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Student Reactions to Learning Theory Based Curriculum Materials in Linear Algebra – A Survey Analysis

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Abstract:
In this report we examine students' perceptions of the implementation of carefully designed curriculum materials (called modules) in linear algebra courses at three different universities. The curricular materials were produced collaboratively by STEM and mathematics education faculty as members of a professional learning community (PLC) over several years. We have described the development and implementation of these materials elsewhere (authors, authors). Our focus here is on more detailed analysis of comments that student participants made about the impact on their engagement, perceived learning, self-confidence, and notions of the broader nature of mathematics.

Key words:
Curriculum Development, Student Understanding, Mathematics

1. Introduction

This report examines students' perceptions of the implementation of a framework for teaching undergraduate mathematics that includes strong, deliberate and sustained collaboration between science, technology, engineering and mathematics (STEM) and mathematics education faculty to develop course materials that are based on the APOS (Action – Process – Object – Schema) learning theory. The Linear algebra In New Environments (LINE) project - partially supported by the National Science Foundation (NSF grant number 0837331), an independent U.S. government agency responsible for promoting science and engineering through research programs and education projects - was designed to promote a reflective, collaborative culture of teaching and learning among STEM discipline faculty. The approach was implemented in linear algebra courses at four different institutions; three of these institutions participated in the study reported here. Each institution had at least one STEM faculty member and one mathematics education faculty member who worked together in the linear algebra courses. The faculties from all four institutions collaborated in one professional learning community (PLC) over several years. The PLC began with an online reading seminar on learning theories and moved to project meetings at each other's campuses at which we used the knowledge of the learning theories to develop carefully planned materials, which we called modules, that were appropriate for the different institutions and their students. The researchers have described the development and implementation of these materials elsewhere (Cooley, Vidakovic, Martin, Dexter, Suzuki, and Loch, 2014; Cooley & Vidakovic, 2013; Cooley, Vidakovic, Martin, Dexter, Suzuki, and Loch, 2013; Martin, Loch, Cooley, Dexter and Vidakovic, 2010). The focus in this report is on a more detailed analysis of written reflections that student participants provided in response to questions about their engagement, perceived learning, self-confidence, and notions of the broader nature of mathematics. Additionally, we include results from some students who participated in videotaped group problem-solving sessions where they negotiated solutions to linear algebra problems. Finally, to triangulate the data, a subset of these students were then interviewed alone with follow up questions about the group work and their experiences.

The LINE project was designed to (a) motivate students by crafting learning experiences that would guide them to think about mathematics more deeply; (b) make a coherent connection between instructional models, learning theories and content; and (c) provide collaborative support for faculty so that expertise in content and pedagogy are used to design effective instructional practice. *A novel characteristic of this project is that the modules, infused by a particular theory of teaching and learning mathematics at the undergraduate level, were different in content and pedagogy at each institution while guided by the same theory*. The students, the mission of the college, and the type of course offered were all considered so that the activities made sense for each institution. Course materials, including a variety of modules incorporating ideas such as student explorations of uses of linear algebra or proof development, were produced which built upon the APOS theory (Asiala, Brown, DeVries, Dubinsky, Mathews, and Thomas, 1996) for concept development as individuals learn mathematics.

While the content area was linear algebra - chosen because of its significance as a core course for both mathematics majors and many other disciplines, the design and collaboration ideas and goals could easily be adapted to other mathematical domains. *Since the participating institutions varied greatly and the linear algebra courses varied in their abstractness, the course materials were developed to accommodate these differences*. Two sample modules are provided in Appendices A and B, along with an APOS *genetic decomposition* - a theoretical model that describes the mental constructions and mechanisms needed to construct and understand a particular mathematical concept (See Section 2 for more details about APOS learning theory). These examples illustrate two different types of modules - one demonstrates a model simulation and the other a more formal treatment of linear transformations.

The LINE project promotes collaborative change in undergraduate mathematics instruction through an interchange of ideas and reflection by mathematicians and educators to strengthen student learning in linear algebra. Four objectives for student learning that the authors seek to promote through this work are:

1. Students will gain more conceptual understanding of mathematical content.
2. Students will be more actively engaged in their own learning.
3. Students will gain self-confidence in their capacity to do mathematics.
4. Students will develop a broader understanding of the nature of mathematics.

Student reflections were examined through analysis of confidential post-course survey responses about their learning and experiences in linear algebra courses taught at participating institutions from 2011 – 2013. The data were then triangulated using the analysis of videotaped group problem solving sessions in linear algebra followed by individual interviews of a subset of these group members at one institution to verify students' reflections on their experiences.

This paper is part of a series of research reports about LINE outcomes. These include papers on the learning of linear algebra from an APOS perspective (Meagher, Cooley, Martin, Vidakovic & Loch, 2006), coordinating learning theories with linear algebra (Cooley, Martin, Vidakovic & Loch, 2007; Martin et al., 2010), the effect of the simultaneous study of linear algebra and learning theories (Vidakovic, Cooley, Martin, Meagher, 2008), and integrating learning theories in applications for linear algebra (Cooley et al., 2014).

2. Background

How students learn undergraduate mathematics has been a focus of research for many scholars for decades (Cooley, Trigueros & Baker, 2007; Loewenberg-Ball, 2003; Bransford, Brown & Cocking, 1999; Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 1996, Novak & Gowin, 1984). In particular, improving the teaching and learning of linear algebra has been a focus of various groups of mathematicians and mathematics educators (Arshavsky, 1999; Britton and Henderson, 2009; Carlson, 1997; Carlson, Johnson, Lay, and Porter, 2002; Cooley et al., 2014; Cooley & Vidakovic, 2014; Dorier and Sierpinska, 2001; Martin et al., 2010; Sierpinska, Dreyfus, & Hillel, 1999; Sierpinska & Nnadozie, 2001; Watkins & Watkins, 1993). A variety of resources contain materials related to content and pedagogy, including instructional technology, levels of abstraction and rigor, applications, trends in teaching, student comprehension of particular topics, student diversity, and connection to other STEM disciplines (Chang, 2011; Halmos, 1995; Leon, Herman, and Faulkenberry, 2003; Pecuch-Herrero, 2000; Sierpinska, Trgalova, Hillel, and Dreyfus, 1999). A number of scholars have conducted research that points to specific difficulties that students encounter while learning linear algebra concepts and offer some explanations for these difficulties (Sierpinska & Nnadozie, 2001; Todorova, 2012; Trigueros & Lozano, 2010; Wawro, Sweeney, Rabin, 2011). Some of these studies take the form of manipulating linear algebra pedagogy in the classroom and studying the effects of these changes (Martin et al., 2010; Possani, Trigueros, Preciado & Lozano, 2010). The researchers of this report build on these efforts in developing rich classroom experiences and incorporate them into the discourse of our professional learning community.

To connect learning theory, mathematical content and student reflection, the LINE approach relies on an instructional methodology based on the Action-Process-Object-Schema (APOS) theory (Arnon, Cottrill, Dubinsky, Oktac, Fuentes, Trigueros & Weller, 2013; Asiala et al., 1996). APOS is a well-developed theory for understanding how students develop mathematical concepts; in addition to a model of cognition, it provides an instructional model intended to facilitate student development of richer and more sophisticated understandings of mathematical concepts. APOS is the result of qualitative research based on theoretical ideas from Piaget (1972) and Piaget & Garcia (1989) concerning reflective abstraction in the context of college level mathematics (Asiala et al., 1996). The research approach based on APOS theory has three components: It rests on a *theoretical analysis* (resulting in a genetic decomposition) of what it means to understand a concept and how learners construct new concepts. This leads to the design of *instructional treatment* focused on these mental constructions suggested in the genetic decomposition. Then, as they teach, instructors *gather data*, which are analyzed in the context of the theoretical perspective and used to revise both theory and instruction, as needed.

Briefly, the APOS perspective on a learner's construction of new concepts is as follows: An *Action* is a transformation of a mathematical object according to an explicit algorithm. It may be a manipulation of objects or acting upon a memorized fact. When a learner reflects upon an action, constructing an internal operation for a transformation, the action begins to be interiorized, becoming a *Process*. Each step in a process may be described or reflected upon without actually performing it; processes may be transformed through reversal or coordination with other processes. To construct an *Object*, a learner becomes aware of the process as a totality, reflecting on actions applied to a particular process and realizing that transformations (whether actions or processes) can act on the process. When the learner is able to actually construct such transformations, s/he has reconstructed a process as a cognitive object and the process has been *encapsulated* into an *Object*. Finally, a *Schema* is a collection of actions, processes, objects, and other previously constructed schemata, coordinated and synthesized to form mathematical structures. Schemata evolve as new relations between actions, processes, objects and other schemata are constructed and reconstructed. In our discussion of instructional practice, below, we relate the development of our modules to this APOS perspective.

The faculty in the LINE project worked jointly as a professional learning community (Hamos, Bergin, Maki, Perez, Prival, Rainey, Rowell & VanderPutten, 2009; McLaughlin & Talbert, 2006; Vescio, Ross & Adams, 2008). They began the project with a reading seminar in which the faculty jointly examined mathematics learning theories - including APOS and other theories of scientific learning as well as cognitive difficulties related directly to linear algebra (Bransford, Brown & Cocking, 1999; Dorier, 1995; Harel, 1989; Sierpinska, Drefus & Hillel, 1999; Viholainen, 2008) and collaborated in the initial development of modules for linear algebra using approaches based on APOS. The authors of this paper had experience for many years in the use of APOS theory and built their work on this learning theory as it was the one that was designed specifically for higher order mathematical learning and thinking. The professional learning community continues to foster faculty development through a combination of activities among the different institutions using videoconferencing and working meetings at each other's campuses. Faculty have also co-taught the modules, videotaped each other while teaching, observed student engagement, and used these videotapes and observations to reflect on the learning and teaching processes in the context of learning theories.

In accordance with the LINE perspective, at least two faculty members developed and implemented the modules at each site; one faculty member is a mathematics educator and the other is a STEM faculty member. The STEM faculty members are all mathematics professors except for one computer science professor. The faculty members worked collaboratively at each site, and across sites, to develop the modules and integrate them into classes. Most importantly, these modules shared a common aim: to introduce topics in ways that bring students both to reflect on what they are learning and to actively participate in the process of learning and potentially readjust their understandings as they are exposed to new concepts, building on what they know and using that knowledge to grasp new ideas. The linear algebra courses at each institution come at varying points in the students' mathematical education. Thus, the modules played differing roles, depending on the student populations and prerequisites for the class. For example, at one college the prerequisite for linear algebra is calculus I while at another college, students were expected to have completed a matrix methods class as well as an introduction to proof class. The linear algebra class at the second institution is closer to the linear algebra II class at the first. *Therefore, while the modules differed at the colleges, there was a joint collaboration in developing these modules so as to structure activities that would actively assist students in reflecting on the concepts and developing their understandings in terms of APOS.* The three institutions involved in this report are described below along with the prerequisites for linear algebra, how the modules were administered and what effect they had on students' grades.

Institution A is a four-year liberal arts public urban college (with some master's degrees) within a university system with enrollment of about 15,000 students. The Mathematics Department offers undergraduate degrees in mathematics in several tracks (that is, applied math, financial math, mathematics teacher, and theoretical math tracks). About 50% of its undergraduate majors are enrolled in the Grades 7-12 mathematics teacher track. The linear algebra course in this study is a sophomore level course with a variety of majors, such as mathematics, computer science, physical sciences, economics, and finance. The textbook used in this course was *Linear Algebra: A Modern Introduction*, 2nd edition, by David Poole. The prerequisite is calculus I. Students completed three modules in pairs and jointly handed in their work. One lesson was devoted to each module and each module counted as 5% of the grade (total of 15%). The following lesson would begin with a whole-class discussion of the module, what they had learned, and the main concepts of the module.

Institution B is a state land grant university with enrollments of over 14,000. The Mathematics Department offers undergraduate and graduate programs through the PhD. In addition to its own majors, it has an important undergraduate service role for science and engineering programs. The linear algebra course that was the focus of this study is a 400-level upper-division undergraduate course that also can be taken for graduate credit at the 600-level (it is not eligible for use in mathematics master's programs); the textbook used in this course was Sheldon Axler's *Linear Algebra Done Right*, 2nd Edition. Its prerequisites are the regular three-semester mathematics, science, and engineering calculus sequence and the 200-level transitional proof course, "Introduction to Abstract Mathematics." Students are also expected to have a background in matrix algebra such as provided in the 100-level lower division non-calculus based "Matrix Algebra" course. The majority of students in this linear algebra course, which is required for mathematics and secondary mathematics education students, are mathematics majors with other students from statistics, computer science, and physical science and engineering departments. In this course, students developed their responses to the modules outside of class and then presented their results in class, with discussion among students and the instructor. The four modules counted for 15% of the students' grades.

Institution C is a public urban research university in the Southeastern region, with enrollments of over 30,000. It offers more than 200 fields of study in 50 accredited programs awarding bachelor's through doctoral degrees. The Mathematics Department offers undergraduate and graduate programs through the PhD and provides undergraduate service to various departments, including K-12 teacher preparation programs. Linear algebra content is taught through two courses, "Introduction to Linear Algebra," a 3000 level course, and a dual (undergraduate/graduate) 4000/6000 level course titled "Linear Algebra." The LINE project was implemented in the Introduction to Linear Algebra course, which has as its prerequisites multivariable calculus and "Introduction to Proof", Math 3000, course. The textbook used in both courses is *Linear Algebra and its Application, 3rd Ed.,* by David Lay. At this institution, modules were used to introduce new ideas, with students using them to grapple and reflect before in-class discussion. Students were asked to complete the modules before the particular content was taught/discussed in class. Usually, the modules were distributed one week in advance and students were required to bring their written work on the day the content was taught. The lesson would start with a whole-class discussion about each question in the module. Through this discussion, the instructor had the opportunity to 'teach' the lesson making sure that all identified misconceptions about the content were addressed. The modules counted towards the homework component of their grades, about 5% of the total grade.

3. Methodology

3.1 Surveys

The focus of this paper is on analyzing student responses to survey questions about their experiences after the conclusion of the courses and triangulating this data with observations of group problem solving sessions followed by individual interviews of the students in the groups at one institution. Open-ended, short answer surveys are useful as data collection tools for this particular kind of research (Creswell, 2003). The researchers wanted to gauge in a qualitative manner how students reacted to the modules incorporated in their classes. As the modules varied at each campus and the researchers had studied their effects in other papers, this particular research was designed to study the success of the approach across campuses in terms of student satisfaction, reflection on their own learning, and perceptions of the nature of mathematics so that the theory-based pedagogical approach may be modified and improved. Surveys were the appropriate method to gather these kinds of data.

The survey used for this study was refined over two years. Earlier versions of the survey were found to be too general and students answered either very generally or focused on the new linear algebra content and not on their own understandings or experiences. The surveys were reviewed and revised with more specific questions that solicited students' reflections. The purpose of the survey was to gain student feedback about their classroom experiences and group work, the perceived impact on their learning, and whether the modules were engaging. Students completed the confidential end-of-semester questionnaires either in hard copy or online via Google surveys.

The survey questions were:

1. In linear algebra, we worked with matrices throughout. In addition to working with, and the deep use of matrices, what would you say are the three most significant ideas that you learned in this class? Why do you think so? Please explain clearly. (If you want to put four, it's okay.)
2. We know that linear algebra is a topic in itself and you may not have studied this subject before. So, other than the actual linear algebra content, was this mathematics class different than other math classes you have taken? If so, in what ways? Please explain clearly.

3. Other than the applications contained in them, in what ways were the modules (quizzes) helpful as a learning tool for you personally? If you think they were not helpful, please explain why.
4. What do you believe was the most striking aspect of the course that will stick with you after the class is over? Please elaborate with a few sentences.
5. In what way did working with a partner help or hinder your learning? What was the most useful part of the modules? Would you like to have more assignments in module form with a partner?

A grounded theory (Glaser & Strauss, 2009) approach was implemented to analyze student responses. While there were specific goals and objectives for student learning in this project and course materials were developed to reflect the APOS theory, students in the courses were not informed of the specific curricular development framework nor the project goals and objectives.

Analysis of the survey responses of 71 students proceeded in three distinct phases at three project meetings, each lasting several days, over the course of a year. There was no pre-existing list of themes that students were expected to mention. Given this, all of the written comments were individually reviewed by each researcher to create a list of the themes that appeared and then those lists were consolidated. As a group, the researchers reviewed the list, discussing how specific comments were seen to reflect an identified theme, and eventually agreed on a preliminary draft list of themes that they all had observed.

In the second phase, each researcher used these draft themes to again review each survey individually and classify student written comments by the agreed upon themes, also indicating comments that could not be classified or when the classification was questionable. The researchers then again compared the individual classifications as a group, revising the themes (both the term used and its definition) to better match the variety of student reflections.

During the third project workshop the researchers again reviewed student comments using the revised themes to classify comments and to finalize the definitions. The researchers sought themes and definitions that could independently be used to achieve high levels of consistent classifications. When that consistency was not achieved, student comments were left as un-codable. The developed themes reflected the negotiated categorization of the student survey responses. Only after this process did the researchers move to relate the classifications to the project goals and objectives.

With these results, the researchers seek to describe the *students' perspectives* on their learning and experiences, so the authors emphasize that the codes presented below reflect student perceptions rather than some preconceived conceptions of the researchers. The terms and definitions used for the codes reflect the research team's negotiated *interpretation of students' perceptions*. The researchers recognize that many of these terms may be interpreted differently in other settings. The final codes and their descriptions for the purposes of this paper are:

C: Abstraction - Students identified abstraction in linear algebra, discussed conceptual understanding, recognized recurring mathematical ideas across multiple contexts and/or explicitly mentioned the theoretical nature of what they had learned.

N: Nature of mathematics - Students discussed how their perceptions of mathematics as a whole have developed in the class. They may have indicated new insights into a broader interpretation of mathematics or discussed how what they have learned fits in with their prior knowledge.

M: Metacognition - Students reflected on their own understandings, especially how they have changed or been influenced by the class.

L: Language - Students specifically discussed, and perhaps confessed to some confusion with, language, vocabulary or terminology.

A: Applications - Students discussed the value *to their learning* of applications or modeling real life situations.

G: Group work - Students discussed benefits of collaborating with other students, such as talking over ideas, solving problems jointly, or the interactive dynamics.

R: Responsibility for learning - Students discussed their agency in their own learning. They may have mentioned the need for stronger study skills, new habits of thought, or increased self-motivation.

T: Traditional learning styles - Students indicated a preference for traditional models of teaching and learning. They may have identified difficulties with working with others, problem solving in the modules, or a preference for a textbook, teacher-centered, or lecture type of classroom experience.

P: Procedural - Students emphasized the computational, mechanical, or algorithmic aspects of linear algebra.

We related these codes to our four objectives for student learning as follows:

Objective 1. Students will gain more conceptual understanding of mathematical content. *Applications* and *Abstraction*.

Objective 2. Students will be more actively engaged in their own learning. *Applications*, *Group work*, *Metacognition*, and *Responsibility for learning*.

Objective 3. Students will gain self-confidence in their capacity to do mathematics. *Metacognition* and *Responsibility for learning*.

Objective 4. Students will develop a broader impression of the nature of mathematics. *Nature of mathematics*, *Language*, *Procedural* and *Traditional learning styles* (these last two codes were used to indicate Objective 4 had ***not*** been met).

3.2 Group Problem Solving

The analysis of the students' perceptions included a series of videotaped interviews with groups and also with individual students. The largest sample population was at Institution A and so we drew upon this population to conduct observations of their group problem solving as well as individual interviews (described below). The students were chosen based on their success in the class - 11 students had achieved a grade of B or higher; their availability and their consent to participate. Eight of these 11 students participated in this phase. The problem solving and interviews supplemented the data on (a) student performance on modules, (b) their individual course results, and (c) their survey responses. Groups of 4 students were given a set of 3-4 questions to solve during an hour-long interview. To illustrate, the problem below was designed to provide opportunities for the researchers to observe conceptual understanding of course content in an unfamiliar context along with group interactions:

> Group Interview Problem 3. Reminder: A Fibonacci Sequence is a sequence of real numbers in which each number equals the sum of the two preceding numbers. The first two numbers are arbitrary. Here are some examples of Fibonacci Sequences: $\{1, 1, 2, 3, 5, 8, 13, 21\ldots\}$, $\{3, 4, 7, 11, 18, 29\ldots\}$. Is the set of all Fibonacci Sequences a vector space? Assume regular vector addition and scalar multiplication. Elaborate on your answer.

Students in each group worked collaboratively to solve problems, discussing their reasoning and working the problems on a large board. These problem solving sessions were videotaped and analyzed afterward.

3.3 Interviews

After the group sessions the researchers held individual interviews with two of the students. All of the students were invited to participate in the interview but only these two were available to complete the interview. During the interview, the students were shown selected video clips of the group session. They then answered questions about their experience in the group, thoughts about the mathematics, and their level of confidence in the work the group produced. They were also asked about their reactions to, and reflections on the modules in the linear algebra course. These interviews were conducted by a LINE faculty member from another institution who did not know the students beforehand. The researchers sought to observe the relation between actual student problem-solving performance and their comments about the experience and the course, and the utility of our classification scheme for student reflections on their learning.

4. Results

4.1 Surveys - Some student reflections were more extensive than others and, therefore, touched upon more than one theme. While most comments were coded with just a single code, some were coded with two or three codes. There were 71 surveys in total. Table 1 summarizes the distribution of codes.

Table 1 – Student Survey Results – Frequency of Codes* in Student Responses

Code	Single Frequency	**Dual Code**	Dual Frequency (appeared together)	**Triple Code**	Triple Frequency	Combined Single and Multiple Frequency
A	19	**AC**	6	**ACM**	2	32
C	21	**AM**	2	**AMT**	1	38
G	18	**AR**	2	**CMR**	1	20
L	6	**CL**	1	**CNR**	1	10
M	13	**CM**	3	**LMR**	1	35
N	9	**CN**	2			13
P	9	**CR**	1			9
R	12	**GM**	2			24
T	6	**GT**	1			11
subtotal	***113***	**LR**	1			
		LM	1			
Uncoded	119	**MN**	1			
		MR	5			
		MT	1			
Total	232		**29**		**6**	**192**

*The 71 students provided a total of 267 comments. Of these, there were 113 comments that were characterized as representing a single theme; 29 comments that received two codes; 6 comments received three codes; and 119 were considered un-codable. The last column is the most important, since it lists how many student comments were characterized as representing a given theme. For example, the code A was given to a comment that referred to the value of applications in the course: A appeared 19 times as a single code, 6+2+2=10 times along with a second code, and 2+1=3 times with two other codes, in a total of 32 comments.

Below we provide more details for each of the four objectives and the three sources of data – surveys, group problem solving, and individual interviews.

4.2 Objective 1: Students will gain more conceptual understanding of mathematical content. Codes A (applications) and C (conceptual nature of the material, the most frequent comment) were two of the three most commonly mentioned ideas by students about the different linear algebra courses. Over a third of the coded responses - 70 out of 192 (36%), indicated that students believed that applications improved their learning or that the course had deepened their conceptual understanding of linear algebra. They appeared together as a double code - in other words, student responses included indications of both ideas, more than any other pair (8 times out of the total of 35 multiple coded items). The following response was coded both A and C:

"As much as I think some people had problems with the modules (they take up time that could be spent being 'more productive'), the modules are definitely beneficial for the class since they show a real world application of the ideas we're learning. Surprisingly, that is something so foreign to many classes. We usually learn concepts to apply them to constructed textbook problems. It is refreshing to see your work conquer a real problem, and it gives you a greater sense of usability and relevance to the concepts you are learning."

This next example was just coded A (Applications):

"The problem with academia is that we learn a lot, but most of the time knowledge seems sterile and not applicable in real-life. It makes a course so much more interesting, if we know where we can use all the new ideas that we are learning. Undergraduate students do not have any significant work experience. It's only after years as professionals that they understand what they were learning is useful!"

4.2.1 Group Problem Solving - In the analysis of the group work videos, it was observed that all students actively participated by sharing their answers, asking questions, offering their thoughts, making observations, and working out solutions. The ability of these students to coherently discuss each other's solutions, explanations or comments and to build on each other's thinking provided evidence of their conceptual understanding of the linear algebra topics involved.

4.2.3 Interviews – During the interviews, both students pointed out that the modules and the group work had helped them in thinking about linear algebra in ways that pushed them beyond what they would have done in a more traditional setting of homework and lecture. They also discussed the value of working in the group problem solving session and how their interactions with other students and hearing other students' ideas had broadened their own understandings.

4.3 Objective 2: Students will be more actively engaged in their own learning.

4.3.1 Surveys - For the second learning objective, we mapped the codes M (metacognition), G (group work), A (applications) and R (responsibility for learning). Of the 192 codes awarded, 111 or 58% were in these four categories. The survey questions seemed to parse out different aspects of the engagement in learning; most of the responses were labeled with single codes. M (35 occurrences) was the second-highest code used; it relates to Objectives 2 and 3. M was a code we took to indicate that students described being more actively engaged. Code A (applications helped learning) was the third most frequent, indicating the perceived value to their learning of modules that applied linear algebra concepts. Twenty student comments were about positive effects on their learning via group work during the course. Below are sample student reflections that we coded M, A, G or R:

R (responsibility for learning), A (applications): "I discovered that I am still interested in math but that I do not invest the time I should into the mathematics as I should. I also did see what I was doing with math. And learning on how I can apply math helps & motivates me to better understanding math. Talking!"

G (group work), M (metacognition): "Even though working with a partner or partners slow my completion time, I would like to have more assignments in module form with a partner/partners. Since they count for a small portion of the grade, it's really more about learning the ideas. Working with a partner will slow myself down and make sure I really understand what's going on, because I learned that if you can't teach it to other people, you probably don't really know what you're talking about. Also I'm all for having students communicate with each other."

4.3.2 Group Problem Solving - The analysis of videos from the group problem solving sessions revealed active engagement in problem solving. Students alternatively took the lead in writing the solution on the board. They shared their own solutions, commented, questioned, built on other members' ideas and offered alternative solutions. The researchers observed high levels of focused, on-task interactions in the group, indicating that students were cognitively engaged; they were able to follow somebody else's solution, ask questions, or explain their own solutions in different ways.

4.3.3 Interviews - Students confirmed the benefit of group work on their own understanding during the individual interviews through comments such as “everyone has something to add to the group and everyone has a different way of thinking,” “if they didn’t understand you’d have to figure out a new way to explain to them,” or “other people were pointing out some properties of matrices that I did not look at before or I didn’t think about when I was solving the problem.”

4.4 Objective 3: Students will gain self-confidence in their capacity to do mathematics.

4.4.1 Surveys - Students mentioned their confidence in solving problems as a result of the module experience in various ways. Persistence and success after persistence were a part of the theme for these comments. Of the total 192 codes, 59 (31%), related to this objective. The comments below included the two codes for Metacognition and Responsibility:

M (metacognition), R (responsibility for learning): “I learned so much, I would say the approach to proofs is where I learned the most. I developed a new way of think[ing] which was much more deductive and solid compared to what I had used in most of my other courses. I learned that in order to really understand a concept you have to think about different approaches, and maybe do some research on the side.”

M: “Being able to write down what I already know in words and describe my thought process which is a lot harder than it seems.”

M: “Linear Transformations, seems at first a very simple concept. However the deeper we got into the study of them the more careful I had to be. I wouldn’t say I don’t understand linear transforms, but that [sic] particular questions confuse me at times. The full importance of this kind of concept is hard to talk about, however linear transforms are required in many types of applications and studies from programming to problem solving.”

R: “That even though the material may seem different and difficult at first, that if I stick with it, it won’t be as difficult as it seemed to be.”

R: “I discovered that if I really sit and think about a problem, I will eventually come to an answer. Obviously this will help me in the future, and has increased my motivation to really work at a problem.”

A few students voiced a preference for a more traditional instructional approach, preferring more teacher or book-centered lessons or preferring lectures rather than collaborative peer group study. These comments were a minority: Only 11 responses (6%) were coded T for a preference for traditional classroom experience. For example:

T: “I felt the homework assignments and the book gave me a better understanding of the material.”

T: “Different: The biggest difference between this class and others was the presentation component. When I took the class I didn't mind, but I don't think it's the best way to run a math class. While being able to effectively communicate your ideas is important, it can also be really intimidating to some people. Another thing I noticed was that most of the assignments had one or two problems that were an application of the concepts instead of just purely abstract. I did enjoy having these to give a clear example of how to make use of the theorems and ideas.”

Finally, only nine students said that procedures were a focus of linear algebra, in contrast to the more common survey response (mentioned 38 times) that this course focused on concepts and ideas rather than procedures. In terms of APOS, a focus on procedures matches the lower level *Action* and *Process* levels of the theoretical framework. Here is an example of a response coded P:

P: “I think the three most important ideas learned in class are row reducing matrices, multiplying matrices and transformations. Row reducing a matrix really helps you understand if the ‘vectors’ contained in the matrix are linearly dependent or independent. Transformations were important as well since they cover how multiplying a matrix with a vector changes the vector, and leads to eigenvalues/eigenvectors. Multiplying matrices is just fun to be honest.”

4.4.2 Group Problem Solving - The analysis of group videos confirmed that students were confident in their mathematical abilities and, in particular, in their solutions to specific problems. Their flexibility in understanding and accepting a different way to solve a problem, arguing or explaining their thinking, re-wording their explanations when necessary, and active engagement in the process of writing the solutions on the board were indicators of their mathematical confidence.

4.4.3 Interviews – The above was confirmed during students’ individual interviews when they reflected on the group’s solutions and were asked how confident they were about them. In addition to stating that they felt confident, they also provided reasons why they felt confident such as referring to multiple methods by different members of the group that gave the same result. In some instances, they pointed out their particular contribution to the group’s solution - to something they had suggested or shared that no other group member had offered prior to that point.

4.5 Objective 4: **Students will develop a broader impression of the nature of mathematics.**

4.5.1 Surveys - Twenty-three coded comments (12%) were determined to be about students' broader views of mathematics. Students mentioned that their views of the nature of mathematics had been extended in some way by the class or they discussed the implications of language in mathematics based on their experiences in this class:

L (language), R (responsibility for learning): "The aspect of explaining everything that you do, mathematical vocabulary needed as well as the need to always be ahead and prepared."

N (nature of mathematics): "Also, the ideas behind what is done with the matrices probably pops up throughout math classes all the time, and they will most likely prove useful."

This theme did not arise in the student problem solving sessions nor in the student interviews.

5. Discussion and Conclusions

Our research agenda is based on the notion that the APOS framework provides a sound basis for designing instruction that promotes deeper conceptual student learning of mathematics. The framework is familiar to many mathematics education researchers; it is less well known by mathematicians who teach undergraduate courses. We believe that collaborative development of instructional materials by mathematicians and education researchers can facilitate instructional materials and practices that reflect this theory of the nature of deep conceptual understanding of mathematics. Such instruction should then lead to improved learning - both in terms of mathematical content and of student attitudes about the nature of mathematics. The focus of this paper is on the latter: We sought evidence that student attitudes and perceptions reflected this richer and more conceptual view of linear algebra. It is important to emphasize that while the instructors and researchers were aware of the theoretical framework, students did not have that information. Therefore our analysis of their comments reflects their perceptions of what they gained during the course without recourse to the language and framework that served as the structural basis of our development of course modules.

It must be reemphasized that this research agenda is not simply about the impact of a specific linear algebra module or modules. Our broader concern is how mathematicians and mathematics educators can collaboratively design and tailor instructional materials and practices to reflect the APOS theoretical framework in a variety of differing undergraduate settings. This paper examines the apparent impact of such shared course development practices in different institutional and course settings on student attitudes and beliefs about their own learning of undergraduate linear algebra. The modules in each course were specifically designed in alignment with the course and student body - it was the implementation of the theoretical framework that was shared across institutions and courses. This paper examines how students appear to have responded to those theory-based instructional practices.

As the researchers collected and analyzed student surveys about their experiences in these courses that were developed and implemented using the LINE project framework, the question of their credibility was raised. Would students be honest in their reflections? How accurate would their impressions of their own learning be? To help address such validity questions and to provide some context for the analysis of the survey responses, additional data were included, based on student performance in group problem solving sessions and on video interviews to gain a sense of student learning and performance in relation to their impressions.

Students at Institution A where the group and individual video interviews were conducted completed three modules during the semester. The class median score on the three modules was 95% and the mean was 91%. The two students who were interviewed alone following the group work had module mean scores of 96% and 99%. The eight students who worked in groups on the problems had a module mean score of 95%. The mean score of the final exam in this class was 81%. These scores seem to indicate that students who participated in the interviews, along with all students in the course, did as a group have a reasonable level of success learning linear algebra. Although one cannot directly tie responses on the anonymous survey to a specific student's work, this does provide increased confidence in the validity of student comments about benefits of this approach.

The videotaped interviews allowed the researchers to directly connect actual student performance on important conceptual problems with their reflections in the interviews - at least for a selected sample of students. Students normally worked in pairs on the modules at this institution while the instructor walked around the room and answered any questions. Students were observed problem solving in a group to better ascertain the group dynamics and how students were able to explain their thinking. These observations and any new insights were used to better understand the student reflections from the survey data.

For example, this representative comment was cited previously:

> "I learned so much, I would say the approach to proofs is where I learned the most. I developed a new way of think[ing] which was much more deductive and solid compared to what I had used in most of my other courses."

The course performance data and video analysis provided increased confidence that students were accurately and honestly claiming that "I learned so much." These additional analyses increased the confidence that students provided honest and accurate reflections in the surveys on their class experiences. Student reflections strongly supported the impression that three of the four project objectives for student learning had been promoted by the LINE framework, namely, (a) Students will gain more conceptual understanding of mathematical content, (b) Students will be more actively engaged in their own learning, and (c) Students will gain self-confidence in their capacity to do mathematics.

For these three learning objectives, survey results contained a high number of comments strongly suggesting that this experience helped them to grow in these ways. This has several implications for pedagogy. Although many professors are concerned that introducing such projects may take away from the time needed to cover topics, student responses indicate that to the contrary, they believed their understanding had been deepened by engaging in these problem-solving situations. At one site the professor introduced ideas via modules prior to class; students in that class commented that this approach helped them learn the material and that they felt better prepared for continuing study of the new material.

The 11 responses in which students professed a preference for traditional teaching approaches - that is, more teacher or book-centered, is a small subset of the overall student group (6%). The authors believe that this is important because the problem-solving approach was designed to push students beyond a procedural perspective to a deeper conceptual view and understanding of the material, corresponding to the *Object* and *Schema* levels of APOS (Arnon et al., 2013; Asiala et al., 1996). Student comments suggest that this approach did promote a more conceptual perspective.

Students made few comments that represented the codes addressing Objective 4 that *students will develop a broader impression of the nature of mathematics.* This does not necessarily mean that they did not broaden their view of the nature of mathematics, just that they did not mention it on the survey or the interviews.

At the three colleges, the LINE project increased the recognition in the mathematics departments that mathematics educators had developed useful research frameworks that could be used collaboratively to help core mathematics faculty design curriculum and instructional practices that would lead to deeper conceptual understanding of important core concepts by undergraduate mathematics majors. As a result, a broader group of mathematics faculty are enthusiastic about working with mathematics educators on new efforts to more effectively assess learning as candidates progress through the undergraduate program. Using a framework for analysis of student ability to comprehend, use and write proofs, faculty at two of the colleges' mathematics departments (Colleges B and C) will apply mathematics education research to improve the teaching and learning of proof and mathematics reasoning. Plans for this implementation include both the transitional proof course and the use of formative and summative assessments during and at the end of the program for all undergraduate mathematics majors.

This paper adds another dimension to the LINE project body of work. The authors have successfully facilitated an effective collaboration of mathematics, computer science and education faculty who used the LINE framework to design and implement a variety of modules in different linear algebra course settings (Cooley et al., 2014; Martin et al., 2010; Cooley et al., 2007). They were able to collaborate to develop instructional strategies and materials that had a positive impact on student mathematical learning and attitudes. Faculty and students have reported beliefs about the benefits of these new environments in their linear algebra courses: They not only said that conceptual learning took place, but also expressed positive impressions about the modules, their interactions with others in the class, and their broader mathematical experiences. The authors are convinced by this research that a course development framework that integrates research-based education theory with core mathematics content in the undergraduate curriculum can be implemented widely by teams of STEM and education faculty and is not limited to linear algebra (See also Arnon et al., 2013; Asiala et al., 1996).

Appendix A – Sample Module 1

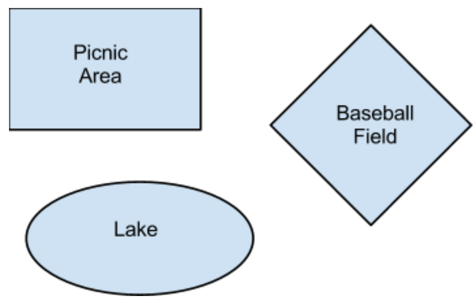
This module from Institution A was one of three completed in a semester, with one lesson dedicated to each one. Students worked in pairs and were allowed to use their notes and textbooks. Each pair wrote their solutions collaboratively and handed them in jointly. The following class, their work would be returned and there would be a whole class discussion about the module and what they had learned. As a reminder, the course prerequisite is calculus I.

Traffic Models

Complete this module with your partner. Write out all of your answers in complete sentences on a separate sheet of paper or NO credit will be given. Hand in your answers with both names on it.

Introduction

Suppose you have a park with three areas: the picnic area; the lake; and a ballpark. If you know how many people are in each location at a given time, can you predict the number of people in each location at a later time?



Assumptions

In order to model the number of people at each location, we'll have to make some assumptions on how the people move. For now, we'll assume time is measured in hours, and make the following assumptions:

- Of those in the picnic area at time t, 70% will be at the baseball field one hour later; 20% will be at the lake one hour later; and the remaining 100 - 70 - 20 = 10% will still be at the picnic area one hour later.
- Of those in the baseball field at time t, 20% will be at the picnic area and 40% will be at the lake one hour later, with the remainder staying at the baseball field.
- Of those at the lake at time t, 30% will be at the baseball field one hour later and 50% will be at the picnic area one hour later, with the remainder staying at the lake.

Questions – Important! Explain your answers in each question with full sentences or NO credit will be given:

1. Suppose that, at noon, there are 100 persons in each area. How many will be in each area one hour later? Two hours later?
2. Suppose there are x persons at the picnic area, y persons at the baseball field, and z persons at the lake. How many persons will be in each of the three areas one hour later? Express the total as a sum, using x, y and z.
3. Suppose that, at noon, there are 100 persons in each area. How many persons were in each area one hour *before* noon? (It's okay to get a negative value here.)
4. Is it possible for the number of people in each location to be the *same* at time t and time t+1? If this is not possible, explain why; otherwise, determine how many people would be in each area.

5. Express the number of people in each area at time t + 1 as a linear transformation of the number of people in each area at time t. In other words: Suppose the number of people in each area at time t is (x, y, z). What linear transformation A(x, y, z) will give you the number of people in each area at time t + 1?

6. How would you change the model to reflect that people both leave and enter the park?

APOS Genetic Decomposition of Sample Module 1

Here, we explain how APOS principles were used to construct the 6 questions of the module.

1 – This requires an *Action* conception of transformations. Students are given specific values and can rely on previous algebraic knowledge to carry out basic manipulations.

2 – This question requires students to utilize both *Action* and *Process* conceptions of transformations as students need to build on their conception from (1) that required them to compute movement on a particular data set and now express this transformation in a generalized system.

3 – In this question, students again need to draw on *Action* and *Process* conceptions. Asking student to think about how to unpack the concept of transformations will require a *Process* conception.

4 – This question uses a *Process* conception of transformations, assuming they can imagine the transformation while incorporating a different, but related idea of eigenvalues and eigenvectors. This broadening of a process conception of transformations to consider the concept in a more abstract way, while incorporating other concepts into their developing schema, and pushes students towards an *Object* conception of transformations. They had studied eigenvalues and this is designed for them to incorporate the idea in this situation.

5 – Students will need to reflect on their *Process* conception of transformation and unpack the basic tenets thus building an *Object* conception of linear transformations. They are asked to formulate the movement as a linear transformation while also asking them to think more broadly of the idea (object).

6 – This question needs an *Object* conception of transformations that will need to be de-encapsulated to analyze its component processes and repacked to apply the concept to a new situation.

Appendix B – Sample Module 2

This module from Institution B was the second of four completed during the semester. As a reminder, students completed the modules prior to class and presented their solutions. There would be a whole class discussion of each question. The course prerequisites were calculus III, the *introduction to proof* course and an expected introductory knowledge of matrix algebra.

Linear Maps

Make sure that you explain all your answers. Your solutions must be written up **clearly, legibly, in complete sentences,** primarily focusing on **explaining your reasoning**. For this particular assignment, most credit will be awarded for satisfying these conditions. As usual, $\mathbb{F}$ will denote either $\mathbb{R}$ or $\mathbb{C}$.

1. a. Give an example of a linear map $T: R^2 \to R^2$ such that $T(2,3)=(7,8)$ and $T(3,4)=(10,11)$. (If such a linear map does not exist, explain why.) For example, does there exist $(a,b) \in R^2$ such that $T(a,b)=(1,1)$?

   b. How many linear maps $T: R^2 \to R^2$ with $T(2,3)=(7,8)$ and $T(3,4)=(10,11)$ exist?

2. a. Give an example of a linear map $T: R^3 \to R^2$ such that $T(2,3,1)=(7,8)$ and $T(3,4,2)=(10,11)$. (If such a linear map does not exist, explain why.)

   b. How many linear maps $T: R^3 \to R^2$ such that $T(2,3,1)=(7,8)$ and $T(3,4,2)=(10,11)$ exist?

3. Let U, V be finite dimensional vector spaces over $\mathbb{F}$ and let $(u_1,\ldots,u_n)$ be a list of linearly independent vectors in U and $v_1,\ldots,v_n \in V$. Prove that there exists a linear map $T: U \to V$ such that $T(u_i)=v_i$ for all $i=1,\ldots,n$.
4. Let U, V be finite dimensional vector spaces over $\mathbb{F}$ and let $(u_1,\ldots,u_n)$ be a list of vectors in U. Assume that for every $v_1,\ldots,v_n \in V$ there exists a linear map $T: U \to V$ such that $T(u_i)=v_i$ for all $i=1,\ldots,n$. Prove that the vectors $u_i,\ldots,u_n$ are linearly independent.
5. Let U, V be finite dimensional vector spaces over $\mathbb{F}$, W a subspace of U, and let $T: W \to V$ be a linear map. Prove that there exists a linear map $S: U \to V$ that extends T. (S extends T means $S(w)=T(w)$ for all $w \in W$.)

APOS Genetic Decomposition of Module 2

The focus of this discussion is on the development of a student's concept of linear maps in relation to this module and the APOS framework. We recognize that the module requires action, process, object and schema conceptions of proof and introductory matrix algebra, but do not discuss those here.

Problem 1a primarily requires an *Action* conception of linear maps. Students are given specific vectors and use prior knowledge of functions and systems of equations to carry out simple computations.

Problem 1b requires both *Action* and *Process* conceptions of linear transformation because the student must be able to imagine the effects of linear maps in general on given vectors while building on their conception from part (a) that a map can be computed from the given vectors.

Problems 2a and 2b again have students draw on *Action* and *Process* conceptions in a setting that is less determined by the provided information. Again, students will need to use knowledge from prior work with systems of equations to solve the problems. Parts 1b and 2b provide a transition to the upcoming purely *process* conception needed in problem 3 by referring to a specific *action* that the students have carried out.

Problem 3 requires a *Process* conception of linear maps: the proof asks students to perform a more or less mechanistic analysis of the definition of linear maps.

Problem 4 also uses a *Process* conception of linear maps. Additionally, students need an *Object* conception of linear independence, that is, they need to be able think abstractly about what constitutes linear independence and to apply it in this proof situation.

Problem 5 asks students for a proof that draws on an *Object* conception of linear mapping that must be *de-encapsulated* to analyze its component processes.

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