

Common Visual Representations as a Source for Misconceptions of Preservice Teachers in a Geometry Connection Course

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In this paper, we demonstrate how atypical visual representations of triangle, square or a parallelogram may hinder students' understanding of a median and altitude. We analyse responses and reasoning given by 16 preservice middle school teachers in a Geometry Connection class. Particularly, the data were garnered from three specific questions posed on a cumulative final exam, which focused on computing and comparing areas of parallelograms, and triangles represented by atypical images. We use the notions of concept image and concept definition as our theoretical framework for analysis of the students' responses. Our findings have implication on how typical images can impact students' cognitive process and their concept image. We provide a number of suggestions that can foster conceptualization of the notions of median and altitude in a triangle that can be realized in an enacted lesson.

Keywords: visualization, misconception, median, altitude, area, preservice teachers.

The use of visual representations of mathematical objects has been an integral part of the process of learning mathematics (Battista, 2007; Presmeg, 2006; Arcavi, 2003; Duval, 1999), and can help with mathematical reasoning (NCTM, 2000). Visual objects are sometimes necessary to articulate the need for new mathematical notions, especially when we lack more precise analytical description of those notions. The visual objects may also be used as an intermediary between symbolic and verbal representations of some mathematical objects, and will strengthen learners' ability to manipulate and express in an analytical way some of the properties of the objects they represent (Arcavi, 2003; Brown, 2008; Duval, 1999). They help in placing the mathematical objects they represent in the appropriate category of objects with similar or same characteristics.

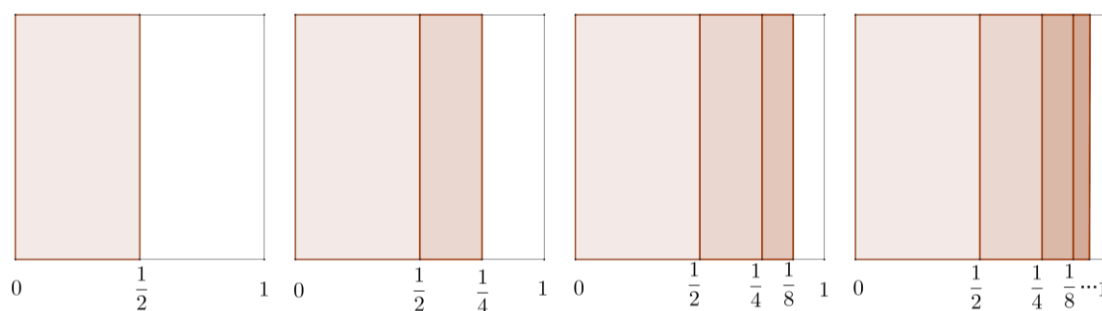


Figure 1. This sequence of figures suggests the possibility of adding infinitely many numbers (areas of corresponding rectangles) and obtaining a number (the area of the square).

As an example, presenting an infinite sum of positive rational numbers as an infinite union of non-overlapping rectangles whose areas correspond to the given numbers, is a

case of an intuitive thinking that justifies the search for a proper analytic context and grounds for the notion of “adding infinitely many numbers” (Figure 1).

In geometry, visual objects are tools that can support conceptual understanding of a mathematical idea being presented (Lowrie & Diezmann, 2007; Hershkowitz, Haim, Holes, Lappan, Mitchelmore, & Vinner, 1990, p. 94; Hershkowitz, 1989). Specifically, images of geometric objects help in solidifying the abstract nature of mathematical notions, they can guide learners’ intuition and provide means on how to solve a given problem, or how to develop mathematical arguments (Brown, 2008, p.200; Arcavi, 2003; Diezmann & English, 2001).

Given that diagrams play an important heuristic role in guiding students’ intuition when constructing viable arguments (proofs), or when introducing mathematical objects, careful thought ought to be given as to what aspects of a diagram construction or interpretation of a diagram should be emphasized. A visual image depicting a mathematical object should express the arbitrariness of the object by excluding details that may derail an intended interpretation, and it should not create a categorical property by mere coincidence in the choice of its components (Dimmel & Herbst 2015). Therefore “It is important for students not only to understand a feature of a diagram when their attention is called to it, but also to recognize on their own that a diagram may contain information needed for the solution of a problem, and to develop a habit of looking to diagrams as a source of such information” (Zimmerman, 1991).

Considering the importance of images in a learner’s conceptualization of geometric notions, we sought to unpack the phenomena on how images can influence middle school preservice teachers’ notions about geometric concepts and can be a source of misconceptions. We will also analyse the degree to which a visual image can impact preservice teachers’ (PST’s) computational skills of area of a parallelogram. Specifically, in this article we describe misconceptions middle school mathematics preservice teachers exhibited when analysing relations between altitude and median in a triangle and the area formula of a triangle. Furthermore, we look into the difficulties PST exhibited when a given image of a mathematical object is in conflict with the concept image of the same object. We see our investigation as an initial step into a more systematic study that will examine the effect of visual representations on the relation between students’ concept image of an altitude in a triangle or a parallelogram and the concept definition of these objects. Looking from this perspective, our research question sought to answer the following question:

How do atypical representations of geometric objects affect middle school preservice teachers’ accuracy and reasoning in solving geometry problems?

Following Cannon and Krajcevski (2018), a *typical image* of a particular mathematical object is a visual representation of that object that is drawn a certain way in the majority of instances with no content-based reason. As an example, a right triangle represented so that one of its legs is parallel with the line of the text, is an example of a typical image. Cannon and Krajcevski (2018) show evidence that majority of images of parallelograms and triangles in high school geometry textbooks are typical images. How these images affect middle school PSTs has not been systematically researched, and although the constraint of our small sample size is difficult to generalize, we believe that our study will provide impetus for more detailed and robust findings when conducted over a bigger sample of middle school mathematics preservice teachers.

Literature Review

Visual images, representations or experiences that are often recalled for a particular concept or for a mathematical notion are referred to as a *concept image*, while the verbal definition that can be recalled about specific concept is referred to as a *concept definition* (Vinner, 1983; Tall & Vinner, 1981). For individuals with inadequate concept image, geometrical arguments are based on properties of a prototype, and they generally reject examples that do not reflect their perceived prototype. Individuals with a somewhat stronger concept image make arguments based on prototypical images and some additional mathematical properties; while individuals that have a completely developed concept image, have a large variety of examples, and an in-depth knowledge of the relevant properties that align with the examples (Vinner, 2002). Therefore, images presented during enacted lessons or images that are within geometry textbooks, can potentially affect preservice teachers' conception of a geometrical concept and their internal visualization of these mathematical objects. In many problem-solving situations, it is also important to introduce some auxiliary elements in the given visual representation thereby further increasing the visual content and reducing the cognitive challenge of the problem (Polya, 1957, Kaufmann & Helstrap 1985; Uygun & Akyuz 2017).

Pedagogical practice shows that many high school and collegiate geometry students do not make the distinction between a mathematical object (notion) and their physical realization in the form of a visualization object or picture (Brown, 2008a Herbst at al. 2017). Just as an illustration (Figure 2), if AH is the altitude from vertex A in the triangle ABC assuming that the angle at the vertex C is an obtuse angle, for majority of geometry students, the altitude AH will not exist, or it will not be introspectively visualized, unless drawn on the paper or a whiteboard. Introducing the altitude AH as an auxiliary element in the image visualizing of the triangle ABC , will provide valuable insight on how to apply the basic formula for the area of a triangle if we take side BC to be a base of the triangle (Polya, 1957, see p. 47; Brown, 2008, see p.105).

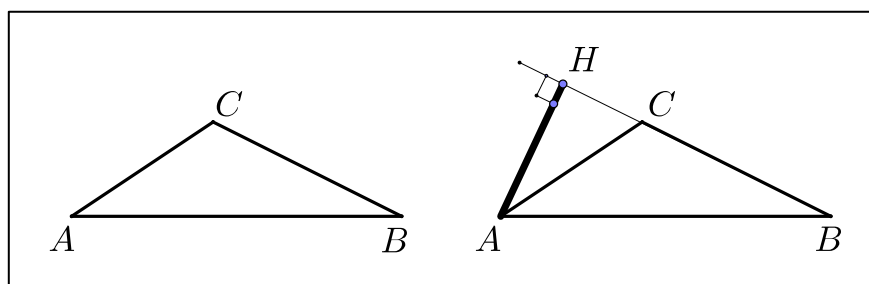


Figure 2. The existence of a mathematical object in the educational practice is often established after this object has been drawn on the whiteboard.

Even though preservice teachers may be able to provide a concept definition of a geometrical notion, they may still experience difficulty in constructing the correct concept image, especially when the illustration used is not a typical representation of the geometrical notion in question (Cunningham & Roberts, 2010; Ward, 2004; Blanco, 2001). The ability of a mathematical illustration to bend the structure of the cognitive process has been indicated by many researchers. As noted by Fischbain (1993) and later by Herbst at al. (2017), when students engage in reasoning or proving they have to reconcile conceptual and figural aspects of an illustration. Gutiérrez and Jaime (1999) analysed preservice

teachers' understanding of altitude and found that the preservice teachers' concept image was slightly better than their students' concept image. Their concept image was based on prototypical figures, and the presence of a concept definition did not influence their responses when asked to identify a particular geometrical concept. Blanco (2001) found that preservice teachers were able to define an altitude correctly; however, they drew the altitude and the orthocenter incorrectly. Ward (2004) points to preservice teachers' difficulty applying verbal descriptions of mathematical objects even though they gave correct mathematical definition of a shape in question. Other researchers indicated that preservice teachers have difficulties stating precise mathematical definitions as a result of their limited ability to use the mathematical language (Gokbulut & Ubuz, 2013).

Some of the difficulties students experience when solving geometry problems relate to figuring out when and where to add auxiliary elements on a given figure. Most of the time this will be helpful in choosing a productive path of inquiry and solving the question correctly. An auxiliary element (in a drawn figure, or given data) can be defined as an element that is not present (either drawn or part of the given data) and its introduction will further the solution of a given problem. As indicated by Polya (1957, p. 46), without introducing them, we cannot make any concrete use of the definition of altitude or area in our case. Senk (1985) has found that many students had difficulties with auxiliary lines (segments) and suggested that students need to be taught how, why and when they can transform a diagram. According to Yerushalmy and Chazan (1990), adding auxiliary lines helps students access their prior knowledge. In our research, students were expected to draw the joint altitude of the two triangles in the first problem as an auxiliary segment, a segment that will make connection between the new problem situation and previous knowledge. This will contribute towards building stronger concept image and it will improve students' spatial ability. (Nemirovsky, R., & Noble, T., 1997).

Knowing how to use PST's visual representations to connect to their reasoning in problem solving situations requires careful examination of PST's justifications when providing arguments (proofs).

The Method

We collected data from one cohort (n=16) of middle school mathematics preservice teachers enrolled in the Geometry Connections course in 2015 academic year at a research one university. The data were garnered via preservice teachers' response to a two hours in-class final exam and analysed thematically. In the subsequent paragraphs, we describe the context of the study, participants and the process used to collect and analyse the data.

Content Coverage and Organizational Structure of the Geometry Connections Course

The Geometry Connections course is a course that has been offered by the Department of Mathematics and Statistics from Spring 2014 annually. This course is designed to present elementary geometry content, which middle school mathematics preservice teachers need to know, following the recommendations from the Common Core State Standards for Mathematics and Standards for Mathematical Practice. The course's focus is on developing specialized content knowledge (Hill, Rowan & Ball, 2005) relative to geometry at the middle grades level. The course content includes the structure of logical arguments, axioms and basic propositions of Euclidean geometry, brief introduction to analytic geometry and few elements of modern (transformational) geometry. Our goals in

this course were to straighten students' ability for deductive reasoning and build up their appreciation of an axiomatic system in mathematics. Course objectives were to enable students to justify main results (theorems) of Euclidean geometry in a deductive way as a logical consequence of previous results or the adopted axioms. There was no designated book for the course although students were encouraged to consult other geometry books or online sources. They were provided with weekly handouts, gradually advancing through the syllabus. For example, the two-page handout entitled "Area" (see Appendix 1), after providing a definition of a polygonal region, specifies the axioms every area function must satisfy (congruent triangles have equal areas, additivity of the area function in case of non-overlapping polygonal regions, and area of a rectangle is a product of its side lengths). The handout is closing with four theorems stating the areas of a right triangle, arbitrary triangle, parallelogram and trapezoid, without providing proofs. Students were able to prove that the area of a triangle is the semi-product of (the length of) a base and corresponding altitude. Following the axioms and the previous theorem they also proved the area formula for a parallelogram. There were no illustrations in this handout other than picture of a polygonal line and visual examples of polygons (octagon and concave quadrilateral) and non-polygons. The ubiquitous association of the area formula with the typical visual representation of a triangle was not given.

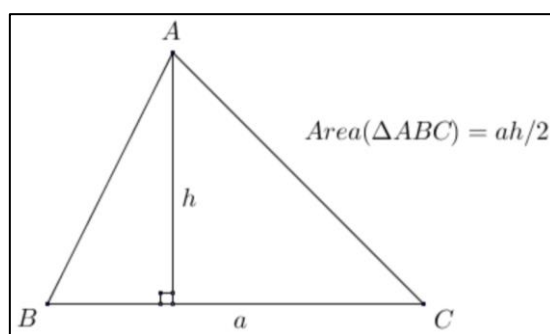


Figure 3. Typical visual representation of area formula for a triangle (see Cannon & Krajcevski, 2017).

To structure the class, the instructor employed direct instruction, facilitated class discussions, and engaged preservice teachers in solving geometry problems on the printed course materials. Most of the time new concepts were introduced in a visual way, avoiding mathematical symbolism at the moment of their introduction, highlighting connections between the concepts. The next step consisted in building adequate scaffolding for students' attempts in providing logical arguments for their claims. The first author, who was the instructor on record for this course, also made an effort to emphasise the relationship between the visual representation of mathematical notions and their symbolic/analytic description. Visualization was presented as a way of modelling mathematical structure and we strived to emphasize the distinction between a mathematical object and its visualization as a material object. During the semester, preservice teachers were assessed via two tests, a final exam, weekly homework assignments that typically comprised of two to three problems, mostly related to proving some of the propositions in the handouts. If one is to characterize broadly the learning outcomes of this Geometry connection course, then this is a course upon whose conclusion students will be able to prove the most basic theorems in (Euclidean) geometry.

Participants

Study participants were 16 middle school mathematics pre-service teachers (PSTs), who were enrolled in an innovative standards-driven STEM middle school teacher education program (for a detailed explanation of the Helios Middle School Residency program, see Ellerbrock, Kersaint, Smith, & Kaskeski, 2016). During the time of the study, the PSTs were in the second semester of their junior year and were enrolled in the Geometry Connections course concurrently to the Algebra Connections course. In previous semesters, they had taken College Algebra, Calculus I, Number Connections and Introduction to Probability courses and had been exposed to the core strands of mathematical practice. These mathematics courses collectively satisfied 18 hours of mathematics content coursework needed to satisfy the state certification requirements for middle grades teachers. Thus, PSTs have been exposed to the core strands of mathematical practice prior to the data collection period for this study.

Data Collection

Collected data was from preservice teachers' responses to an in-class proctored final exam (see Appendix 2). This exam consisted of 6 questions in which PSTs were asked to: (1) state a definition of a geometric notion (chose three from: an angle bisector, a minor arc, an altitude, and a quadrilateral), (2) write a contrapositive and converse of two given propositions, (3) prove two of the given three theorems whose proofs were provided in the class notes (one of the theorems was that the area of a parallelogram is a product of the lengths of its base and the corresponding height), (4) construct a circle tangent to two parallel lines and passing through a given point, (5) determine equations of the lines bisecting the angles between the lines $y=2x$ and $y=0$, and (6) compute the area of a rhombus of side a and one interior angle of 30° , with accompanied figure given below the text of the problem (see Figure 4).

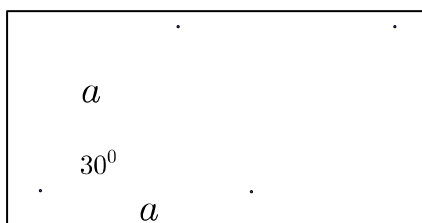


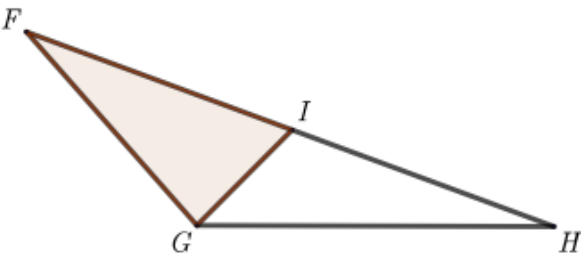
Figure 4. Problem #6: Compute the area of the above pictured rhombus.

Additionally we posed two bonus multiple-choice problems (see Figures 5 and 6) with an additional open-ended free response question asking students to provide justification as to why they have selected a particular response for each of the bonus problems 1 and 2. The points gained from the correct answers on these problems were credited as bonus points that can potentially improve their overall standing in the course. As previous research shows, extrinsic motivation can enhance students' learning experiences (Middleton, J.A., & Spanias, P., 1999). Thus, the bonus points sought to motivate the preservice teachers to challenge themselves and to do well on these questions.

We will briefly comment on the problems 1,3,6 and we will give more detailed analysis of students' response to the two extra problems indicating a common thread in all these problems.

In the first of the bonus problems students were asked to compare the areas of two triangles (see Figure 5). Previously in class, student derived the basic formula for the area of a triangle (one half the product of the lengths of a side and corresponding altitude) from the axioms about area function of a polygonal region. There were few elements of the question designed so that the preservice teachers do not simply use mnemonic $bh/2$ (where b represents a base, and h represents the corresponding altitude) for the area of a triangle, but they have to demonstrate understanding gained from the proof of this formula. Also, labels on the vertices of the triangle were not the familiar ABC labels, hence, none of the sides had label “ b ” or “ a ”, and the common altitude to the sides IF and HI (see Figure 5) from the vertex G , has not been drawn. What was presented was the median of the side HF , which does not play a role in the basic area of a triangle formula (meaning, students have not been acquainted with the proposition that every median divides a triangle into two triangles of equal area).

Extra problem 1. In the obtuse triangle $\triangle FGH$ on the figure below, the point I is the midpoint of $\overline{FH}$ and $GH > GF$.



Which of the following is true?

- a) The area of the triangle $\triangle GIF$ is greater than the area of the triangle $\triangle GIH$.
- b) The area of the triangle $\triangle GIF$ is less than the area of the triangle $\triangle GIH$.
- c) The area of the triangle $\triangle GIF$ is equal to the area of the triangle $\triangle GIH$.
- d) There is not enough information to compare the area of the triangle $\triangle GIF$ with the area of the triangle $\triangle GIH$.

Figure 5. Bonus problem #1.

There is also one unnecessary piece of information as a decoy, namely it is given that the side GH is larger than the side GF . Hoping to see a correlation between preservice teachers' ability to apply formula for the area of a triangle and the typical illustration accompanying this formula, in the accompanied figure, we positioned the key elements for the solution of this problem in an atypical way to what is commonly represented within geometry textbooks. The relevant joint altitude of triangles GFI and GIH when internally

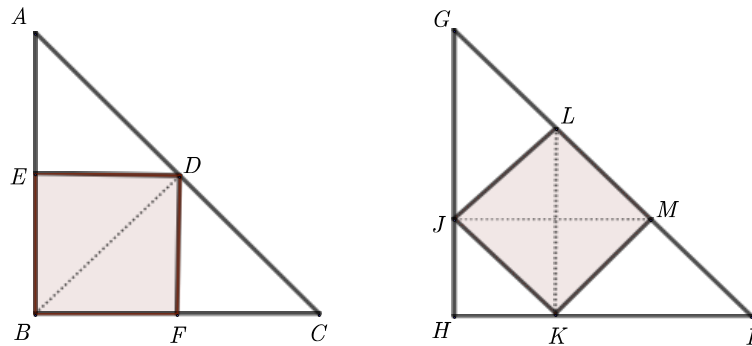
visualized, will not appear perpendicular to the horizontal line determined by the text and similar remark applies to the sides IF and HI of both triangles, not being drawn parallel to the line of the text of the problem. Of course, the correct choice in the first problem is the third option (c) because IF is congruent to IH and the triangles GIF and GIH have the same altitude from the vertex G (not sketched), thus they have the same area.

In the second problem (Figure 6) students were asked to compare the areas of two squares inscribed into two congruent isosceles right triangles. The notion of a square inscribed in a triangle (or more general notion of a polygon inscribed in another polygon) has not been previously defined in a formal way. Nevertheless, we believe that for the given problem an image will be sufficient in clarifying this concept. The difference of the areas of the two inscribed squares is less than 2.8%, which is difficult to perceive from the images of these squares inscribed in congruent triangles and use it as a hint for the correct answer. Again, we see that the inscribed square in the triangle on the left being in a typical position (with its right angle coinciding with the right angle of the triangle), offers straightforward analysis. In the triangle ABC , the segment BD is an angle bisector because it is diagonal of the square $BFDE$, and it is also an altitude and a median in this triangle. (Figure 6).

Notice that two sides of the square and the diagonal BD partition the triangle ABC into four congruent isosceles right triangles, making the area of the square one half the area of the triangle. Similarly, the altitudes of the triangles GJL and KIM from the vertices L and M respectively (these are not pictured as segments in the triangle GHI) will complete the partitioning of the triangle GHI into nine congruent isosceles right triangles, making the area of the square $4/9$ the area of the triangle. Therefore, the correct choice for the second problem is (a).

At the end we asked PSTs to explain their reasoning for the choices they have made on the last two problems. Not every student provided justification for the choice he/she made. When students were using auxiliary drawing to explain or support their justification, we examined these drawings and determined the nature of student's justification based on these drawings.

Extra problem 2. Squares have been inscribed in congruent isosceles right triangles ABC and GHI as shown in the figure below



Which of the following is true?

- ☒ a) The area of the square $BFDE$ is greater than the area of the square $JKML$.
- ☐ b) The area of the square $BFDE$ is less than the area of the square $JKML$.
- ☐ c) The area of the square $BFDE$ is equal to the area of the square $JKML$.
- ☐ d) There is not enough information to compare the area of the square $BFDE$ with the area of the square $JKML$.

Write a justification for each of the answers in the previous two problems.

Figure 6. Additional Problem #2.

Data Analysis

The data obtained from preservice teachers' responses were analysed and reported by using descriptive statistics and content analysis methods. We calculated frequency and percentage of their choices for each option in extra problems 1 and 2. In four cases, although students did not provide justification, they used the images as a basis for their analysis by producing some auxiliary drawings to support their choice. To analyse preservice teachers' responses in their justifications, qualitative content analysis method was used. Therefore, data were presented in words and themes, which made it possible to interpret the results (Porter, 2002). Following this methodology, the responses for the bonus problems were sorted relative to the students' choices. For example, all of the responses of individuals that selected (a) were grouped together for each problem, and all of the responses of individuals that selected (b) were grouped together, etc. The grouped data were coded for the pre-set codes (altitude and median of a triangle) and emergent

codes relative to the use of the words, altitude, height, isosceles triangle, equal length measures, square, congruent and area formula.

The data were then organized in a tabular format, which indicated the percentage of preservice teachers that selected a corresponding item. For the inter-coder reliability of the research, researchers coded the data separately, and there was an acceptable agreement between the coders. Subsequently, authors met to discuss the themes and their implications as to the nature of preservice teachers' conceptual understanding of the concept of altitude and median of triangles.

Results

We will summarize the accuracy of students' responses on problems 1,3 and 6, and we will give more detailed analysis of students' response to the two bonus questions, indicating a common thread in all these problems.

With exception of one student (whose choice included ii) from problem 1 every other student chose to provide the definitions of angle bisector, an altitude and, quadrilateral, although only 4 of them (25%) gave *mathematically* correct definition of an altitude. This definition stipulates that one of the end points of an altitude may not be a point on the side of the triangle opposite the other end point, but on the *line* generated by this side (see Figure 2. as an illustration of this case). This is one of the key components of the solution to the first of the bonus questions.

For problem 3, majority of students (12 or 75%) chose the combination 3.1 and 3.3, in which they had to prove that in every parallelogram the opposite sides are congruent, and that the area of a parallelogram is a product of its base and the corresponding height. Only 4 (25%) chose the combination 3.1 and 3.2. in which they have to show that an angle inscribed in a semicircle is a right angle. Out of 12 students 2 (less than 17%) provided a proof that was not discussed previously in class (these two students used one of the parallelogram's diagonals to divide the parallelogram into two congruent triangles, and then used one of the axioms for area function of a polygonal region and already proven formula for the area of a triangle). All students used a correct mathematical notation and each of them had drawn a picture of a parallelogram indicating the altitudes to the base that will be used in their proof of the area formula. Additionally, each of the drawn parallelograms was tilted to the right as illustrated on Figures 7a and 7b, and none of the students considered the case when the altitude will fall at a point that does not belong to the base of the parallelogram. The students did not consider this as a separate case that had to be taken into consideration, and consequently did not provide justification.

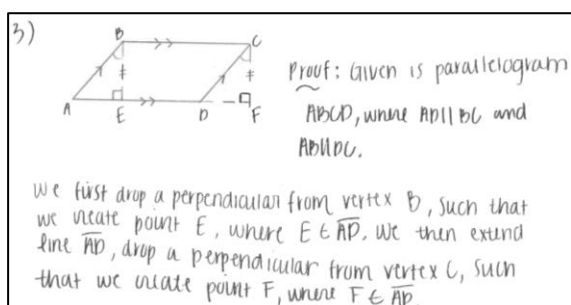


Figure 7a.

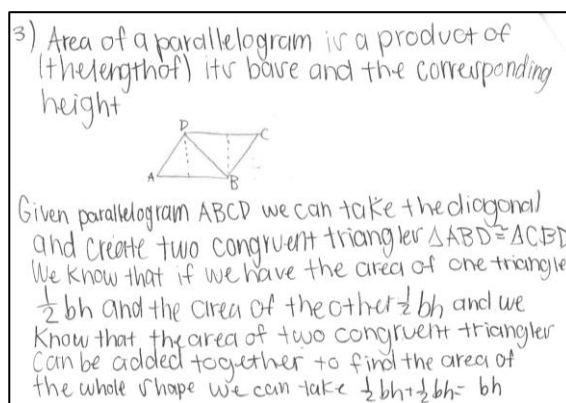


Figure 7b.

Problem 6 was attempted by every student with 8 (50%) of the students attempted to solve the problem algebraically by recalling the Pythagorean theorem. This problem was designed as a two step problem in which students should notice that: 1. The right triangle formed by the two sides of the rhombus (making the angle of 30°) and an altitude of the rhombus will create a 30° - 60° - 90° right triangle, so the side opposite the 30° -angle is “half the hypotenuse” of that right triangle, or $a/2$. Therefore, the area of the rhombus will be $a(a/2)$, or $a^2/2$. Only 7 (44%) of the students in their auxiliary drawings indicated the angles of this (auxiliary) right triangle but none of the pre-service teachers answer this question correctly.

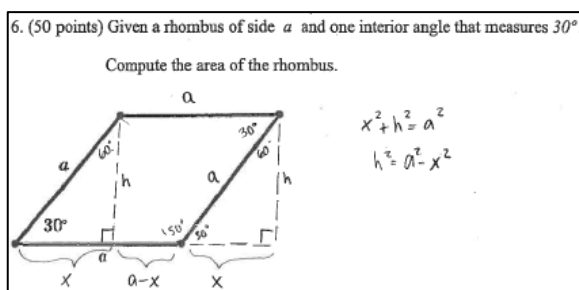


Figure 8a.

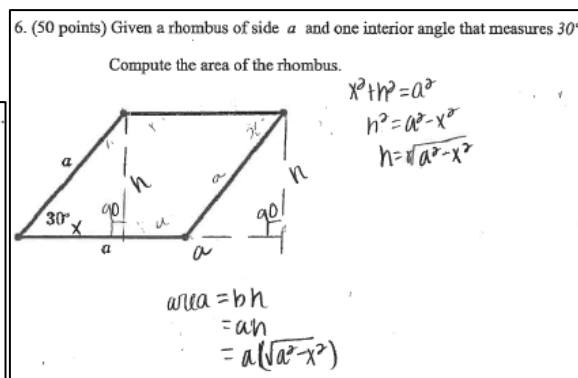


Figure 8b.

For the purposes of our research we will provide more detail into the solutions of the bonus questions. Although preservice teachers were more successful on the second of these questions, the reasoning provided on both tasks suggest the visual representation have been interpreted incorrectly and not fully conceptualized. The justifications provided by the preservice teachers, in both problems, add credibility to our hypothesis that preservice teachers were challenged to arrive at the correct solution due to an atypical visual representation of the elements critical to the solution of these problems.

In the first of the bonus questions, students had to compare the areas of two triangles that are positioned unlike most of the triangles they have encountered. Table 1 summarizes students' responses, and their justifications for choosing one of the multiple-choice options. For this question, two of the eight students that chose D as an answer did not provide a rationale for their choice. One student suggested that there was a lack of information relative to the sides of triangles FIG and GHI, and no information presented relative to the size of the angles of the triangles, as a reason for not being able to compare the areas of the two triangles. The most prevailing argument for students who chose D as their response can be illustrated with the following two quotes from students:

- There is not enough information because the area of a triangle is $\frac{1}{2}bh$ and we don't know the heights of both triangles.
- There is not enough information because we do not know the height at base of the triangles; if GH and GF are the bases then $\triangle GIH$ will have a greater area than $\triangle GIF$. However, we do not know this for sure.

We summarize the justifications for the choices in bonus question 1 in the following Table 1:

Table 1
Responses provided to bonus question 1

Selected response	Number of selected response (N =16)	Summary of the key arguments provided
A	2 (13%)	No rationale provided, only auxiliary drawing as justification of this choice.
B	7 (44%)	GH>GF. GI is an altitude. Altitude of $\triangle GFI$ < Altitude of $\triangle GIH$.
C	1 (6%)	Only one student answered correctly.
D	6 (38%)	Don't know the heights of both triangles. $\triangle FGH$ is not isosceles triangle. No heights or angle measures are given.

We see that students have not internalized the notion of an altitude in a triangle and by referring to “the height” of the triangle we can speculate that their concept image of a height is one that is represented by a segment drawn perpendicularly to “the base” of the triangle given horizontally. The altitude from the vertex G that will be the joint altitude for both triangles (but cannot be positioned vertically), is not part of the students’ concept image of an altitude, although they know the concept definition of an altitude in a triangle. Students are recalling the standard formula of the area of a triangle ($\frac{1}{2}ah$), although none of the initial letters of the alphabet labelled the vertices of the triangle, and the letter ‘a’ labelled none of the sides of the triangles. This further indicates that students’ concept image associated with the area of a triangle formula is a triangle whose chosen base for the area formula is “horizontal”, which is the typical image of a triangle. We present two examples of auxiliary drawing of altitudes in the Figures 9a and 9b.

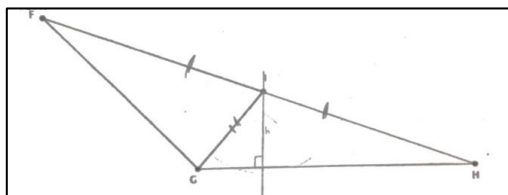


Figure 9a.

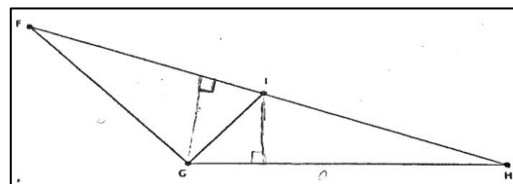


Figure 9b.

None of the two students who chose (a) as an option have provided justification for their choice. These students did not give written justification for their choice but they

explained their thinking by modifying the provided image in the question, one of which we present as Figure 6b. Notice how the student sketched the altitude from the vertex D : it is “almost perpendicular” to the side GI , but it is not accepted as an altitude of the triangle GIH because of the presence of “real altitude” that is positioned perpendicularly to the horizontal base of the triangle.

Almost all students who chose (b) as an answer provided $GH > GF$ as a justification for their choice. For those students, assumption that GH is larger than GF is the dominant piece of information from the question. Here are two citations from students:

- If $GH > GF$ then altitude/height of triangle $\triangle GIF$ would be less than triangle $\triangle GIH$ which would make that if GIF 's altitude was multiplied by the base of GI then it would be less than if it were multiplied by the height of GIH making the area less.
- Since $GH > GF$ the triangle $\triangle GIF$ is smaller the $\triangle GIH$, which means its area is less than GIH .

The second problem presented a different challenge for the students. What is atypical for the second of the bonus questions is not the position of the right triangles but the position of the inscribed square.

Here are the results of the students' responses to the second problem.

Table 2
Responses provided to Problem 2 and Justifications Provided

Selected response	Number of selected response (N =16)	Summary of the key arguments provided
A	1 (6%)	No justification. Only $BFDE = 1/2$ of $\triangle BCA$, and $JLMN = 4/9$.
B	0%	- We can rotate $JKML$ and get $BFDE$. Because the triangles are congruent. $BF = KM$.
C	12 (75%)	Pythagorean Theorem. The heights of both triangles are congruent. Area = s^2 .
D	3 (19%)	We don't know BC & HI . Don't know the length of GL or MI .

Only one student correctly identified the right answer using the provided figure to do a short computation without justifying his/her answer. (Figure 10).

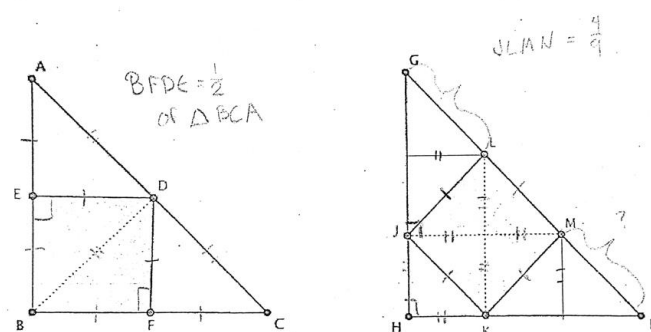


Figure 10.

Majority of students (12 students) chose C as an answer to the second bonus question, providing variety of justifications or no justification at all (4 students). There is no common theme among these justifications. We will illustrate with the following example:

We know that square[s] inscribe[d] in the same triangle will always be congruent because of the Pythagorean Theorem. Because the two squares are congruent they will then have an equal area.

Justifications for choosing D as an answer to this question are similar to the justifications for the first bonus problem. Students seek numerical values associated to given notions, in order to compare them. Here are two typical examples of this type of thinking:

- Since we do not know the length of GL or MI , it is not safe to assume that $LM \cong ED$ because we cannot assume that if the figure looks congruent to another then it is congruent. Although figure $JKLM$ appear smaller than $BFDE$ we do not know the side lengths.
- There is not enough information because we don't know for sure the side lengths of the two inscribed squares and you need to know them in order to compute the area of a square. Area of square: (side length) 2.

Thus, it appears that the preservice teachers frequently asserted lack of information as a rationale for the limited response they provided.

Discussion

Our findings confer with previous research indicating the difficulties PSTs have in developing a working knowledge of the basic geometric concepts (Fujita & Jones, 2007). We hypothesize and attribute this to their inadequate concept image of altitude and median in a triangle and insufficient practice with using and modifying visual information. As pointed by Hershkowitz (1987) a reason for having a poor concept image of a mathematical notion are students' over-exposure to typical images in textbooks and/or lessons. Because the main focus of the Geometry Connection course was on proof and proving, we take that few images in the handouts for this course did not significantly affect students' concept images of an altitude in a triangle. Our research also supports Presmeg's (2002) prospect that overuse of some images of mathematical objects within geometry textbooks, may influence students' interpretations of these objects in a way that is not a logical consequence of their mathematical definitions.

Considering the difficulties PSTs experienced in providing correct answer for problem 6, we point again to the power of a visual interpretation in disrupting the logical chain of cognitive inferences students make in problem solving situations. The altitude of the

rhombus that creates the right triangle, does not “look like” being half of the hypotenuse because of the visual representation of the angle of 30° , creating a deception in student mind that the adjacent side in the right triangle is half of the hypotenuse. As a consequence, students attempted to find the length of the altitude of the rhombus using Pythagorean theorem.

Taking into account that our PSTs had difficulty conceptualizing definitions of median, altitude, or angle bisector of a triangle it is suggested that PSTs be provided opportunities to observe differences between these notions. For this, different types of triangles (acute, obtuse right, isosceles...), positioned differently within the natural coordinate system of a textbook aligned with the text should be presented to students in their learning environment. For instance, the base of a triangle usually tends to be parallel with the horizontal direction determined by the text in the textbook can be slightly rotated. This image and few other examples of triangles with various side lengths can provide students with opportunities to see a variety of possible triangular shapes, other than those usually presented. This will allow them to recognize certain properties of an object (say altitude of a triangle) when this object is presented in an atypical setting.

Use of computer environment can further increase opportunities for exploration. In an attempt to get closer to the dynamic representation in the previous suggestion, one can begin with an isosceles triangle and the angle bisector or median BM and create a set of triangles $ABC_1, ABC_2, \dots$ as shown on the following Figure 11.

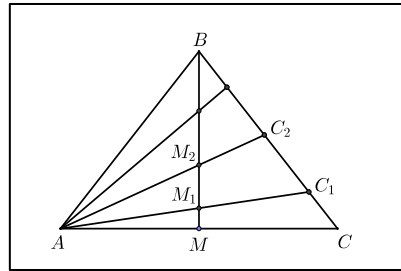


Figure 11. Segments $BM_1, BM_2, \dots, BM_n$ are angle bisectors in triangles $ABC_1, ABC_2, \dots, ABC_n$ respectively, but points $M_1, M_2, \dots, M_n$ “clearly” do not represent midpoints of the sides $AC_1, AC_2, \dots, AC_n$.

Emphasising the importance and demonstrating the usefulness of auxiliary drawings in problem solving situations is another way of helping prospective teachers conceptualize the medians or altitudes in a triangle. Looking at the second bonus question, one can notice that students could have been successful in answering this question correctly if they introduced auxiliary altitude from the vertex L in the triangle GJL perpendicular to the side GJ and the altitude from the vertex M in the triangle MKI . Research shows that the auxiliary elements have positive effect on students’ problem solving skills (Uygun, T., Akyuz, D. 2017).

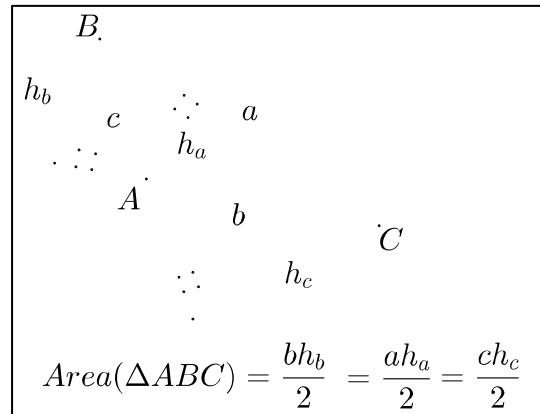


Figure 12. Different algebraic representations of the area formula for triangle ΔABC and atypical position of the triangle help in establishing a correct concept image of the area formula and the notion of altitude in a triangle.

Moreover, there is a need to examine the nature of teaching interventions that are used to support PSTs development of a conceptual understanding of geometrical concepts. The studies ought to reflect on how theorems and visual representations are introduced, used during enacted lessons and subsequently assessed. For example, a future study may seek to unpack how faculty facilitate mathematical instructions relative to the theorem asserting that in an isosceles triangle the median associated with the vertex of the triangle is also the bisector of the angle at the vertex and the altitude from this vertex. The study can also consider how the accompanying visual representation of this theorem is utilized during instruction. Particularly, it can be beneficial to understand how the persuasiveness of the visual representation of a scalene triangle with one of its medians (Figure 13 a) interferes with the logical chain of deductions, and infer false implication of an altitude being a median or median being an angle bisector in this arbitrary triangle (See Figure 13).

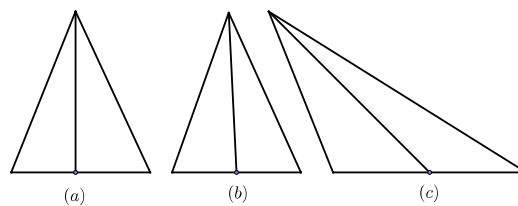


Figure 13. Triangle (a) depicts an altitude that can be seen as angle bisector. Triangle (b) depicts a median that can be depicted as altitude. Triangle (c) is more suitable image representing a median to the chosen base.

Although erroneous assumptions that median and altitude in an arbitrary triangle coincide can be more difficult to make if this arbitrary triangle is presented as in Fig. 13(c), it may still be present if the median we choose for the triangle in Fig. 13(c) is the one associated with the side opposite the obtuse angle. The notions of median, altitude or angle bisector in a triangle, when represented visually by triangles that can be seen as isosceles triangles by the learner, may be difficult to separate and special care should be taken.

Conclusion

In this paper, we have demonstrated two main obstacles to a productive use of visual representations of mathematical notions. While problem 6 demonstrates the dominance of a visual illustration in a geometry problem over textual information, in the first bonus question we see the effect of atypical presentation on students' cognition. Analysing PSTs' justifications for the second bonus question, we realized that for many students, comparison of the areas of two geometric figures could be done only if defining elements of the given figures were presented numerically. This finding implies that PSTs' approach to the concept of area relies solely on a memorized algorithmic procedure of computing area of a triangle or a parallelogram. Seeing the standard area formula accompanied by a visual representation of a triangle whose base is horizontal (aligned with the text) and corresponding altitude vertical, over and over, creates an obstacle for true conceptual understanding of relation between the visual representation and its analytic counterpart. We are in agreement with Hiebert and Carpenter (1992, p. 78) when they indicate that "evidence suggests that learners who possess well-practiced, automatized rules for manipulating symbols are reluctant to connect the rules with other representations that might give them meaning". Making connection between these two types of knowledge is one of the requirements for attaining mathematical expertise.

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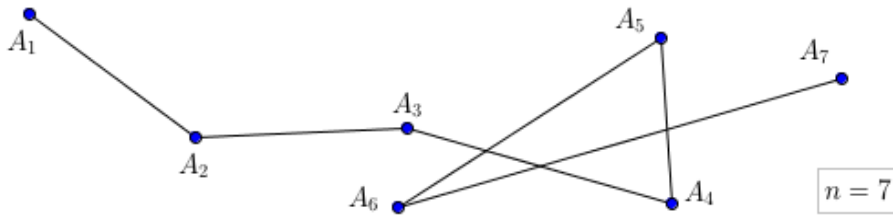
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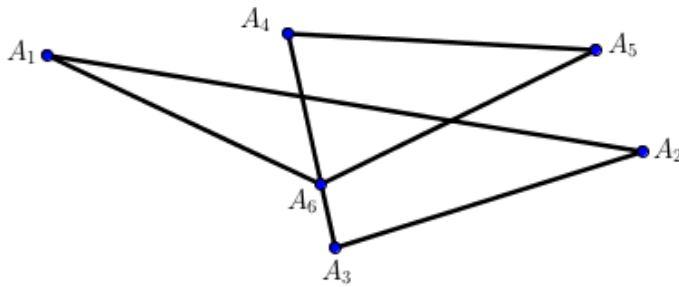
Appendix One

AREA

Given n points $A_1, A_2, \dots, A_n$ in the plane, a polygonal line $A_1 A_2 \dots A_n$ is the union of the segments $\overline{A_1 A_2}, \overline{A_2 A_3}, \dots, \overline{A_{n-1} A_n}$ for $n \geq 2$.



Points $A_1, A_2, \dots, A_n$ are called *vertices* of the polygonal line and segments $\overline{A_i A_{i+1}}$ are called *sides* of the polygonal line. Points A_1 and A_n are called *endpoints* of the polygonal line. A polygonal line is *convex* if it lies on each side of each of its segments (More precisely, on each side of the line determined by each of its segments). A polygonal line is called *closed* if its endpoints coincide. Looking at the following figure



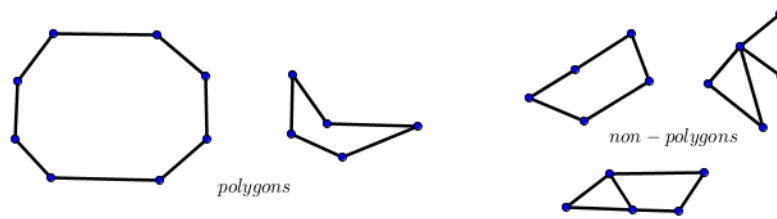
we see that a polygonal line may have self-intersections. So, here is a challenge:

PROBLEM. Define what does it mean for a polygonal line to self-intersect.

A figure formed by a closed non-self intersecting polygonal line, together with the part of the plane bounded by this polygonal line is called *polygon* if no three consecutive vertices A_i, A_{i+1}, A_{i+2} $i \geq 1$, (counting cyclically) are on the same line.

Of course, triangles and quadrilaterals are polygons following this definition.

□



Sometimes we'll refer to a polygon as a polygonal region, and the closed

"area" can be assigned to some regions in the plane.

we'll focus on defining this notion for polygonal regions in the plane. Following



Appendix Two

MTG 3207 GEOMETRY CONNECTIONS, SPRING 2015

FINAL EXAM

This Final Exam has 6 problems and will be graded out of 320 points. There are also two additional multiple choice problems (each graded out of 20 points). Points for each problem are indicated. Please write in a clear and precise manner and justify your answers. In the case of a construction, follow the required steps. Partial credit will be awarded in case your attempt for a solution has some elements of a solution to the problem.

1. (30 points) (Choose only three of the following four) State the definition of:
i) angle bisector ii) minor arc iii) an altitude iv) quadrilateral
2. (40 points) Write the contrapositive and converse of each of the following two propositions:
 - I. Diagonals in a parallelogram bisect each other.
 - II. A point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment.
3. (50 + 50 points) Prove two of the following three theorems:
 - 3.1. In every parallelogram, the opposite sides are congruent.
 - 3.2. An angle inscribed in a semicircle is a right angle.
 - 3.3. Area of a parallelogram is a product of (the lengths of) its base and the corresponding height.
4. (50 points) Given two parallel lines and a transversal, construct a circle that is tangent to all three lines.
5. (50 points) Given lines $a_1 : 2x - y = 0$ and $a_2 : y = 0$ determine the equations of two other lines, each bisecting the angles between a_1 and a_2 .
6. (50 points) Given a rhombus of side a and one interior angle that measures 30° .
Compute the area of the rhombus.

