Teaching Mathematics for Understanding in Primary Schools: Could Teaching for Mathematising be a Solution?

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This paper argues for *Teaching for Mathematising* as a pedagogy that supports teaching mathematics for understanding in primary school. Mathematics education in Australia currently emphasises teaching for mathematical understanding, a shift that redirects children's learning from merely memorising computation procedures to helping children construct knowledge of the mathematics that informs mathematical *concept and processes*. The shift to teaching for understanding however is not reflected in students' item responses in international and national achievement studies. This paper provides one example that illustrates a learning trajectory for supporting mathematical understanding of both conceptual and procedural knowledge through teaching that builds on the framework of *Teaching for Mathematising*. A corresponding field-test will then be used to illustrated how students learn pragmatically.

The importance of students developing number sense continues to gain traction as research reveals its short and long term implications for mathematics learning generally (e.g. Clements & Sarama, 2014). An emphasis on the development of early number sense in Western countries can be found in early year curriculum where mathematical content for teaching identifies core number concepts and accompanying processes such as problem solving and reasoning needed to develop mathematical understanding (e.g. ACARA, 2017; DoE, 2013; NCTM, 2000). Mathematics texts for classroom educators also reflect a pedagogical focus on teaching mathematics for understanding, with this philosophical underpinning often found in the opening chapter (see for example Bobis, Mulligan & Lowrie (2013) in Australia, Haylock (2010) in the UK and Van de Walle, Karp and Bay-Williams (2015) in the USA). Building on the work of researchers such as Skemp (1976), Shulman (1986), and Ball, Thames, and Phelps (2008), teaching for understanding acknowledges the connections that need to be made between mathematical concepts and mathematical processes in order for students to develop critical mathematical relationships such as numerical structure and pattern (Mulligan & Mitchelmore, 2013). Conceptual understanding is one of the four proficiency strands in the Australian Curriculum (ACARA, 2017) and a learning aim in both the English national curriculum (DoE, 2013) and the NCTM (2000) Principles and Standards for School Mathematics. Understanding can therefore be viewed as a mathematical action or aim for students' mathematics learning. The current emphasis on 'understanding' in curriculum, theory and practice has implications for pedagogical choices for the early mathematical learning experiences.

A shift in focus has led to a move away from teaching number computation as presenting algorithms created by teachers, to engaging students in investigative activities such as problem solving in order to construct conceptual understanding for themselves. This shift redirects students learning from merely memorising computation procedures, for example, in multiplication, to supporting students to construct knowledge of the mathematics that underpins the *concept and algorithm* of multiplication for themselves. A key part of teaching for understanding is encouraging students to develop their conceptual knowledge and gain mathematical understanding. Teachers play a key role in scaffolding students to find and justify mathematical connections and links. Open-ended questioning and discussion can be used to focus students' attention on making the mathematical concepts visible when they are prompted to look for structures in their representations, such as groupings, and to give mathematical reasons for their answers. Examples of the types of questioning that can support such reasoning can be found in the field-testing data in the section of Field Test. Although there has been an increased pedagogical emphasis on students' reasoning and justification for their mathematical thinking that has helped move teaching away from purely procedural teaching, it appears that more is needed to develop students' mathematical understanding.

Different international studies on primary student's mathematics achievement such as TIMSS (Beaton et al., 1997; Mullis, Martin, & Foy; 2008; Mullis, Martin, Foy, & Hooper; 2016) and PISA (Lemke et al., 2004; Fleischman, Hopstock, Pelczar, & Shelley, 2010; OECD, 2013) indicate that Australian students are underperforming particularly in text items that are specifically concerned with the four operations (addition, subtraction, multiplication and division) of whole numbers (see for example, Mullis, Martin & Foy, 2008, p.67 - 70). The results suggest that the shift to teaching for understanding is not visible in students' item responses in these types of high stakes testing measures, and so the question arises: is there still a disconnection between mathematics teaching and learning that is not supporting students' development of mathematical understanding? Could changing the approach to the integration of conceptual and procedural knowledge in classroom pedagogical practices improve students' mathematical learning and understanding? How can the concept of integrative teaching of conceptual and procedural knowledge be translated into classroom activities that bring about effective teaching? We proposes, as more or less Bruner (1977) did more than thirty years ago, that more research and empirical work should be done to answer **how** the development of conceptual and procedural knowledge can be carried out in ways that integrate their development in the classroom, and **in what way** the hierarchical structure and nature of mathematical knowledge can be fully respected and emphasized in instructional design. This paper attempts to provide one example that illustrates an instructional design for supporting procedural and conceptual understanding through teaching that builds on the framework of *Teaching for Mathematising*. A corresponding field-test will then be used to illustrate how students organise their thinking and reasoning pragmatically.

Integration of Conceptual and Procedural Knowledge

In the past decades, there has been a conceptual and procedural knowledge dichotomy in Western mathematics education. Conceptual knowledge is usually understood as "knowledge that is rich in relationships" (Hiebert & Lefevre, 1986; p.3). It is an "understanding of the principles that govern a domain" (Rittle-Johnson, Siegler & Alibali, 2001; p.346). In short, conceptual knowledge involves knowing the relationship between related concepts (Wearne & Hiebert, 1988), and understanding why a procedure works (Hiebert & Wearne, 1986) and whether a procedure is legitimate (Bisanz & Lefevre, 1992). In contrast to conceptual knowledge, procedural knowledge is defined as a "familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols" (Hiebert & Lefevre, 1986; p.7). Wearne and Hiebert (1988) describe it as syntactic processes which involve symbol-manipulation and routinizing the rules for symbols. Procedural knowledge involves knowledge of the rules and procedures (Hiebert & Wearne, 1986; Skemp, 1976), or the steps taken to complete a mathematics task (Fuchs et al., 1997). There is a general perception that conceptual knowledge of mathematical symbols and rules should be constructed prior to practicing the rules (Li, 2006). Likewise, procedural knowledge has been deemed less important than conceptual knowledge (Star, 2005). Those who argue against procedural knowledge are concerned that procedural knowledge leads to the development of isolated skills and rote knowledge, and therefore does more harm than good in learning (Star, 2005).

In a current review of the literature on the relationship between conceptual and procedural knowledge, there has been a widening of the horizon with respect to both the nature and understanding of different knowledge types. There is general agreement that both conceptual and procedural knowledge are necessary for concept acquisition (see: Baroody, Feil & Johnson, 2007; Rittle-Johnson & Koedinger, 2009; Rittle-Johnson, Siegler & Alibali, 2001; Schneider, Rittle-Johnson & Star, 2011; Star, 2005; Star & Stylianides, 2013). Silver (1986) argues that mathematics competence rests on both developing and linking the two. Studies by Rittle-Johnson, Siegler and Alibali (2001), and Rittle-Johnson and Alibali (1999) point out that procedural and conceptual knowledge influence one another and may develop iteratively, with gains in one leading to gains in the other, which in turn trigger new gains in the first. Building on the idea of positive correlation of conceptual and procedural knowledge on mathematics learning, Rittle-Johnson and Siegler (1998) state that articulating how concepts and procedures interact is critical to an understanding of how knowledge is constructed. Though, competing theories on the developmental precedence of one type of knowledge over the other has been hotly debated (Rittle-Johnson, Siegler and Alibali, 2001), Star (2000) argues that the outcomes of different research hardly provide a consistent answer simply because “it depends”.

In this paper, we agree with the position that conceptual and procedural knowledge are learned in tandem rather than independently, and that this holds for acquiring a majority of mathematical concepts. The fundamental mathematical ideas that characterise mathematics, in the sense proposed by Bruner (1977), are applied repeatedly and consistently, creating networks of concepts and algorithms that are intrinsic to its structure and amalgamation, forming a hierarchic knowledge system. The integration of procedural and conceptual knowledge is not however a particularly powerful principle for informing mathematical pedagogy and instructional design, nor does it automatically translate into designing tasks that could bring about the kind of effective teaching that most teachers look for. What is missing here is a way to approach instructional design that supports the construction of both conceptual and procedural understanding of mathematics as an academic discipline. Freudenthal’s work in *mathematising*, achieves this construction through a re-invention process, in which both types of knowledge (conceptual and procedural) are intrinsically linked, and in which both the mathematical products (i.e., formulas, theorems and algorithms) and their evolution and refinement are highlighted, provides access to ‘how’ and ‘in what way’ that the mathematics is developed. Freudenthal’s re-invention process however, is often too distant from the classroom teaching reality.

What is Mathematising?

The concept of mathematising was introduced by Freudenthal (1973) as a way of supporting teachers and students to understand mathematics through organising subject matters to make it more mathematical. Mathematising literally is to mathematise something that is “non-mathematical or something not yet mathematical enough, which needs more, better, more refined, more perspicuous mathematising” (Freudenthal, 1991; p.66). Thus, Freudenthal states, mathematics has developed as the result of human activity. From this perspective, mathematics learning (and therefore teaching) should also reflect how mathematics was developed, and emphasise the process of mathematical development that creates the mathematical products we work with. Freudenthal's view was that complex mathematical products, for example definitions, theorems, formulas, and algorithms, are meaningful to very few people. He believed that mathematics educators should focus their attention on not only mathematical products, but also on the *developmental processes* of mathematics, as our understanding of mathematics is built through the processes of actively working with and developing mathematical concepts so that conceptual knowledge is gradually developed with procedural knowledge. It is not possible to repeat the historical processes that developed mathematics in the classroom; students do not live in an ancient world. Freudenthal argued therefore that the core work of mathematics teaching should be to emphasize a process of guided mathematical re-invention, so that both the final mathematical products and their developmental processes (including mathematical refinements we now have) are visible in the model's students develop when working through contextual problems. This approach emphasises the structure of mathematics and how it has developed. Through this approach, the integrated nature of conceptual and procedural understanding of mathematics can be made visible and reiteratively developed.

Freudenthal (1973) noted that the mathematical content knowledge in primary and secondary schools is often simplistically identified, with an emphasis on procedural and rote learning, a focus that ignores the underlying conceptual knowledge and how the formulas/algorithms have evolved and refined over time (i.e., the developmental process of mathematics). Fung (2004) argues that learning final mathematical 'products' without understanding the evolution process that has developed them, is as incomplete as reading the final chapter of a novel without ever bothering to read its beginning. Current curricular structure such as that found in the Australian Curriculum, appears to slice mathematics up into 'learning outcomes' or 'objectives' (Askew, 2012) which can make it difficult to gain a sense of mathematics as a unified and integrated whole. Inevitably, when mathematics is presented in this way, classroom teaching will focus on product outcomes mapped from curriculum requirements. Learning in this environment is built on a student's introduction to, and rote learning and application of formulas and algorithms to different mathematics problems. This is done without the student understanding why, that is, without understanding the underlying mathematical concept (i.e., conceptual knowledge) being used, how it was developed, and why it is being used. Teaching mathematics in this way does not attend to developing students' understanding of how mathematics has developed which is needed to understand the mathematics they are working with. This divide or 'gap' between mathematical product and the developmental process of mathematics which depicts the interlocking relationship between conceptual and procedural knowledge, appears as fissures in a mathematical learning journey (Fung, 2004), as illustrated in Figure 1. The 'learning gap' fragments mathematics by fragmenting mathematics teaching and learning.

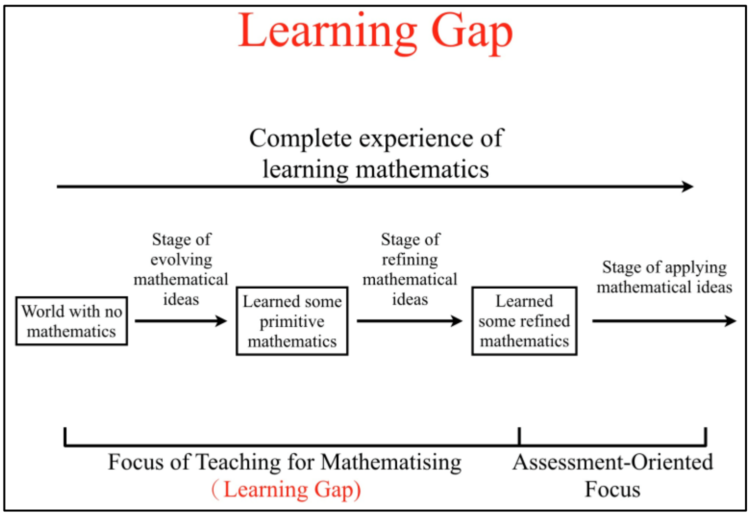

Figure 1. The Learning Gap (modified from Fung, 2004).

What is missing in mathematics classrooms we suggest, is an instructional design that supports the construction of both conceptual and procedural understanding of mathematics, through which the developmental process of mathematics is exemplified, but not in the form commonly found in most curriculum documents. We argue that mathematics teaching should be in a form in which fundamental mathematical ideas are applied repeatedly and consistently across learning experiences so that the logic and structure of mathematics is made visible and students are supported to develop maps of relationships that can be used to navigate mathematics concepts effectively. If mathematics teaching, and therefore learning is approached in this way, the construct of both mathematical processes and its products is embedded and integrated into a structured, amalgamated and hierarchic knowledge system of mathematics. Ultimately, conceptual and procedural understandings of mathematics are developed naturally.

Teaching for Mathematising – An Instructional Design

Teaching for Mathematising (Fung, 2004) emphasises the importance of designing learning trajectories that are built on students' concrete (hence local) experiences, thereby engages them to learn mathematics through re-inventing the mathematics themselves. *Teaching for Mathematising*, on one hand, inherits the goal of Realistic Mathematics Education (such as Treffer, 1987; Van den Heuvel-Panhuizen, 2003) that values students' re-inventing mathematics, and on the other hand adopts an iterative approach to the development of learning trajectories where instructional designs are created through thought experiments, tested by different scale and scope of implementation, and finally brought to a higher level of professional effectiveness. *Teaching for Mathematising* does not claim to provide flawless lesson plans, or that the initial instructional design, demonstrations and assessment found in the lesson plans work perfectly (although the readers do). Rather, *Teaching for Mathematising* intends to offer "provisional instruction activities and a conjectured learning process that anticipates how students' thinking and understanding might evolve" (Gravemeijer & Cobb, 2006; p.19) when activities are enacted in actual classrooms. More importantly, *Teaching for Mathematising* highlights the process of analysing and improving the conjectured instructional design so that students' learning can emerge through students' invention of their own mathematical solutions and that their participation and learning are sustainable. Similar to RME, mathematical activity in *Teaching for Mathematising* should be experientially real to students in the sense that they can immediately and naturally engage themselves in the process of negotiating and generalizing informal ways of mathematical reasoning (Cobb, Zhao & Visnovska, 2008). *Teaching for Mathematising* also focuses on the means of supporting this process (refer to Freudenthal, 1991). Cobb, Zhao and Visnovska (2008) suggest that mathematical activities may "involve making drawings, diagrams, or tables, or it could involve developing informal notations or using conventional mathematical activity" (p.110) so that students' participation in the mathematical activities is substantial.

This paper provides an example of an approach to multi-digits multiplication to illustrate an instructional design built on the framework of *Teaching for Mathematising* (Fung, 2004) in which both conceptual and procedural knowledge are integrated. A corresponding field-test will then be showcased to provide evidence of how students are guided to re-invent the distributive law for developing the multi-digit multiplication algorithm and how their thinking and understanding are evolved.

Multi-digits multiplication

One of the year 5 learning objectives in the Australian Curriculum: Number and Algebra strand is to "solve problems involving multiplication of large numbers by one- or two-digit numbers using efficient mental, written strategies and appropriate digital technologies" (ACARA, 2017, ACMNA100). This objective follows "recall multiplication facts up to 10 × 10 and related division facts" (ACARA, 2017, ACMNA075). Learning multiplication of multi-digit numbers from a curriculum perspective is therefore built on both the fundamental concept of multiplication as repeated addition, and recalling multiplication and division facts ('times tables'). From a mathematising perspective however, to understand multiplication, a student must also understand the distributive law and Base 10 as an efficient way of managing multiplication problems. A student could rote learn the procedure for multiplication using times tables and use this knowledge to solve the repeated addition of twelve four times resulting in an answer of 48. Distributive thinking however moves multiplication of twelve times four as a possible operation using known Base 10 relationships. Using this thinking, 'twelve' can be distributed, enabling the one unit of ten in 12 to be multiplied by 4 (40) and the 2 ones to be multiplied by 4 (8), and both these products then combined to result in (48). When multi-digit multiplication problems are provided that cannot be solved using times tables, such as the algorithm '*26 multiplied by 3'*, students are usually taught a standard algorithm (Figure 2), illustrating the distributive law. Students then rely on their knowledge of times tables up to 10 to 'solve' the problem, inevitably calculating with tens as if they were ones.

$$\begin{array}{lcc} & 2 & 6 \\ \times & {}_{1} & 3 \\ \hline & 7 & 8 \end{array}$$

Figure 2. Standard algorithm of 26 multiplied by 3.

Using Freudenthal's approach, the aim would be for students to experience the re-invention of the multiplication algorithm early in the development of multi-digit operations and in ways that reveal the efficiency of using Base 10 in computation. The aim is for Base 10 to be understood as an effective and efficient result of the development and refinement of mathematical processes over time and to also develop a deep understanding of the structure and legitimacy of arithmetic processes in multiplication; that is the development of conceptual knowledge. We therefore need to perform a "thought experiment" on how this algorithm could reasonably evolve. As it is a thought experiment of a teaching trajectory, the design will be primarily built on the researchers' conjecture of a learning process that is based on the mathematical structure of multi-digits multiplication in base-10 system. Thus, this thought experiment presented in the following section is inevitably looked like a teacher-led activity - knowing the breadth of inherent mathematical relationships and being anticipate and plan for the teaching experience.

Connecting to and building on the idea of repeated addition, students could initially represent 26×3 in the array diagram as shown in Figure 3: three rows of 26 dots in each row.

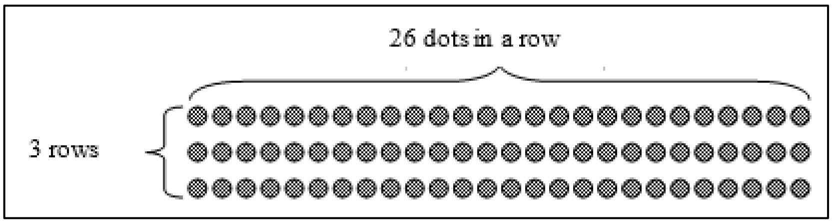

Figure 3. 26×3 in array diagram.

Using multiplication facts from the ten times table, students next need to consider how to break or partition 26 into numbers not larger than 10, so that this knowledge can be applied to the problem. The principle at work here is that the quantity '26 x 3' is broken apart in different ways to create partial products that are easier to work with both mentally and conceptually. Logically and pragmatically, students should, in the course of experimenting with distributive possibilities, come to see initially that the least number of partitions reduce cognitive work. The following diagrams illustrate some of the possible groupings created through 'splitting' the 26 x 3 array. Figure 4 shows 3 groups beginning with the largest number (using 10): 10, 10 and 6. As an algorithm, this could be represented as: 26 × 3 = (10 + 10 + 6) × 3 = 10 × 3 + 10 × 3 + 6 × 3.

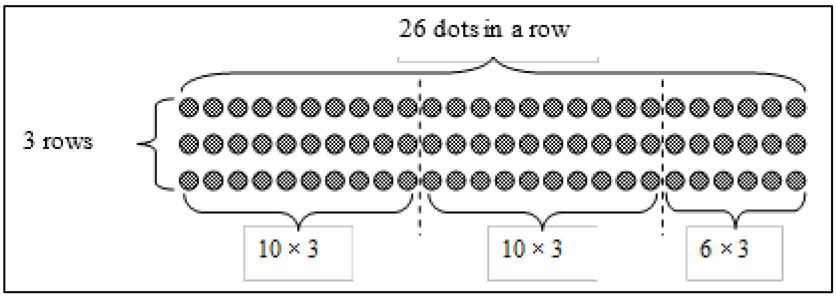

Figure 4. 26 grouped as 10, 10 and 6.

Figure 5 shows another 3 groups: 10, 9 and 7 which can be represented as 26 × 3 = (10 + 9 + 7) × 3 = 10 × 3 + 9 × 3 + 7 × 3.

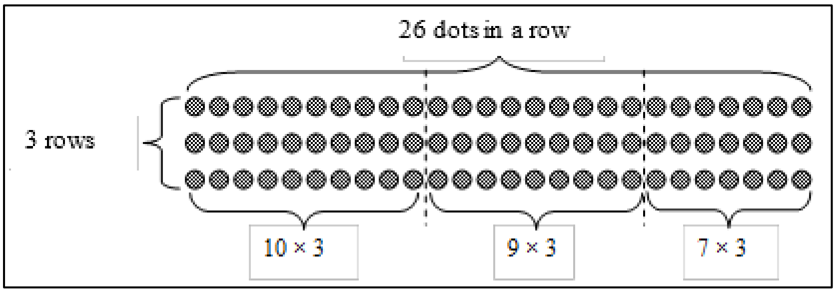

Figure 5. 26 grouped as 10, 9 and 7.

Figure 6 shows a further possible 3 groups: 10, 8, and 8, which can be represented as 26 × 3 = (10 + 8 + 8) × 3 = 10 × 3 + 8 × 3 + 8 × 3.

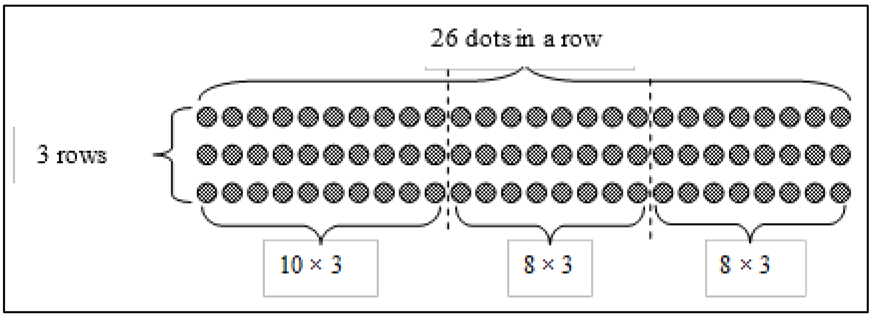

Figure 6. 26 grouped as 10, 8 and 8.

Figure 7 shows another a further possible 3 groups: 9, 9, and 8, which can be represented as 26 × 3 = (9 + 9 + 8) × 3 = 9 × 3 + 9 × 3 + 8 × 3.

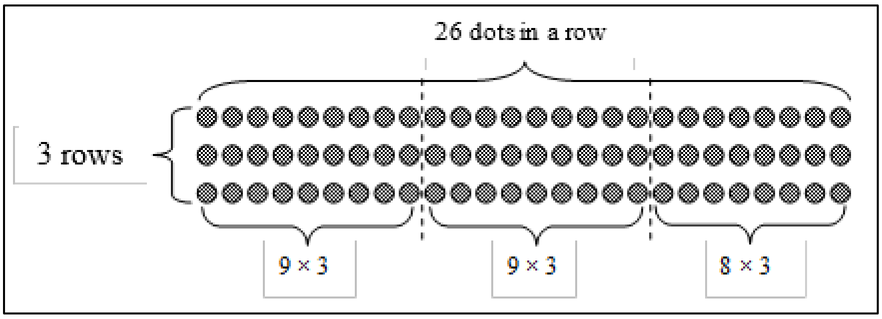

Figure 7. 26 grouped as 9, 9 and 8

The above four distributive strategies all provide mathematically correct answers for the problem 26 x 3. The importance of partitioning by decades however has not emerged naturally or automatically from the different groupings, although all of the strategies displayed clearly make use of the distributive property. A teacher would need to distinguish the “By Decades” strategy out of the many different strategies students create to draw attention to the Base 10 concept grouping rather than the computation performed through the alternate groupings. To highlight the increased complexity of working numerically with higher powers of ten, teachers can ask students to compute some larger numbers such as 47 x 3 and 89 x 6. Students will begin to see as they work that it is both difficult and tedious to break apart and re-group large numbers without a Base 10 “strategy”. In the meantime, teachers can support students to represent each group of ten dots as a ‘long’, a rectangle that represents ten single dots that connects to MAB block representations of 10 as a bar with marks to indicate ten individual ones. Figure 8 illustrates this “By decades strategy” for 47×3 as an example.

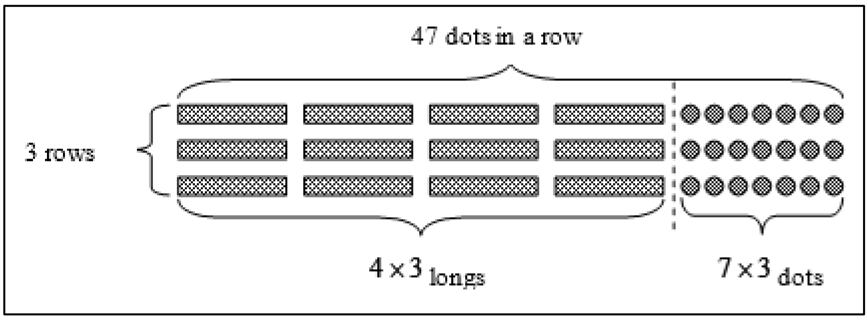

Figure 8. 47×3 illustrating a ‘By decades’ strategy.

Having each group of ten dots represented by a single shape reduces the computation process to a two-step multiplication. The problem can now be seen as ‘4 longs’ in each row for 3 rows. When calculated, the 12 ‘longs’ represent 120 dots that is, 12×10. Combining the 7 dots in each row for 3 rows (i.e., 21 dots), there are altogether 141 dots which can be represented as $141 = 12 \times 10 + 7 \times 3$. Through the “By Decades” strategy, the standard algorithms can be seen to evolve naturally as shown in Figure 9.

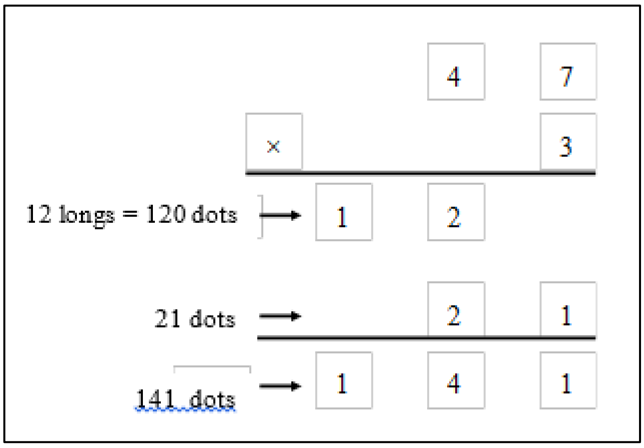

Figure 9. 47×3 illustrating a ‘by decades’ strategy in a standard algorithms.

The above algorithm is then further refined to represent a more sophisticated and simplified standard algorithm as shown in figure 10.

$$\begin{array}{lccc} & & 4 & 7 \\ \times & & {}_{2} & 3 \\ \hline & 1 & 4 & 1 \end{array}$$

Figure 10. 47×3 presented as a simplified standard algorithm

Field-Test on Multi-Digits Multiplication

Method

Background of Field-Test. The thought experiment described above is theoretically built on the mathematical structure of multi-digits multiplication in base-10 system. Whether this thought experiment is compatible with students' way of organising their mathematical reality is yet to be confirmed by actual implementation. The rationale for a field-test method is that a field-test on a thought experiment is expected to cast insights into a better understanding of various pragmatic issues that emerge from students cognitive engagement with the mathematics in the way it is presented. The rationale of a field-test method is a characteristic of the Realistic Mathematics Education approach, where the teaching experiment methodology is extended to fine-tune the mathematising process - the process of guided mathematical re-invention. Thus, the primary purpose of field-test is to study the process by which the participating students organise their mathematical reality so that a retrospective analysis of the thought experiment can be conducted for developing/refining the instructional designs that can possibly be used to steer the mathematical learning of other students. Ultimately, field-tests aim to "co-participate in the process of formulating, testing, and revising designs before we became able to develop adequate designs for supporting students' mathematical learning" (Cobb, Zhao, 2008 & Visnovska, 2008; p.108). It is worth noting here that Fung's field-test work in Hong Kong has been conducted in collaboration with classroom teachers in a highly localised setting where language, culture, curriculum, and teachers' knowledge are contextual parameters affecting the process, yet different from those of the English-speaking communities. Reports of this kind on different curriculum topics are written in Chinese (See for example Chan, 2007; Chan, 2010; Chan & Fung, 2006; Chan, 2012; Chow, 2006; Fung & Ip, 2004; Fung, Ip, & Lo, 2008; Fung, Wong, & Yeung, 2014; Lok, 2000; Tang, 2000; Wang, Fung, & Wong, 2012; Yeung, 2007; Yeung & Fung, 2005; Yiu, 2007; Yiu & Chan. 2008).

The Field-Test. The field-test method in this study was adapted from the teaching experiment (Cobb & Steffe, 1983, Steffe & Thmpson, 2000) in which researchers "interact with a small number of students one-on-one and attempts to precipitate their learning by posing judiciously chosen tasks and by asking follow up questions, often with the intention of encouraging the student to reflect on his or her mathematical activity" (Cobb, Zhao & Visnovska, 2008; p.106). It meant that the researchers in this study acted as teachers when conducting this field-test with the primary emphasis on probing students' understanding and reasoning and finding out why they use particular approaches (Gravemeijer & Cobb, 2006).

Using multiplication facts from the ten times table to consider how to break or partition a large number into numbers not larger than 10 is a crucial step for developing the concept of distributive law which in turn for re-inventing the multiplication algorithm. To achieve a better understanding of how students organise their mathematical reasoning and how their thinking evolve, it is necessary to assign this multi-digits multiplication task to students who have learnt the concept of multiplication and times tables up to ten. The goals of this particular field-test was to understand (1) how students break or partition a large number into numbers not larger than 10, (2) students' mathematical reasoning for breaking a large number, and (3) how they arrive at the answer.

Participants

The field-test was conducted with two participants, a 9-year old girl (Year 3) and 7-year old boy (Year 2) doing Year 3 mathematics in the same class in a primary school located in South Australia. University ethics, parental approval and the children's assent were obtained for the field-test described here. The data in this section is drawn from this experience with the children. The boy was currently working with mathematics one year beyond his school level, and the girl was working at her year level in mathematics. Both students had learnt their times tables by rote and were able to provide fluent answers for all recall multiplication facts up to 10×10.

Task and Participants' Episodes

In the following excerpts, pseudonyms are used for the purpose of keeping participants' identities confidential. Joyce (the 9-year old girl) and Peter (the 7-year old boy) were asked to use MAB blocks and/or pictures to calculate 26×3. Initially, Peter and Joyce were asked to use the MAB blocks to represent 26. Both students used two longs (i.e., ten) and six cubes (i.e., ones) to represent 26 and were confident that it was a good representation of 26. The following excerpts set out each student's actions and mathematical reasoning.

Peter's excerpt. When asked to work out the answer for 26×3, Peter drew a picture of 26 groups of three as shown in Figure 11.

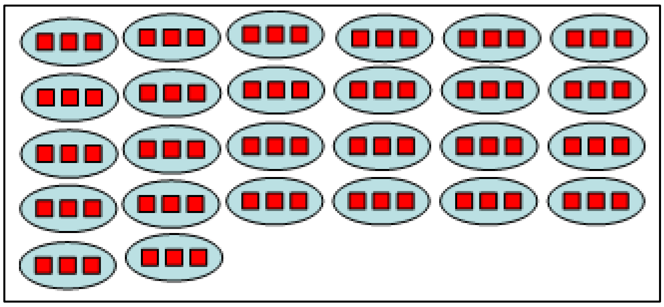

Figure 11. Peter's pictorial representation of 26 groups of 3.

Researcher: Could you explain your drawing to us?

Peter: There are 26 groups of three (his figure is pointing at the groups)

Researcher: Right, you are thinking of 26 groups of 3. Can it be 3 groups of 26?

Peter: Hmm… yes, but you are asking 26 times 3. So, I think it should be 26 groups of 3.

Researcher: How about if it is 3 times 26?

Peter: Then it is 3 groups of 26.

Researcher: So, does the order matter?

Peter: I think so.

Researcher: Could you work out the answer from your drawing?

Peter then grouped three groups of three in rectangles in his drawing as shown in Figure 12.

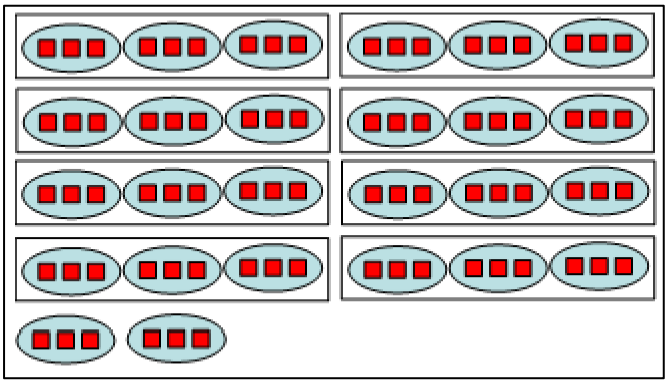

Figure 12. Peter's pictorial representation of 8 groups of 9 and 2 groups of 3.

Peter: I put 3 groups of 3 together and it becomes 9 in a group. Now, I have eight groups of nine plus 3 and 3. Eight times nine is 72 plus three plus three is 78.

Researcher: Why did you put 3 groups of 3 in a bigger group? Has it to be 3 groups? I mean why not 2 groups or 4 groups or even bigger number?

Peter: Hmm … cos … the times tables is up to 10 times 10. I mean …… if the times are within 10 times 10, I then can use the times tables to work out the answer. So, 3 groups of 3 is just right. (Peter says it confidently)

Joyce's excerpt. When Peter was explaining his reasoning to the researcher, Joyce was working on drawing her representation as shown in Figure 13.

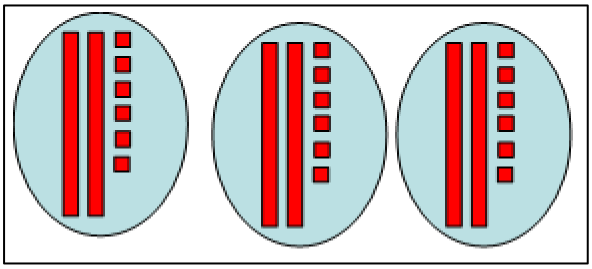

Figure 13. Joyce's pictorial representation of 3 groups of 26.

Researcher: Could you explain your drawing to us?

Joyce: There are 3 groups of 26.

Researcher: Can it be 26 groups of 3 like what Peter did?

Joyce: Yes it can. They are the same but I just think 3 groups of 26 is easier. I don't know. Am I right? (Joyce is puzzled)

Researcher: How could you work out the answer with your 3 groups of 26?

Joyce wrote down 20+20+20+6+6+6 = 20×3+6×3 = 60+18 = 78. As she was writing she said the following:

Joyce: Two tens in one group and there are three groups (she is writing 20+20+20). Six ones in one group and three groups (she is writing +6+6+6). Now three 20s (She is pointing at =20+20+20) is 20 times 3 (She is writing =20×3). The same to three 6s (She is writing +6×3). Last, 60 plus 18 is 78 (She is writing = 60+18 = 78). My answer is the same as Peter.

Researcher: Yes, right! Both of you got the right answer but used different methods.

Peter was impressed by Joyce's approach and appeared a bit disappointed with his not 'clever' method.

Researcher: Peter, what do you think about Joyce's method? Do you like her method?

Peter: Oh yes, probably hers is better.

Researcher: Why is Joyce's better? Joyce, do you think yours is better too?

Joyce: I don't know but I like my method. (She says confidently)

Researcher: Peter, what do you think?

Peter: Joyce's method is easier as there are less groups; less drawing too haha!

Researcher: Why is less groups better? Better for what? Easier to get the correct answer?

Peter: Probably! There are two tens and six ones in a group and three groups. So, you can just think of 20+20+20 then 6+6+6. So it is easier.

Researcher: Quicker too right?

Peter: It is quicker because it is multiple of ten. So two times three is six then put a zero at the back and got 60. You then 3 times 6 got 18 plus 60 is 78. Just easy! (He appears to be surprised)

Peter's actions and reasoning show an initial connection to his knowledge of multiplication gained from his times tables up to 10. In this task, he does not show an understanding of the commutative properties of multiplication ("the order matters"), however he does show that he has conceptual knowledge of number relationships (e.g. nine is the closest number to 10 that is possible when working with groups of three) and of the processes of addition and multiplication, and he was able to use these to calculate and explain his answer. His calculation however, illustrates the tediousness and difficulty of partitioning and distributing large quantities without a Base 10 strategy. Peter has not made use of conceptual connections to grouping by ten when faced with a task requiring him to work with multiples of ten.

Joyce's calculations however illustrate how Base 10 partitioning can emerge from groupings, and that the process of representation and reasoning to others enables the efficiency and viability of her mathematical processes and conceptual knowledge and their connections to be made visible to other students. Peter's explanations and representations illustrate the way he organized his thinking to enable him to work out an answer successfully. By partitioning using grouping based on ten he found a more efficient way of managing the calculation both conceptually and procedurally, and this organisation followed his observation of Joyce using the base-10 system. His existing knowledge of ten was now visible in different ways and using a mathematising analysis, Peter refined his mathematical ideas and reorganized his thinking to enable him to flexibly reason and partitioning in ways that were logical and pragmatic and so solve the task. He has therefore shown that he has extended his knowledge of the concept of multiplication, structure and pattern in number, and the processes of partitioning and distribution and has used this to solve the task.

Both children gained confidence and their thinking, reasoning and therefore understanding of the mathematical concepts and processes they were working with was developed. Although a small vignette, by contrasting the two ways of calculating 26 x 3, the development of a place value system in general, and a Base 10 positional numeral system in particular, as an efficient means of managing large quantities emerged upon which, the distributive law of multiplication could hopefully be re-invented through careful deliberation.

Conclusion

Planning for ways to engage students in the re-invention of mathematics requires careful design and patient attention to fine details as illustrated by the above example. In the multi-digit multiplication example, the core mathematics content to be learned is clearly visible and connects to prior knowledge students have worked with. The thinking processes students need to engage are exemplified and used repeatedly. Importantly, in working with the tasks, students are reconstructing and reorganising the mathematics in ways that are authentic and mirror the developmental path of the mathematics they are using. This approach to task design and teaching seeks to guide students into the re-invention of essential mathematical knowledge as a basis for developing a deep understanding of the 'why' and also the 'how' of mathematics in solving.

> "In a sense youngsters should repeat history though not the one that actually took place but the one that would have taken if our ancestors had known what we are fortunate enough to know. (Freudenthal, 1981, p.140)"

To make such learning happen in mathematics, support is needed at the research and professional development level to both create and field-test instructional designs and teaching that are grounded in mathematising.

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