# Elementary Teachers' Planning for Mathematical Reasoning through Peer Learning Teams

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Many elementary teachers find the complexity of understanding and teaching mathematical reasoning challenging. Teachers can benefit from professional learning (PL) programs designed to develop strategies to identify reasoning and implement it in mathematics lessons. This paper reports on a PL program designed to support a Peer Learning Team (PLT) of elementary teachers in Canada who were assisted by a researcher to peer-plan, peer-observe, and reflect on lessons fostering reasoning. Recorded data from PLT meetings were analysed against a planning framework to study the teachers' growth in understanding reasoning. The findings revealed teachers engaged in a PLT which plans together to embed reasoning in a lesson, followed by peer-observation and reflection is a powerful and effective model of PL for understanding mathematical reasoning and pedagogical approaches that foster students' reasoning.

## Introduction

Reasoning underpins mathematical learning and is required for the validation of mathematical ideas (Kilpatrick, Swafford, & Findell, 2001). They referred to reasoning as "the glue that holds everything together, the lodestone that guides learning" (p. 129). Its purpose is to convince others of the validity of our claims and to solve problems (Brodie, 2010). This is consistent with the British Columbia's curriculum where "[s]tudents are expected to reason mathematically ... to solve problems" (British Columbia Ministry of Education, 2017, para. 4). Whilst mathematical reasoning is emphasised in curriculum documents, it is not widely understood by teachers (Brodie, 2010; Stacey, 2010; Dreyfus, Nardi, & Leikin, 2012; Lobato, Hohensee, & Rhodehamel, 2013; Stylianides, Stylianides, & Philippou, 2007; Loong et al., 2017). Reasoning is difficult to define for teachers (Jeanotte & Keiran, 2017) and often confused with problem-solving (Herbert, Vale, Bragg, Loong, Widjaja, 2015). Without a clear understanding of how to define reasoning teachers are unable to recognise reasoning in action in the classroom (Bragg, Herbert, Loong, Vale, & Widjaja, 2016). Teacher knowledge of the content of their teaching is crucial in students' learning (Darling-Hammond, 2000). The lack of understanding of reasoning and manifests in many teachers struggling to incorporate reasoning into their practice (Stacey, 2010) including planning to embed reasoning in their lessons (Davidson et al., 2018). This limited teachers' understanding of reasoning impacts opportunities for students to develop reasoning.

Shifts in pedagogical practice to incorporate reasoning require an impetus for change, such as engagement in PL programs (Wood, 1999; Liljedahl, 2010). More recently, there has been an upsurge in PL opportunities devoted to strengthening the crucial proficiency of reasoning (Rasmussen & Marrongelle, 2006; Stacey & Vincent, 2009).

This paper reports on Phase 2 of the Mathematical Reasoning Professional Learning Research Program (Bragg et al., 2016). In Phase 1, teachers observed, and trialled lessons prepared and taught by the researchers. Findings from Phase 1 indicated that whilst some success in teachers' capability to identify reasoning was noted (Loong et al., 2017), limitations of this approach included a lack of teacher autonomy and financial constraints. As a result, an alternative mode of PL was sought to address these concerns by encouraging teacher autonomy in planning their own lessons for reasoning and reducing financial costs by scaling down the number of staff required for time release to engage in this program. In Phase 2, a PLT was formed with teachers from the British Columbian elementary school involved in Phase 1 of the project. A PLT, in this case, was a small group of teachers working together, planning, peer-observing, and discussing pedagogical practices to promote reasoning, thereupon providing opportunities for constructive feedback and critical reflection (Eri, 2014). It was expected that the PLT approach would foster teachers' growth in understanding the teaching and learning of reasoning. Despite general support for PLTs (Nickerson & Moriarty, 2005), it is relatively unexplored for embedding reasoning in mathematical lessons. Consequently, this study has the potential to add to understanding the effectiveness of a PLT to support elementary teachers to embed mathematical reasoning into their practice. Hence, our research question is: *In what ways can a PLT support teachers in planning for reasoning?*

## Teaching Mathematical Reasoning

Reasoning is "the line of thought, the way of thinking, adopted to produce assertions and reach conclusions" (Lithner, 2000, p. 167). In elementary schools, students apply reasoning when making sense of mathematics and solving an unfamiliar problem (Lampert, 2001). Stylianides and Stylianides (2006) comment when discussing Reasoning and Proof (RP) is reflective of the view of reasoning in many curriculum documents:

> [A] typical structure of students' engagement in sense making is to first explore mathematical phenomena to identify patterns and make conjectures, and then investigate with arguments and proofs the truth of the conjectures to establish new knowledge … Because students' engagement in mathematics as a sensemaking activity is a high-priority goal of school instruction, and because of the intimate relation between sense making and RP, many researchers and curriculum frameworks… recommend that RP become central to all students' mathematical experiences across all grades (e.g., Ball & Bass, 2003; NCTM, 2000; Yackel & Hanna, 2003) (p. 202).

Much of the research into mathematical reasoning has focused on four actions of reasoning: analysing (e.g., Kilpatrick, et al., 2001); generalising (e.g., Ellis, 2007; Radford, 2008); justifying (e.g., Lannin, Ellis, & Elliott, 2011); and proving (e.g., Stacey & Vincent, 2009). Together, these actions encompass all components of mathematical reasoning described in the new British Columbia Curriculum (British Columbia Ministry of Education, 2017). At all elementary levels, students are expected to "[c]ommunicate mathematical thinking in many ways … [and] [e]xplain and justify mathematical ideas and decisions." At Kindergarten, students "[u]se reasoning to explore and make connections" whilst at Year 7 students are expected to be able to "[u]se reasoning and logic to explore, analyze, and apply mathematical ideas, [and] [u]se tools or technology to explore and create patterns and relationships, and test conjectures" (British Columbia Ministry of Education, 2017, p. 15).

Analysing occurs when students draw on prior knowledge to seek particular examples to solve a problem (Vale, Widjaja, Herbert, Bragg, & Loong, 2017). Comparing and contrasting examples to explain similarities and differences enables students to generalise across several examples and form conjectures (Kaput & Blanton, 2000). Justifying involves verifying a claim (Sowder & Harel, 1998) using a logical argument constructed on taken-as-shared concepts and procedures (Mata-Pereira & da Ponte, 2017). Logical argument uses evidence to convince (Brodie, 2010) and may include: verifying and validating the claim by showing how the rule works for all cases; refuting a claim by providing a counter example; showing that two statements are not equivalent, related or connected; or a generic proof showing the logical steps required to validate the claim for all cases (Lannin et al., 2011).

Bruce, Esmonde, Ross, Dookie, and Beatty (2010) found that teachers who listened to students, elicited their mathematical reasoning, and focused on deep understanding of concepts, were more successful in increasing student achievement. However, identifying and developing understanding of conjecturing, verifying, and refuting, is not easy for teachers and curriculum designers (de Villiers, 2010). To facilitate and develop their students' mathematical reasoning teachers select suitable tasks which have the potential to foster reasoning (Long, DeTemple, & Millman, 2012), such as, inquiry-based activity to generate and validate new knowledge involves generalising, justifying, and proving (Stylianides, 2010; Goos et al., 2017). In teaching mathematical reasoning teachers may employ students' contributions in orchestrated discussions (Stein et al. 2008). They proposed five practices:

1. anticipating likely student responses to cognitively demanding mathematical tasks;
2. monitoring students' responses to the tasks during the explore phase;
3. selecting particular students to present their mathematical responses during the discuss-and-summarize phase;
4. purposefully sequencing the student responses that will be displayed; and
5. helping the class make mathematical connections between different students' responses and between students' responses and the key ideas (p. 324).

The teacher's role in creating a classroom culture that provides a safe environment for students to express their reasoning (Nardi & Knuth, 2017) is crucial for providing opportunities for students to communicate and refine their reasoning. Classroom cultures foster the development of mathematical reasoning when they value students' ideas and provide opportunities for students to explain and justify their thinking and evaluate others' ideas (Long, DeTemple, & Millman, 2012).

Teachers' knowledge of reasoning and its teaching is critical in creating the kinds of classrooms Long et al. (2012) describes. First teachers must notice reasoning in students' actions then employ appropriate prompts to elicit reasoning to move students from explaining what to explaining why and justifying choices (Loong et al., 2017). However, teachers' focus on mathematical content knowledge and/or procedural fluency can diminish their noticing of and support for students' reasoning (Bragg et al., 2016)). This section has discussed relevant issues in teaching mathematical reasoning such as defining reasoning and teachers' knowledge and noticing of reasoning. Research (Brodie, 2010; Stacey, 2010; Dreyfus et al., 2012; Lobato et al., 2013; Stylianides et al., 2007) has identified challenges faced by teachers in understanding various aspects of reasoning and its pedagogy. Supporting teachers to know more about reasoning will assist them in developing their students' reasoning. Consequently, the next section presents literature on PL.

## Professional Learning

This section draws attention to the importance of PL for teachers, its challenges, and various forms it may take. The aim of PL is to produce positive pedagogical learning outcomes for teachers and their students (Du Plessis, 2019). Liljedahl (2010) proposed shifts in pedagogy require Reification, that is, rapid changes to teachers' practice after producing objects, tasks, or lessons, that exemplify different approaches; and Leading Belief Change, that is, reorganising beliefs about mathematics and its teaching that result in significant change in teaching practice. To facilitate significant changes in teaching practice, Bruce et al. (2010) emphasised the importance of engagement in practice over time and connected with teachers' natural contexts and practice. In their Canadian study, Bruce et al. (2010) found that features of effective PL include sustained collaboration with classroom embedded support leading to substantial gains in student achievement as a consequence of improvements in teaching quality. Similarly, Liljedahl (2010) reported that teachers' autonomy in designing and testing tasks in their classrooms resulted in considerable changes in teaching practice. The design of our study draws on this advice with the PL program embedded in school practice.

One avenue of providing the sustained collaborative PL embedded in classroom practice is the formation of professional communities (Nickerson & Moriarty, 2005; Wilson, 2013; Johnston & Cornish, 2016). Professional communities have potential to support teachers wanting to implement changes in their practice (Nickerson & Moriarty, 2005) through giving and receiving constructive feedback from colleagues (Wilson, 2013) and reducing teacher isolation (Johnston & Cornish, 2016). A focus on good practice through discussion and reflection can facilitate shifts towards improvements in teaching (Byrne, Brown, & Challen, 2010). Members of the PL community who are champions of change "inspire and enthuse others…[and] gain the commitment of others to support [their] innovation" (Howell & Higgins, 1990, p.320). Such champions of change exhibit characteristics such as persistence; high energy; the ability to motivate; self-confidence; and teaching experience (Howell & Higgins, 1990). Consequently, in this study a PLT approach was investigated for its efficacy in building teachers' knowledge of mathematical reasoning, and the pedagogies that support it.

Further, this study was informed by research regarding peer-observation (Wood, 1999; Borko, Jacobs, & Koellner, 2010) where a teacher observes a peer's teaching. Effective peer-observation occurs when both the observer and the observed critically reflect on the lesson observed (Eri, 2014). Observing colleagues teach lessons allows teachers to focus on the student-teacher interactions without the teaching responsibility or managing the classroom (Eri, 2014). Teachers could attend to the flow of the lesson and types of questions asked (Lesseig, 2016) and focus on students' thinking more deeply. In our study, teachers planned together, observed each other teach, and reflected together on the observations. To guide reflection the teachers were given a reflection and debriefing form (see Methodology).

In summary, reasoning underlies mathematical thinking and is essential for students' deep conceptual understanding. Increasingly, reasoning is the focus of curriculum documentation and PL programs. However, despite the importance of reasoning, it is still understood superficially by many teachers and enacted poorly in the classroom; impacting both on planning for reasoning and consequently on students' reasoning capabilities (Herbert et al., 2015; Davidson et al., 2019). Research of PLTs in mathematics has focused on how to build students' conceptual understanding of content knowledge or changing teachers' pedagogical practice, rather than a specific focus on the proficiency of reasoning. Therefore, Phase 2 of the project explored the role played by planning for reasoning in PLTs.

## Theoretical Framework

Sullivan, Borcek, Walker, and Rennie's (2016) planning framework (see Figure 1) provided a structure to support our understanding of the interactions in the PLTs planning and subsequent implementation of peer-planned lessons specifically designed to embed reasoning. It includes three nodes: Beliefs, values, and attitudes; Knowledge of mathematics and pedagogy; and, Opportunities and constraints, which together influence Planning Intentions, and, in turn, influence Classroom Actions.

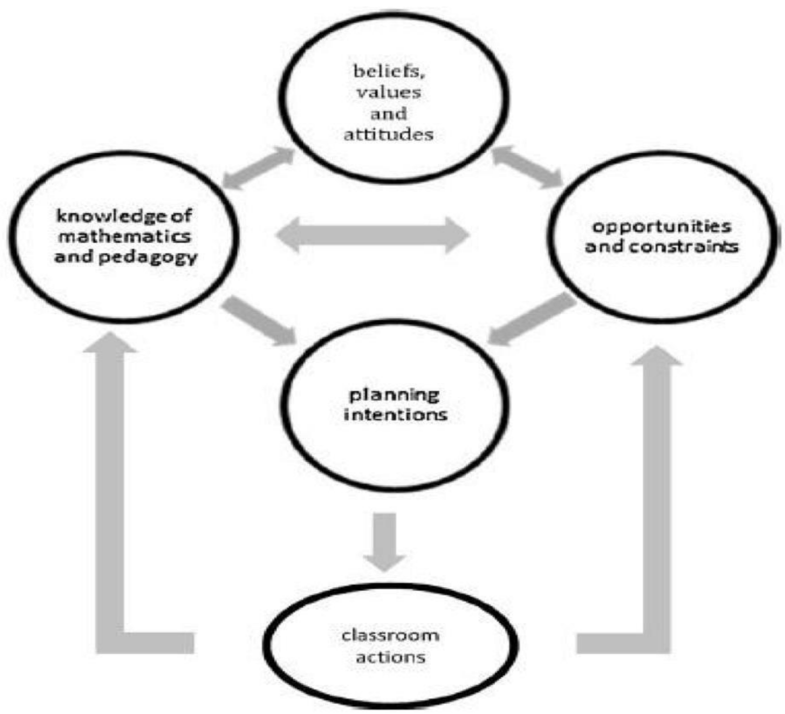


*Figure 1.* Planning framework (Sullivan et al., 2016, p. 161).

Teachers' approaches to teaching mathematics are influenced by their *beliefs, values, and attitudes* about mathematics; the teachers' roles; and students' capabilities (Kilpatrick et al., 2001). Beliefs, values, and attitudes can be aligned with the construct of orientation, calculational or conceptual (Thompson, Philipp, & Boyd, 1994). A teacher's orientation influences their teaching of mathematics, and consequently have impacts on student outcomes (Attard, 2013; Beswick, 2012; Carmichael, Callingham, & Watt, 2017; Forgasz & Leder, 2008; Golafshani, 2013). Teachers with a calculational orientation value and encourage the use of formulae and procedures to produce numerical answers. They focus on transmitting the 'correct' application of procedures and 'correct' solutions. Conversely, teachers with a conceptual orientation value and encourage ways of thinking that develop ideas and understanding of procedures and calculations, that is, using reasoning. They expect students' active engagement in tasks designed to foster these ways of thinking. These differing orientations may be expected to result in different approaches to teaching mathematics and the design of tasks and lessons. For example, a conceptual orientation may be witnessed in the use of open-ended tasks and a focus on problem-solving, whereas classroom activities related to a calculational orientation may feature practicing procedures. Teachers' beliefs and the implementation of mathematical practices form a complex relationship (Beswick, 2012).

The *knowledge of mathematics and pedagogy* node of this framework refers to how lesson structure is informed by teachers' knowledge including both teachers' subject matter knowledge and pedagogical content knowledge (Hill, Ball, & Schilling, 2008) and impacts on planning decisions, such as a teachers' choice of tasks. Shulman (1987) described three categories of teacher knowledge: content knowledge; pedagogical knowledge; and pedagogical content knowledge. In mathematics teaching content comprises the knowledge of mathematical concepts. Pedagogical knowledge relates to knowledge of teaching, classroom management and organisation. Pedagogical content knowledge is a combination of content knowledge and pedagogical knowledge. It includes knowledge of the curriculum; knowledge of learners and how they learn; and knowledge of how to function within their educational context. The specific pedagogical content knowledge required for teaching mathematics is complex and has been the focus of much research (Ball, Thames, & Phelps., 2008; Chick, Pham, & Baker, 2006; Rowland Turner, Thwaites, & Huckstep, 2009; Liljedahl, Durand-Guerrier, Winsløw, et al., 2009).

The third node, *opportunities and constraints*, reveals teachers' anticipation of the influences on their planning that either advantage or disadvantage the learner. As Clark and Peterson (1986) suggested, constraining factors which inhibit, or support teachers may include: curriculum considerations; school leadership; and the physical environment of the school setting. Circumstances may impact a teacher's decision making with or without their awareness (Shavelson & Stern, 1981), such as, an understanding of students' capabilities, behaviors, and participation in the learning environment. In a study of 62 elementary teachers' planning decisions, Davidson (2010) found that the opportunity to team plan had a positive bearing on the planning process. Further advantages which positively impact planning are support of PL programs (Goos, Dole, & Geiger, 2011) and in-school numeracy leadership; however, PL may be constrained by financial costs and personal discomfort for some teachers (Bragg & Herbert, 2017).

*Planning intentions* for reasoning involve: knowledge of mathematical reasoning which goes beyond explaining what to include other reasoning actions such as analysing, generalising, justifying and proving; knowledge of pedagogies that support students' development of reasoning; selection of suitable tasks with potential to foster students' reasoning; and lesson structure that allows time for discussion (Bragg, et al., 2016).

Together teachers' beliefs, values, and attitudes about mathematics; their knowledge of mathematics and pedagogy and various opportunities and constraints influence their planning intentions, in this case for mathematical reasoning. Consequently, they play a part in teachers' *Classroom actions* when teaching lessons focusing on mathematical reasoning. Peer-observation of the participating teachers' classroom actions to identify the implementation and impact of reasoning in peer-planned lessons was incorporated into the PLT approach utilised in this current study. The next section describes the data collection methods and the process of analysis.

## Methodology

This study was the second phase of a larger study of a PL program developed to support elementary teachers' understanding of mathematical reasoning and its facilitation in the classroom. In Phase 1 of this program, twenty-four teachers at four Australian schools and one Canadian school engaged in a PL program based on demonstration lessons (see Loong et al., 2017 for more detail). These findings indicated that while some shifts in the understanding of reasoning were made by these teachers as a result of engaging in demonstration lessons, exploration of a more school-based PL program offering teacher autonomy was required so that a PL program could be scaled up for implementation in more schools.

Phase 2 explored collaborative planning of two lessons with embedded mathematical reasoning, peer-observations of purposefully designed lessons, with post-lesson discussions to follow. These post lesson discussions were guided by the Reflection and Debriefing form mentioned earlier. It was adapted from Stepanek et al. (2007) and included the following questions:

1. To what extent were the goals of the lesson achieved? Please provide supporting evidence.
2. Which instructional decisions might have attributed to helping students meet these goals? Explain.
3. What aspects of the goals were not reached? Please provide supporting evidence.
4. Which aspects of the lesson plan should be reconsidered based on this evidence?
5. Additional thoughts about the lesson

This phase of the program was developed to supplement and enhance teachers' knowledge of reasoning through a more autonomous, cost-effective, and feasible approach. Phase 2 participants were a PLT from the same Canadian elementary school involved in Phase 1 of the program: a Grade 2/3 (Amy[1]), a Grade 3 (Kate) teacher, and a researcher (second author). Both Amy and Kate had over 20 years of experience as teachers and held senior leadership roles in the school. Amy's approach to teaching mathematics focused on children working through a textbook and obtaining the correct responses. Kate's approach to teaching focused on unpacking children's mathematical thinking through tasks that invited discussion and collaboration to expand their understanding of mathematics. When the school was approached to undertake Phase 2 of the project, both Amy and Kate had voiced a keen interest in being involved. Amy was unable at the time to participate in Phase 1 of the project but had witnessed the positive impact on staff and students. Kate, who had participated in Phase 1, was expected to have an understanding of the reasoning to support and guide Amy in planning and implementing a lesson deliberately designed to foster reasoning.

[1] pseudonyms used throughout

### *The PLT Approach*

Phase 2 commenced with a researcher conducting a whole-school presentation that reported findings from Phase 1 and provided further resource materials to assist teachers in eliciting reasoning, such as, questioning prompts and rich tasks. Next, the researcher joined the PLT planning meeting to design tasks with the aim of embedding mathematical reasoning into the lessons of each teacher's class. The researcher's role in the planning meeting was as a participant-observer, responding to teachers' requests for advice and refocusing the team on reasoning where necessary. The tasks "Is it True?" (Amy) and "15 Cookies" (Kate) were designed by the respective teachers, with guidance from the PLT to ensure there were reasoning opportunities in the tasks.

*Is it True?* The "Is it true?" task (see Bragg & Herbert, 2017) emerged from Amy noticing her students having difficulties with place value and, the addition of two-digit numbers. Amy explained to the PLT that it was common for students to not regroup ones to tens; so that 27 + 34 would be written as 511, instead of 61. Building on Amy's concern, the task commenced with an inaccurate calculation, 27 + 34 = 511, presented to the whole class. Amy posed the questions: "Is it true?" and "Why or why not?" The students worked individually on the task for approximately 5-10 minutes, shared their reasoning in pairs, then shared their reasoning again during a whole class discussion. The reasoning intention was for students to *justify* their position and offer a logical argument as they noticed and explored place value.

*15 Cookies.* Exploring efficient counting strategies was the impetus for the 15 Cookies task. The Grade 2/3 students were shown a cartoon image of a tray of 15 cookies in a 3 by 5 array and asked: How many cookies on the tray? Next, each child was given a photocopy of the image and then asked, "Is there another way you could determine the number?" The students investigated and then discussed the problem. The reasoning intention was for students to *analyse* the problem through the exploration of potential counting strategies this image would provoke and explain their thinking.

### *Peer-observation and post-lesson discussion*

Following the planning meeting, Amy and Kate taught their reasoning-enhanced lessons with the other teacher observing. The researcher elected not to observe the lessons to reduce any impact a perceived "expert" might have on the teachers' pedagogical approach within the lesson. The intent of the peer-observation process was for only the teachers from the PLT to observe each other, a process that could be undertaken in any school without researcher intervention. During the lesson, the observing teacher noted any reasoning actions, and afterwards both teachers completed individual reflective notes. Both lessons were taught and observed on the same day. After both the observed lessons were conducted, the PLT including the researcher, met for a post-lesson discussion to share the teachers' observations of students' reasoning and teachers' actions to support reasoning. A group interview rather than individual interviews was conducted, as the post-lesson discussion was viewed as another opportunity for the teachers to further learn together as a PLT. The Reflection and Debriefing form guided their post-lesson discussion. The post-lesson discussion was held on the same day as the observed lessons were conducted to gain the perspectives of the teachers while the experience was fresh, and observation schedule notes were easier to recall. We acknowledge that note taking using an observation schedule has limitations over video recording devices, as the classroom situation cannot be replayed to notice what may have been missed in the observations in-situ. However, teachers taking anecdotal notes is the most common and widely used practice for recording teachers' observations and is in keeping with the intent of this PL program to maintain a program that is at a low cost and easily accessible for schools.

### *Data collection and analysis*

The purpose of collecting and analysing these data is to explore the impact on planning for reasoning a PLT offers. The planning meeting and post-lesson discussion were video- and audio-taped. The lesson plans, teachers' observation notes, reflective notes, and student work samples were collected. The planning session and post-lesson discussion recordings were transcribed verbatim, checked for accuracy by the researchers, and entered into NVivo (QSR International, 2017) to facilitate data analysis. Sullivan et al.'s (2016) framework—suggesting that teachers planning decisions are influenced by their knowledge, their dispositions towards mathematics and their anticipated constraints—proved to be useful in providing a structure for the analysis.

The process of data analysis included an initial joint reading of the transcript from the planning session by the researchers to identify codes consistent with the nodes in the Sullivan et al. (2016) planning framework: Beliefs, values, and attitudes; Knowledge of mathematics and pedagogy; Opportunities and constraints; Planning intentions; and, Classroom actions, to identify any impact on teachers' planning. In this analysis, both researchers independently coded each transcript according to these nodes; any discrepancies in coding were resolved through discussion. Further scrutiny of the transcripts was undertaken to refine coding and create sub-codes. The findings arising from analysis of the data from peer-planning meeting and post-lesson discussion are presented in the following section.

## Results

The purpose of Phase 2 was to consider the ways a PLT can support teachers in planning for reasoning and consequently build their understanding of reasoning. The results are structured according to nodes of the Sullivan et al. (2016) framework. Broadly, in planning for reasoning, evidence was found of the teachers' exhibiting aspects of each of the three nodes impacting planning purported in Sullivan et al.'s framework. The analysis revealed differences between the numbers of utterances of these nodes. Evidence of comments illustrating each node are presented below.

### *Beliefs, values, and attitudes about mathematical learning*

Whilst this article focuses on the teachers' gains in understanding reasoning during Phase 2, their beliefs, values, and attitudes about mathematical learning are an important consideration. Amy's initial description of her task points to a calculational orientation, voicing, "So I'm just thinking how can I make them more efficient in their adding?" Conversely, a teacher with conceptual orientation values exploration over the final answer and believes that sense-making is enhanced by connections to prior knowledge. As evidenced by Kate's comment below:

> Because I know that the majority of them are just going to count, I want them to brainstorm all these different ways to add. Then you ask them, "Show as many ways as you can to get the number 15?"

Kate's quote indicates her preference to connect with students' prior knowledge in the question that she intends to pose to the students. Kate expects the students to explore the problem and produce multiple possible solutions rather than valuing a single 'correct' answer. This open-ended approach encourages exploring the task from differing perspectives, hence fostering the key reasoning action of analysing. The following interchange between Kate and Amy in the planning meeting further demonstrates their differing orientations.

> Kate: That's what you want to teach them. 9 is really 4 and 5. 5 and 5 is 10. Or you could even do where 5 plus 5 plus 5 is 15. So that's where the reasoning part comes in.
>
> Amy: Okay.
>
> Kate: So they're not just always just counting. There are some strategies. That's what these number talks all do. You know, [when you ask students to] solve 28 plus 14. But getting them try to get as many different ways.
>
> Amy: Well I'm thinking for my lesson, like listening to this, then using the reasoning strategy. I wanted to do just simple, they've been working on place value, they've been working number fluency with addition and subtraction. I don't think they truly understand when they're adding and let's say 27 plus 34 why that's 61 and that's what I want to do.
>
> Kate: So there's number talks on that – this is a great … you can break each number into place value.

In this exchange with Amy, Kate is trying to expand Amy's planning of the task to include more opportunities for reasoning to promote students' reasoning. Kate was an advocate for number talks, which is typically a 5–15-minute class discussion about a mathematical concept to promote number flexibility (Parrish & Dominick, 2016). Kate suggested on multiple occasions in the planning meeting that Amy should utilise number talks to foster reasoning.

> Kate: You start your math every day with this little 10-minute number talk where you present them one of these problems and then you jump into your lesson and you relate it to whatever concept you're doing.
>
> Amy: I like that, it's neat.
>
> Kate: So when you give them the number talk, you give them time on their own to try to come up with at least one way of finding the answer.

Kate values the opportunity these conversations about mathematics which are offered by the regular inclusion of number talks, and Amy is receptive to the number talks pedagogical approach.

### *Knowledge of reasoning and pedagogy*

The data from Knowledge of Mathematics and Pedagogy node, as they relate directly to mathematical reasoning, are presented and discussed. Amy initially chose a task which did not encourage reasoning, that is addition of two-digit numbers.

> Amy: They've been working number fluency with addition and subtraction. I don't think they truly understand [place value] when they're adding …. So, I'm just going to put a question up on the board and write down the question and answer.

Amy considered asking the students to complete closed addition problems for her reasoning task. Conversely, Kate recognised and incorporated some aspects of reasoning into her practice, explaining how this task could be used to draw out students' reasoning.

> Kate: So, I'll ask them how many cookies are on the tray, and ask them, "How did you determine the number of cookies? What strategy did you use?" Trying to get all the strategies and then ask, "Is there another way you could determine the amount?" Brainstorm all these different ways to add; so then you ask, "Show as many ways as you can to get the number 18 and explain why they are all accurate." We have different strategies, so ask, "What's the most efficient way of counting?"

As a participant in Phase 1 of the program, it is not surprising that Kate's comments revealed some depth of understanding of mathematical reasoning. Kate's choice of task indicates her knowledge that open-ended tasks provide opportunities for students' reasoning. Kate suggested ways to open up and embedding reasoning into Amy's task of 27 + 34 =.

> Kate: Yeah you can make the problem. "If X got – 511 what do you think?"

> Amy: Is this answer right; why or why not?

> Kate: Then have to look at it. Try to come up with at least one way of finding the answer.

Kate was aware of setting tasks with the potential for more than one solution when she stated, "at least one way" and expected students to explore various solution strategies. The exchange demonstrates how Kate's pedagogical knowledge aided Amy in shifting her understanding of how to open up a task to develop students' reasoning whilst addressing the content area of adding two-digit numbers. Amy considered appropriate reasoning prompts. Further discussion in the PLT saw Amy coming to a more nuanced understanding the pedagogical approaches to support reasoning.

> Amy: So I'll present it this way; give them this [27+34=511] and then give them some time to break it apart. Then have examples of kids coming up and explaining it; "Who would like to show us how?" I'll have them work on their own first then they can partner up.

Amy shared that she seldom invited the students to work in pairs in mathematics lessons and sharing strategies was not a feature of her class. So, the formation of partners to share ideas was a change in Amy's teaching as a result of Kate's pedagogical knowledge to enhance reasoning. As the planning session continued, in response to suggestions made by Kate and Researcher B, Amy considered the use of more complex reasoning terminology and prompts to foster justification.

> Amy: I want them to understand when they're adding and having to regroup; why they're actually regrouping, why is it actually an 11 and it's not just a 1? "Convince me or explain to me why this is correct" I'm thinking how can I make them more efficient in their adding? "Justify why is this right? Why are we thinking it's right? Tell your buddy why is this right?"

After the teachers had planned their lessons, each teacher peer-observed the other's lesson, taking notes on reasoning actions and prompts used. The PLT met again with the researcher to reflect on the observed lessons. In this post-lesson discussion, Amy's enthusiastic elaboration of her teaching of this lesson illustrated the impact of Kate's pedagogical advice offered on how to structure Amy's lesson and to encourage reasoning through prompts and discussion in pairs, and then the whole group. Amy enacted this approach in her lesson with success.

> Amy: Some knew that it was right but didn't know why it was right. They couldn't tell why – or they would just do the standard version and say "Well the answer is this." So I asked, "Well why is it that?"

> They had to write it down to convince me, Student A had to convince Student B of their answer, Student B had to convince Student A. Then I brought everybody together and asked individuals to share their idea and to convince the group.

Through peer-planning and peer-observing, Amy was exposed to Kate's knowledge of the use of reasoning language. This opportunity to enhance Amy's choice of language in her lesson, from engaging in the PLT, was evident in Amy's reflection on the value of the reasoning prompts she utilised.

> Amy: Really I see the value in it [reasoning] and using your [Kate's] language "Convince me," "What's your evidence," "Justification" and not saying "No you're wrong".

These comments signal the growth in Amy's knowledge of reasoning and its pedagogy gained by working with Kate in a PLT to plan and observe their lessons designed to include reasoning. This experience of the PLT, and those presented in the next section, provides opportunity for developing an understanding of reasoning.

### *Opportunities for developing an understanding of reasoning*

In this study, the opportunity for teachers' growth in knowledge of reasoning lay in teachers working in a team to plan a lesson specifically designed to include tasks which foster reasoning. As expected, further analysis of the data showed that a knowledge of reasoning and pedagogy was more evident in Kate, who had participated in Phase 1, and her impact on her peer is presented below.

Kate was keen to foster reasoning through the use of mathematics stations, which are a series of tasks for small groups of students to engage in and rotate through over a lesson or throughout a week. Kate recognises the importance of reasoning and the need to specifically focus on it in her classroom,

> "I have four stations for math...Now...I'll have a reasoning station."

As mentioned above, Kate revealed her capacity to model how to incorporate the use of the language of reasoning, e.g., convince me, if...then, but, etc., to prompt reasoning. As illustrated in this following exchange:

> Kate: Then convince your partner why [it is right] or why it was wrong.
>
> Amy: I really like the "Convince me" and I like the freedom they had to explore.

Amy acknowledged the usefulness of Kate's suggestions to open the tasks up to move beyond a closed, one answer reply, but rather broaden the children's opportunities to analyse the problem and articulate their reasoning through justification. Reasoning, when verbally communicated, afforded the teachers an avenue to assess the mathematical understandings of students which might have typically not been accessed through written work samples. Kate shared her valuable insights with Amy on the power of listening to the children's reasoning.

> Kate: But they're getting the reasoning and being able to think about it. That's what I learned from their writing is that the kids that struggle, they can never really show you their thinking [reasoning]. But when it's verbal, you understand if they get it. Those kids that struggle can verbally share with you. Where they can't [verbalise their reasoning] if they're just sitting there [with] a piece of paper just trying to write a number down.

Throughout the peer planning meeting, Kate was regularly encouraging Amy to rework her task to encourage exploration of the students' thinking. For example, when Amy wanted her students to provide a correct response to adding two three-digit numbers, Amy planned to ask, "Tell me what it is ..." however, Kate suggested the alternative phrasing, "*Now show me how* …".

Amy reported that her observation of Kate teaching the lesson they had planned together was significant in building her knowledge and understanding of reasoning and its teaching.

> Amy: The neat thing that Kate did as they were doing that is she kept identifying the strategy they were using. So she named the strategy.
>
> Kate: I don't even know I do that.
>
> Amy: It's like "Did you realise you're using multiplication?" "Oh, all right you're using …" so it was good just to identify what it was.

It appears that good pedagogical approaches were such an innate feature of Kate's practice that it had been unrecognised by this teacher within the teaching moment.

The following exchange illustrates Kate's leadership in assisting Amy to come to a better understanding of reasoning. Peer-observation of lessons led to Amy noticing Kate's modelling by using effective reasoning prompts:

> Amy: It's very valuable, but you need to be very calculated in the wording that you choose. Whether it is 'Show me' or 'Convince me' or 'Show me another way', 'What's another strategy?' This was something that I picked up from Kate's lesson because one of her students said, "Is it like when we did this?" Then Kate said, "Oh you made a connection to ..." and built on that strategy. Now I have strategies to build on that we've actually discussed. So the reasoning is really important, and taking the time to find out why students think that.

In the post-lesson discussion, Kate further emphasised the importance of appropriate questions and prompts through indicating Amy's missed opportunity during the lesson to promote reasoning by offering this advice (despite this advice having already been offered in the peer-planning meeting):

> Kate: Yeah that's the one thing I was going to say, because you kept saying "Tell me". And I thought that's where you would have said, "Show me."

However, when reflecting on observing Amy, Kate was able to offer advice for how to promote reasoning in the future.

> Kate: I was interested because you picked the kids that showed their answer and I was like, what if before you did that you say, "Who would like to show me why it's why or why not?"

Peer-observation and post-lesson discussion afforded opportunities for positive change. A further advantage to participating in peer-observations was that this approach offered teachers the opportunity to examine and scrutinise students' conversations that may have previously gone unnoticed during the typical course of teaching.

> Kate: You know what was interesting—you didn't hear it—was one of the conversations the kids were having when they were partnering together…so I listened to each of their reasons.

*Planning intentions for reasoning*

The nodes of Sullivan et al.'s (2016) framework, knowledge of reasoning and its pedagogy; orientation and disposition; and opportunities interacted to influence planning intentions and were evident in our data in the planning of lessons. Towards the end of the peer-planning meeting Amy demonstrated a growing awareness of a reasoning pedagogical approach through encouraging an open dialogue with the students with the use of probing questions.

> Amy: Okay I'll have one or two that will say "You carry your 1" [Student's anticipated response]. "Well how come?" [Amy's anticipated response to the student]. "Because it's the rule" [Student's anticipated response]. "But why is it?" [Amy's anticipated response to the student].

Amy had a knowledge of the students' thinking and anticipated their responses. Much of the focus during the planning for reasoning was in the employment of prompts to elicit reason. This focus on the prompts was led by Kate in response to Amy initially presenting a closed task for her lessons. Amy reflected on the effectiveness of incorporating these prompts in her lesson planning:

> Amy: Because you were allowed to have a completely different opinion; there was no one thing I was looking for. I forget that and I just saw the power of it today, of giving them that freedom. I just thought to myself, "I just saw four or five kids contribute to something that I would never have thought they would have."

*Classroom actions*

As the researchers did not observe the lessons, we have relied on the teachers reporting of the lessons to identify classroom actions. For example, Amy described how she elicited reasoning through setting the scene with an open-ended task:

> Amy: I posed that question [27 + 34 = 511 Is it true? Why or why not?] on the board. Immediately hands shot up to respond. I said "You know everyone has their thoughts, their opinions". Then I walked them through what we were going to do, how to explain if it was right or if it was wrong and how they had to prove it and they had to go back and it was just, it was fascinating.

In this quote, Amy has used the planning session to inform her launch of the problem. When reflecting on her lesson Kate demonstrated her ongoing pursuit to encourage her students to reason which at times was not successful despite her classroom actions to promote dialogue.

> Kate: They could not articulate why at all and even just trying to explain. I said "Well okay, could you help each other out. Because you both agree that this is right". But it was interesting. They couldn't come up with one reason why that was right but they agreed with it. I was thinking they would at least say "Well 2 plus 3 is 5" but no, nothing, they honestly could not come up with a reason why. They were absolutely stuck.

Kate, who's classroom practice included giving students time to share their responses with her, the teacher, found value in listening to students sharing their reasoning with other students. Kate found reflecting on her lesson useful to focus on the reasoning and the conversations that occurred and had previously gone unnoticed.

> Kate: It was interesting to hear one of the conversations the kids were having when they were partnering together … Student A initially said it was right and Student B said that it was wrong. So I listened to their reasoning. Student C just shook his head after he listened to Student B and said, "Boy was I wrong". So I said "Oh so Student B convinced you?", he replied "Yep". I was so surprised to hear them sharing their reasoning together and being influenced by other students' reasoning.

As suggested by Sullivan, et al. (2016), the planning intentions had a direct impact on the classroom actions, and therefore fostered reasoning that was unexpected and yet welcomed in this case. The most striking observation to emerge from the data comparison was that the teachers articulated comments illustrating their knowledge of mathematics and pedagogy substantially more often than their beliefs, values, and attitudes, and opportunities and constraints.

## Discussion

This discussion is structured according to the planning framework nodes in Sullivan et al. (2016): *Beliefs, values, and attitudes; Knowledge of mathematics and pedagogy; Opportunities and constraints; Planning Intentions*, and *Classroom Actions*. These nodes are discussed with reference to the findings and research literature.

Teachers' beliefs, values, and attitudes about mathematical learning impact their teaching in general (Staub & Stern, 2002) and their teaching of reasoning in particular. Their orientation and disposition to mathematics shapes their view of teaching reasoning (Thompson et al., 1994). Beliefs about mathematics and how it should be taught are influential in the tasks chosen and approaches used. In this PLT, Kate is exhibiting a conceptual orientation (Thompson et al., 1994) demonstrated by the task she chose and the strategies she employed in her classroom. Nevertheless, we are mindful of Kilpatrick et al. (2001) caution of maintaining a balance between an emphasis on procedures and on understanding in teaching approaches and acknowledge that a blend of both approaches can be effective in teaching mathematics. Possessing a knowledge of reasoning is not enough. Kate offered Amy practical solutions and ideas for eliciting reasoning, hence evidencing her knowledge of a reasoning pedagogy that reaches beyond the idea of showing or explaining, to develop a more complex level of reasoning by seeking students to *convince* and *justify* their positions through asking the powerful question, *Why?* Providing a space within the task for students to express their reasoning is an important element for promoting a coherent argument to foster students' reasoning (Stiff & Curcio, 1999).

Teachers' knowledge of mathematical reasoning and its pedagogy was the focus of this PL project. We sought to reveal changes in teachers' understanding of reasoning through the PLT approach. The PLT approach for PL about reasoning was designed to provide opportunities for teachers to build their knowledge of reasoning and teaching approaches which foster the development of students' reasoning. It involved teachers working together in a PLT to plan lessons specifically designed to foster students' reasoning, then observe each other's teaching to recognise and record instances of reasoning being fostered by these teachers and demonstrated by the students, and finally reflect on their observations, practice and planning process. This is consistent with the importance of peers working together to improve their practice through discussion and reflection (Byrne et al., 2010). Kate offered guidance in teaching reasoning due to her knowledge of reasoning and modelled successful learning situations. Further, Kate's energy and enthusiasm and her ability to motivate Amy was similar to the positive characteristics noted by Howell and Higgins (1990).

Opportunities and constraints were evident in this project. An opportunity facilitated by this approach was peer-observation. Amy found peer-observation advantageous in offering a space to learn from Kate's pedagogical actions (as suggested by Clarke et al., 2013). Exposure to

Kate’s teaching practices, where discussions were orchestrated (Stein, Engle, Smith, & Hughes, 2008) to elicit reasoning, provided Amy with an opportunity to grow professionally from this PLT experience; hence, providing the chance to implement changes in her practice as noted by Nickerson and Moriarty (2005). Furthermore, Johnston and Cornish (2016) supported the employment of peer-observation where teachers are actively engaged in observing good pedagogical practice as illustrated by Kate. One constraint was the size of the PLT since a larger PLT would provide opportunities for more rigorous discussions. In this school, there were only two teachers in the PLT, containing costs involved in time release. A cost-benefit analysis of different sizes for PLTs could be the focus of further research.

The planning intentions in a PLT was a key feature of this form of PL. Planning a lesson together was unusual at this school. However, Amy and Kate found that planning together facilitated embedding reasoning into their lessons. The interplay between the PLT during the planning revealed prompts that signalled the teachers’ growing awareness of the importance of communicating reasoning and building a dialogue between themselves and their students (Nardi & Knuth, 2017). A particularly satisfying result from the PL program was to hear Amy (whose approach to teaching mathematics mainly consisted of children quietly completing problems from a workbook), hoping to encourage active engagement in reasoning from her students through questioning and inviting students to share their justifications.

Classroom actions exhibited by the teachers were at the forefront of the teachers’ post-lesson reflections. The opportunity to observe each other in action provided a rich context to build on their learning of mathematical reasoning and come to a shared understanding of good pedagogical practice for fostering students’ reasoning through reflection. Critical reflection was essential for the effectiveness of the peer-observation process (Eri, 2014). It was evident from the reporting of the lessons that the teachers came to notice more instances of students’ reasoning and reflect on the opportunities provided by the classroom actions that resulted in that reasoning.

## Conclusion

The data presented supported Sullivan et al.’s (2016) assertion that planning intentions were influenced by beliefs, knowledge, and opportunities. These planning intentions impacted on classroom actions. Peer-observation of team members’ classroom actions supported development of knowledge of mathematical reasoning and the pedagogy required to foster it. Sullivan et al.’s (2016) planning framework was useful in structuring the analysis of the data by paying close attention to teachers’ knowledge, beliefs, and opportunities. Therefore, these key nodes should be considered when developing a PL program that incorporates planning for reasoning.

A program of PL that engages teachers to work in a PLT to deliberately plan to embed reasoning in a lesson, followed by peer-observation and reflection has been shown to be an effective model of PL about the proficiency of mathematical reasoning and pedagogical approaches that facilitate the development of students’ reasoning. Whilst in previous research teachers working together as a team is an effective form of PL of mathematical concepts and related pedagogy, little is known about its effect when focussing on a proficiency strand such as reasoning. In this study, the PLT worked autonomously to design and plan their lesson explicitly for reasoning. Although there are limitations due to the small sample size of participants in this study, the success of this PLT as a collegial, supportive group to advocate pedagogical change is worthy of further exploration.

Whilst Phase 1, based around demonstration lessons, was effective in affecting some growth in teacher understanding of the many aspects of reasoning and appropriate pedagogies, it was costly in researchers' time in preparing and teaching of demonstration lessons, and in time-release of a group of teachers. Our results show that the PLT approach taken in this study is effective in supporting teachers as they learn to embed reasoning in mathematics lessons. This approach is a more cost-effective way of embedding reasoning in mathematics lessons if the PLT can be supported by a teacher with knowledge of reasoning and how to implement it in the classroom. A leader of a PLT to champion reasoning is needed to guide teachers in embedding reasoning into every mathematics lesson, raising the importance of identifying reasoning, and training key people in the school in their knowledge of reasoning and approaches that support students' reasoning. Focused PL for the leader of a PLT has the potential to support many teachers to employ reasoning prompts in every mathematics lesson and warrants further investigation.

These findings add to a growing body of literature on addressing the challenges teachers face in learning about mathematical reasoning and substantially adds to our understanding of the nature of PL programs to facilitate teachers' ability to embed reasoning in lessons. Consequently, we are conducting further research to identify key aspects of effective training of reasoning champions to lead PLTs.

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