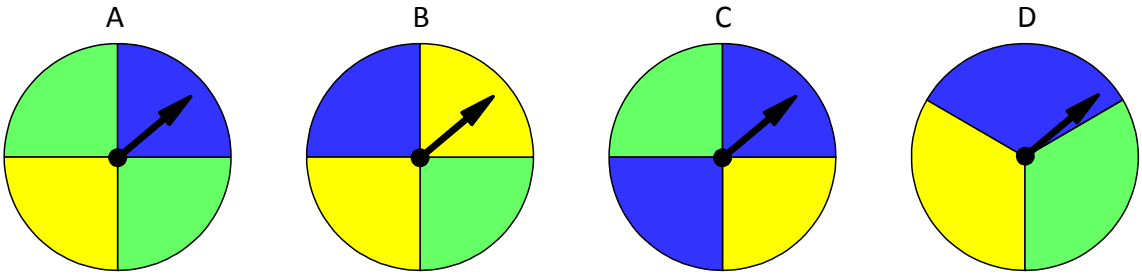
   i. only drinks tea?

   ii. drinks both tea and coffee?

   iii. doesn't drink either?

Probability 3: Mastery questions

1. Taro recorded the results of 30 spins of one of the spinners shown below.



Taro's results

Colour	Tally marks	Total
Green	𝍸 ////	9
Blue	𝍸 𝍸 ////	14
Yellow	𝍸 //	7

Which is the most likely spinner that Taro used?

2. The school hockey team had the following results last season: Won 13 games, drew 2 games, lost 5 games. What is the relative frequency of losing?

A	B	C	D
5	$\frac{5}{15}$	$\frac{13}{20}$	$\frac{5}{20}$

3. The probability of winning a game is 0.6. If you played the game 40 times, how many would you expect to win?

A	B	C	D
4	6	24	40

4. A box contains 50 cubes. If one cube is chosen at random, the probability of getting a certain colour is given in the table below.

Red	Blue	Green	Yellow
0.24	0.3	0.34	0.12

How many blue cubes are in the box?

A	B	C	D
3	15	24	30

Probability 3: Mastery questions

The table below shows the probabilities of scoring 1, 2, 3 or 4 on a biased dice.

Questions 5 – 7 refer to this table.

X	1	2	3	4
P(X)	0.4	0.15	0.25	0.2

6. What is $P(X \neq 3)$?

A	B	C	D
0.12	0.19	0.25	0.75

7. What is $P(X < 3)$?

A	B	C	D
0.2	0.19	0.55	0.8

8. What is $P(1 \leq X \leq 3)$?

A	B	C	D
0.2	0.19	0.55	0.8

9. You roll two fair four sided dice and add up your scores.
   What is the probability that you get a total score of 4?



A	B	C	D
$\frac{4}{8}$	$\frac{3}{24}$	$\frac{3}{8}$	$\frac{3}{16}$

+	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

10. If $P(A) = 0.65$, what is $P(A')$?

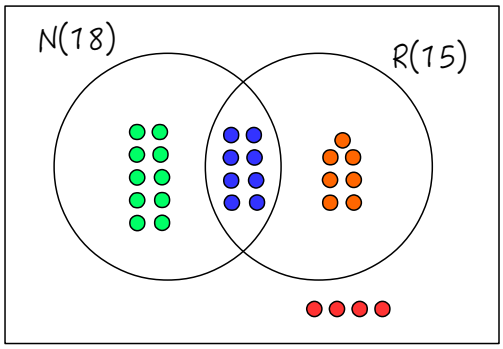
A	B	C	D
0.35	0.4	0.45	0.65

Probability 3: Mastery questions

11. Two fair pound coins are tossed together. What is the probability of getting two heads?

A	B	C	D
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{2}{4}$

12. Look at this Venn diagram. It shows the number of students who played Netball and Rounders for the school team last week.



How many students did not play in the rounders team last week?

A	B	C	D
10	14	18	22

In Questions 13 – 14, you must show all of your working.

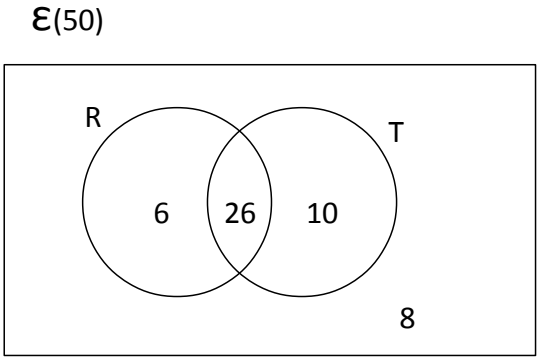
13. List all of the possible outcomes when tossing 3 coins.

What is the probability of

a) Getting 3 heads?

b) Getting 2 heads and a tail in any order?

14. A running club has 50 members. 32 take part in road races (R) and 36 race on the track (T). 8 members do not take part in any running races. This information is shown on the Venn diagram below.



What is the probability that a member chosen at random

a) Runs in road races?

b) Does not run on the track?

c) Runs on the road but not on the track?