From Receiving Mathematics to Negotiating Meaning: Development of Students' Understanding of Geometric Concepts

Michael J. Bossé
Appalachian State University, USA
bossemj@appstate.edu

Anass Bayaga
Nelson Mandela University, South Africa
Anass.Bayaga@mandela.ac.za

Catherine Fountain
Appalachian State University, USA
fountainca@appstate.edu

John Sevier
Appalachian State University, USA
sevierjn@appstate.edu

This study applies a framework for mathematical cognition and learning (Bossé, et al., 2018a; Bossé et al., 2019; Bossé et al., 2018b) to examples of students' comprehension of geometric concepts related to angles. This framework utilizes key concepts from second language acquisition (SLA) to elucidate processes involved in mathematical learning. The present study analyzes concrete examples of student-teacher and student-student discourse to illustrate how students at different stages of development build their understanding of mathematical principles and concepts regarding angles. The findings indicate that students at different framework levels learn, process, and communicate about angles in observably different ways that share similarities with mathematical learning and SLA.

Introduction

Stages in primary and second language acquisition (SLA) theories have been established and understood for decades. Simultaneously, research has developed an understanding of how students learn mathematics and geometry. Recent investigations regarding the intersection of SLA and mathematics learning theories have suggested that mathematics is learned in a manner similar to SLA and has led to the development of the Mathematics Acquisition Framework (MAF), through which four stages of mathematical learning are recognized: receiving, replicating, negotiating meaning, and producing mathematics (Bossé, et al., 2018b).

The MAF has been previously considered concerning the mathematical learning of 5th-7th grade students regarding fraction concepts (Bossé, et al., 2018a) and students from 5th grade considering number ideas, 6th grade considering fraction division, and 9th grade regarding linear equations (Bossé, et al., 2018b). These studies have led to novel findings and implications regarding student learning and instruction. Among other ideas and mathematical concepts, geometry and angles have yet to be investigated through this lens. This study examines students' geometric understanding of different grade levels concerning concepts associated with angles and angle measures through the first three MAF stages. The purpose of this study is to gain greater insights into student understanding of the respective geometric concepts. More importantly, the study aims to further determine the validity and reliability of the MAF as a framework to investigate novel areas of mathematics learning. Through this investigation, student understanding can be recognized in novel ways.

Background Literature

As previously developed in Bossé et al. (2018a), Bossé, et al. (2019), and Bossé et al. (2018b), similarities exist between SLA and mathematics learning theories. Rather than reconsidering these realms independently, in this literature review, we focus on some intersecting ideas in the respective fields, including learning moving from concrete to abstract, students moving from limited understanding and communication to making connections, and students negotiating meaning.

From Concrete to Abstract

Among language acquisition and mathematics learning theories, there exist common themes that students progress from concrete to representational to abstract understanding. Regarding a child's innate cognitive development, Piaget (1972) recognizes the processes of assimilation and accommodation, and Bruner (1966, 1979) notes the progression of Enactive, Iconic, and Symbolic modes. Numerous applied linguists (e.g., Bailey et al., 1974; Dulay & Burt, 1973, 1974; Vygotsky, 1978) agree with Piaget and Bruner and recognize sequential stages of development from concrete through abstract understanding. This includes Krashen and Terrell (1983), who propose sequential stages of primary language: pre-production, early production, early speech emergence, early intermediate, intermediate, and advanced. In respect to SLA, Cummins' (1979, 1984, 1991) theories propose that students progress from a social language regarding cognitively undemanding tasks aided by an abundance of context clues (concrete learning) to an academic language regarding cognitively demanding tasks lacking context clues (abstract learning).

Several recognized mathematics learning theories similarly note the progression of learning from concrete to abstract. For instance, Dienes (1960, 1971) and Dienes and Golding (1971) describe a six-stage Learning Cycle through which learners come to understand mathematics: the three cyclical stages of free play, games, and searching for commonalities and the three stages of representation, symbolization, and formalization. Through their Structure of Observed Learning Outcomes (SOLO Taxonomy), Biggs and Collis (1982) define the sequential mathematics learning stages of prestructural, unistructural, multistructural, relational, and extended abstract. Specifically, concerning geometry learning, Van Hiele (1986) theorizes five levels of understanding: visualization, analysis, abstraction, deduction, and rigor.

Altogether, while considering many different dimensions, numerous theories from cognition, SLA, and mathematics learning point to student learning progressing from the concrete to the abstract. Notably, most theories recognize additional nuances among, and several intermediate steps between, these two poles. Nevertheless, this common appreciation of these poles provides a unifying theme among many theories.

From Limited Understanding and Communication to Making Connections

Language acquisition and mathematics learning theories recognize that students begin with limited communicative skills in a new language or topic and progress to more appropriate and effective communication regarding advanced ideas. For instance, Krashen's Monitor Model (1977, 1982) for language acquisition defines a *silent period* when early learners focus their energy on listening to understand rather than producing the new language. Krashen (1977, 1982) also recognizes the notion of *comprehensible input*, where students simultaneously understand some ideas within the new language and struggle with other ideas slightly beyond their current understanding. Swain (1985) and Swain and Lapkin (1995) add the dimension of *comprehensible* *output*, where attempts at language production reveal that some concepts, while understood, remain slightly beyond the student's ability to verbally articulate the thought. Cummins' (1979, 1984, 1991) theories regarding SLA propose that language learners progress through two simultaneous tracks of progressing from social language to academic language and from cognitively undemanding content to cognitively demanding content. Through these tracks, students naturally progress through ideas they struggle to understand and communicate. A transition from memorized language to greater fluency and command is also found in Krashen's stages of SLA, in which students move from early production, characterized by short, memorized phrases and by errors in production, towards intermediate fluency, characterized by a relatively high level of ability in the second language (Krashen 1982; Cummins, 1986, 1991; Dulay & Burt, 1973; Lightbrown & Spada, 1999).

Regarding mathematics learning, the Learning Cycle (Dienes, 1960, 1971; Dienes & Golding, 1971) notes that students progress from both limited understanding and a limited ability to communicate mathematical ideas to representing and connecting ideas symbolically. In a parallel manner, the SOLO Taxonomy (Biggs & Collis, 1982) reveals that students transition along with the progression of a lack of understanding of a task and attacking the problem inappropriately by oversimplifying a solution strategy (sharing similarities with Biggs & Collis, 1982; Clements, Battista, & Sarama, 2001; Pica, 1996; Selinker, 1992); focusing only on one aspect of the problem; addressing several relevant, albeit disconnected elements of the task; synthesizing relevant elements of the task into a conceptual whole; and generalizing from the previous coherent whole into new topics. Regarding the learning of geometry, Van Hiele's (1986) levels demonstrate, sequentially, that the learner:

- identifies individual shapes, without being able to communicate its attributes (sharing similarities with Burger & Shaughnessy, 1986; Fuys, 1985; Fuys et al., 1988);
- prioritizes properties of a shape but recognizes the properties in isolation and not as a tool through which to compare shapes;
- connects properties and recognizes that particular combinations of properties imply other ideas; and
- uses deductive reasoning, creates simple proofs, and informally and incompletely understands non-Euclidean geometries (sharing similarities with Christiansen, 1997; Fuys, 1985; Fuys et al., 1988); and develops a sophisticated understanding of axiom systems.

The parallels between SLA and mathematics learning theories are salient and sophisticated in this realm. It must be noted that a distinction and connection is made between student understanding and student communication. More precisely, the literature recognizes that student understanding progresses from being limited to making connections and that student communication progresses from being limited to being more fluent. These theories make these arguments in different ways. However, consistent among these is that growing understanding leads to a growing ability to communicate and use ideas in their context.

Negotiating Meaning

Numerous theories regarding SLA recognize that learners personally and interpersonally negotiate meaning in the process of language learning (Christiansen, 1997; Garfinkel, 1967; Krashen, 1977, 1982; Krashen & Terrell, 1983; Pica, 1996). Students individually negotiate meaning as they interpret, analyze, and synthesize ideas, and they interpersonally negotiate meaning as they communicate with others in the learning environment. In the context of interpersonal communication, negotiation of meaning includes specific communicative strategies:

confirmation checks, comprehension checks, and requests for clarification (Pica, 1987). To assist in bridging the gap and addressing comprehensible input and comprehensible output, Selinker (1972, 1992) proposes the learners' use of *interlanguage*, an idiosyncratic and constantly evolving articulation in the process of SLA which allows interlocutors to negotiate meaning resulting in effective communication. Some recognize that negotiating meaning includes students transitioning from teacher-centric to student-centric experiences and from listening to reading and speaking to writing (e.g., Bossé et al., 2018a; Cummins, 1979, 1984, 1991; Krashen & Terrell, 1983; Selinker, 1972, 1992).

Albeit through different nomenclature, theories regarding the learning of mathematics are infused with the notion of students negotiating meaning in the process of learning. For instance, between stages associated with levels of geometric understanding, Van Hiele (1986) theorizes a five-phase sequence through which students transition from any level to the following level: inquiry/information, directed orientation, explication, free orientation, and integration. These phases can be equated to students personally negotiating meaning. Similarly, the six-stage Learning Cycle (Dienes, 1960, 1971; Dienes & Golding, 1971) involves students negotiating meaning in increasingly sophisticated ways as they progress through the cycle. The SOLO Taxonomy (Biggs & Collis, 1982) also implies that students personally negotiate meaning within each level (prestructural, unistructural, multistructural, relational, and extended abstract) and progress to the next level of understanding.

Although the notion of negotiating meaning exists overtly in the SLA literature and can be induced from mathematics learning theories, the latter rarely seems to use this specific phraseology. Nevertheless, the fields share the recognition that students' initial understanding of ideas evolves through internal, personal, wrangling with these ideas to make them fit into their schema.

The Mathematics Acquisition Framework

Combining ideas from SLA and mathematics learning theories, Bossé et al. (2018a), Bossé et al. (2019), and Bossé et al. (2018b) posit the Mathematics Acquisition Framework (MAF) denoting characteristics of a sequence of stages defining the learning of mathematics. Notably, progressing through these stages can be iterative as learners are exposed to new topics, and stages can overlap with students at different stages regarding other topics.

The stages include the following.

Receiving mathematics. Students: listen to teachers' explanations of examples; demonstrate limited comprehension and provide few responses; cannot distinguish valid and misleading information; recognize simple computations and solutions; use imprecise mathematical language. In this stage: the language of mathematics is primarily social replete with context cues; the cognitive level of the instruction and tasks is undemanding; instruction and learning are teacher-centric, and the primary mode of communication is through speaking.

Replicating mathematics. Students: comprehend contextualized information; respond to simple questions; talk and write about mathematical experiences; understand mathematical concepts disjointedly; use imprecise mathematical communication; demonstrate limited mathematical conceptual understanding; are focused on familiar heuristics; attempt to replicate what they observe and read simple contextualized mathematics. In this stage: the language of mathematics is blended between social and academic language, still with a greater focus on the former, with a minimally decreasing use of context cues; the cognitive level of the instruction and tasks remains primarily undemanding; instruction and learning are primarily teacher-centric with occasional attempts at student-centric learning; and the primary mode of communication is through speaking, with some leanings toward reading.

Negotiating meaning. Students: have limited mathematical repertoire; follow simple ideas but struggle to track novel ideas; are growing in proficiency in communicating ideas but struggling with precise mathematical intricacies; practice correctly communicating mathematics; use multiple, albeit disconnected, representations; see mathematical concepts and applications discretely, and have an inability to create novel mathematics. In this stage: the language of mathematics becomes less social and more academic, with decreasing use of context cues; the cognitive level of the instruction is growing toward being demanding; learning is becoming more student-centric; and the primary mode of communication is through reading and speaking, with leanings toward students communicating their ideas and understanding. Notably, two dimensions are associated with this stage: personal versus interpersonal.

Personal. Students: have proficiency in communicating simple ideas and excellent comprehension with those; practice correctly communicating mathematics; apply mathematics to what they know; use multiple, albeit disconnected, representations; follow simple ideas but struggle to track novel ideas; engage independently in mathematical investigations; apply mathematical concepts to their personal interests; have a limited mathematical repertoire; see mathematical concepts and applications discretely; and become more involved in textbook readings and class notes.

Interpersonal. Students: understand most mathematics but struggle with intricacies; gain precision in communication but have difficulty with novel topics; integrate mathematical ideas; discuss mathematical ideas to learn from others; use more formal mathematical language; begin understanding concepts in different contexts; and experiment with ideas provided by others.

Producing mathematics. Students: approach semi-professional fluency; explore multi representational math; use their knowledge to extend to novel ideas; become autodidactic; write mathematics properly; and interconnect mathematical concepts. In this stage: the language of mathematics becomes densely academic, with few context cues; the cognitive level of instruction and learning is demanding; learning is student-centric, and the primary mode of communication is through writing.

The MAF forms a Gestalt of the previously noted SLA and mathematics learning theories. This Gestalt considers a greater number of dimensions than any subset of these learning theories. As a superset, the dimensions addressed in the MAF include students progressing from concrete to abstract understanding, students progressing from limited understanding and communication activity to making connections, and students negotiating meaning. Furthermore, in addition to defining behavioral characteristics associated with stages of mathematical learning, the MAF demonstrates transitions in many characteristic dimensions, including transitioning from social to academic language, cognitively undemanding to demanding, teacher-centric to student-centric, and listening to reading and speaking to writing (Bossé et al., 2018a; Bossé et al., 2019; Bossé et al., 2018b; Cummins, 1979, 1984, 1991; Krashen & Terrell, 1983; Selinker, 1972, 1992).

Research Objectives

As previously mentioned, the initial development of the MAF was provided in Bossé et al. (2018b). Since then, the MAF has been employed to investigate mathematical learning of 5th-7th grade students regarding fraction concepts (Bossé et al., 2018a) and 5th, 6th, and 9th-grade students in the contexts of number ideas, fraction division, and linear equations, respectively (Bossé et al., 2019). These studies have spoken to the educational community through novel findings and significant implications. Thus, it then seemed valuable to consider other levels of students in different mathematics contexts to see how this lens could speak to mathematics learning in other situations.

This study, therefore, selected to build upon and extend previous studies by considering geometry as another mathematical context. Geometry was chosen because it is rich in concepts, vocabulary, and potential misconceptions; it requires significant communication in learning and discussion, and most students consider it in their mathematical pursuits. More specifically, the topic of angles was selected because it is foundational to geometric understanding and elementary in the early learning of geometry. Thus, students would have considered the concept of angle early in any investigation of geometry.

Since students are often introduced to measuring angles in elementary school before more expansive considerations of geometry in high school, this study sought to consider scenarios at each end of this spectrum. Due to student availability, an elementary school student is interviewed, and four 10^{th} grade students are interviewed. Additionally, since the MAF is tied to SLA, it was considered valuable to consider students in those years or ages in which students are commonly learning a second language.

The context of geometry, together with the grade levels of the selected students and the use of the MAF as a framework, significantly extends upon previous research by applying the MAF to a novel field of mathematics and considering students in different grade levels. The MAF is employed in this study to analyze transcripts of student work and articulations to investigate student understanding regarding geometry and angles. As previously stated, the primary purpose of this study is to further determine the validity and reliability of the MAF as a framework to investigate novel areas of mathematics learning. This is captured in the *Informal Question* associated with this study. Simultaneously, this study has a secondary purpose: to gain greater insights into student understanding of the respective geometric concepts. This is captured in the Formal Question associated with this study. These questions are:

> *Forma1 Question.* How does the MAF speak to the work, reasoning, and articulations of 5^{th} and 10^{th} grade students participating in this study as they consider geometry and angles?
>
> *Informal Question.* How do the work, reasoning, and articulations of these student participants speak to the MAF? Might the participants' work provide an additional layer of validation to the MAF and demonstrate that the MAF may be used from increasingly broader audiences considering a broader swath of mathematical concepts?

Methodology

The participants of this study were students in schools in the southeastern United States. The student in Scenario 1 was in the 5^{th} grade and had recently completed two lessons on measuring angles using a protractor. According to the classroom teacher, the student had completed the lessons successfully and could readily measure angles with a protractor. The students in Scenario 2 were all in the same 10^{th} grade geometry class and had previously investigated ideas regarding angles, angle measures, and polygons. They were approximately halfway through a year-long course on geometry. These students were selected based on convenience to the researchers and respective classroom teachers. Nothing unique or distinguishing was considered in the selection of these students.

The qualitative research methodologies employed in this study are case studies together with discourse analysis (Bogden & Biklen, 2003; Creswell, 2003; Miles & Huberman, 1994; Wodak, 2009; Wodak & Meyer, 2009). These methodologies would allow the analysis and synthesis of the transcript data.

In both scenarios investigated (a 5th grade student and four 10th grade students), students were videotaped in their respective classroom environments. The researchers and teachers adhere to all research and ethical protections required by the respective schools and universities. All attempts were made to retain the organic and natural experiences the students often experienced in their classrooms. For instance, in Scenario 1, the classroom teacher often questioned students one-on-one to ascertain understanding. In Scenario 2, allowing students to collaborate in groups, the classroom teacher might interject questions or ideas.

Qualitative research systematically analyzes ideas within qualitative data and unpacks meaning (Bogden & Biklen, 2003; Creswell, 2003). Videotapes of student work and communication were transcribed, and copies of student work were merged with each transcript. Discourse analysis (Wodak, 2009; Wodak & Meyer, 2009) was employed to analyze the transcripts and investigate student mathematical understanding, communication, and behaviors applicable to the MAF (Bossé et al., 2018a; Bossé et al., 2018b). Behaviors from the MAF were sought in the student transcripts. Codes were created regarding the MAF stages of receiving, replicating, and negotiating meaning. Additional codes were developed to observe characteristics regarding a silent period, comprehensible input, comprehensible output, interlanguage, confirmation checks, comprehension checks, and requests for clarification. Coding by different research team members was compared, differences reconciled, and refinements were made. Check-coding (Miles & Huberman, 1994) allowed researchers to reach a consensus of the analysis. In the end, narrative summaries were developed and validated against the transcripts with respect to the MAF.

Notably, the MAF includes the stage of Producing Mathematics. None of the students in this study demonstrated characteristics associated with this stage. Thus, it was not considered a component in the analysis of these students.

Results and Brief Analysis

Scenario 1: Determining the size of an angle

The following example is part of a conversation in which a student is working with a teacher to determine which of the two angles is larger. Notably, due to the length of the transcripts, portions are provided, followed by some analysis.

Teacher: [Considering Figure 1] Which angle has a greater measure than the other?
Student: Angle B is smaller.
Teacher: Why?
Student: It fits inside of A.
Teacher: So, the location of the angles can indicate which is greater than the other?
Student: Sometimes.

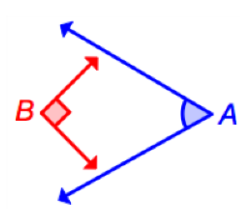

Figure 1

Teacher: If we drew it this way [Figure 2], which angle would have a greater measure?
Student: A is bigger.
Teacher: Why?
Student: Because it's bigger.
Teacher: What do you mean?
Student: [Pointing to the top ray on ∠A beyond the ray on ∠B on Figure 2] See here.
Teacher: You mean that the ray on angle A is longer than the ray on angle B?
Student: Yup.

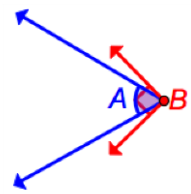

Figure 2

In this exchange, we recognize some behavioral indicators that apply to this research. Notably, the student's communication seems to be characterized by imprecision and a lack of comprehension of many concepts. The student initially provides the incorrect answer, "Angle B is smaller," and soon shows that this answer is based on the misunderstanding that the location of angles can indicate which is greater; this seems to demonstrate limited comprehension and an inability to distinguish between valid and misleading information. When the teacher shows the student a different representation of the same two angles, the student further demonstrates this limited comprehension by stating, "A is bigger." As the teacher asks for clarification, the student's response is repetitious, "Because it's bigger." When pushed to explain further, the student responds simply by pointing and saying, "See here," and then, to the follow-up question, a single-word answer, "Yup."

Comprehensible input is evidenced by the notion of the magnitude of the angle measures, while discussable, being slightly beyond the cognitive reach of the student. The student cannot satisfactorily articulate their ideas regarding the characteristics of the angle demonstrates comprehensible output. The student's use of "bigger" rather than "greater than" demonstrates instances of interlanguage—when language adjustments are made to effectively communicate ideas between interlocutors. Notably, in this snippet of the transcript, the student demonstrates no examples of confirmation and comprehension checks or requests for clarification. The student's very brief responses may represent not fully emerging from the silent period. The student's language is informal (social) and lacks formal precision (academic language). The learning experience is more teacher-centric in form than student-centric.

Teacher: How far do the rays go?
Student: They go forever.
Teacher: So, [pointing to the top ray on ∠B] how far does this go?
Student: [Pointing to the arrowhead] Right to here.
Teacher: Why does it end there, if you said a ray goes forever?
Student: It's an angle?
Teacher: So, if it's on an angle, it doesn't go on forever?
Student: No.
Teacher: Why not?
Student: Then you wouldn't know how big it is.
Teacher: Angle A has an arc to indicate something about the angle, and angle B has a square. Does this mean anything?
Student: It means that angle B is a right angle.

Teacher: That is great. I'm glad you remembered that. And what about the arc on angle A?
Student: It just means that it is not right.
Teacher: Can it be less than or greater than a right angle?
Student: Either.
Teacher: Right. So which one is it here?
Student: A is bigger than B.
Teacher: Why?
Student: This is still longer.
Teacher: But isn't the opening on angle B wider than the opening on angle A?
Student: No. [Pointing at the distances between the respective pairs of arrowheads] This [the distance between the arrowheads on angle A] is bigger than this [the distance between the arrowheads on angle B].

As an example of recognizing simple computations and solutions, the student correctly states that rays go on forever. However, further discourse demonstrates limited comprehension, as the student fails to connect the understanding of rays to the side of an angle; the student thinks that, while rays have infinite unidirectional length, the sides of angles are limited to the length from the vertex to the arrowhead. The student recognizes that the square angle mark indicates a right angle, and the arc indicates a non-right angle. However, the student seems to overlook this understanding, preferring to consider the distance between arrowheads on the respective angles, demonstrating an inability to distinguish between valid and misleading information.

The student's struggle to comprehend and articulate notions connecting rays with angle sides simultaneously demonstrates instances of comprehensible input and comprehensible output. The student uses "wider" as an interlanguage to communicate the idea of greater angle measure. The necessity of the teacher to continually follow up terse and incomplete responses with additional questions may again suggest that the student has not fully emerged from the silent period. The student struggles to reconcile a nascent understanding of angle with the blatant difference in the ray lengths. When challenged, the student seemingly steps back to something about which they believe they are more confident (i.e., the ray lengths), limiting the student's exposure to potential errors by giving terse answers. Notably, still, the student does not exhibit instances of confirmation and comprehension checks or requests for clarification.

Teacher: If I drew it this way [Figure 3], which angle has a greater measure?
Student: B is bigger.
Teacher: Why?
Student: The lines look close to the same, but angle A is inside angle B.
Teacher: You mean that the sides of the angles look like they are the same length?
Student: Yes.
Teacher: Can I show you one more example? What about this one? [Figure 4]

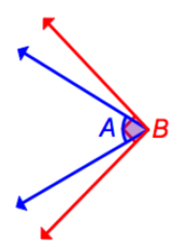

Figure 3

Student: B is bigger.
Teacher: What if I told you that, in every picture, the measures of both angles stay the same.
Student: [Confused look]
Teacher: Angle B stays as a right angle in all pictures. And angle A stays the same measure in all the pictures.
Student: [Confused look] But angle B here [pointing to Figure 1] is bigger than here [pointing to Figure 4].
Teacher: But they are both 90 degrees.
Student: Yes, but this [angle B in Figure 4] one is a bigger 90 degrees.

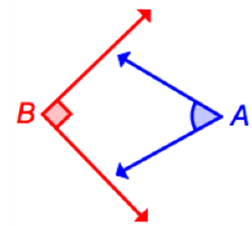

Figure 4

This portion of the transcript may most clearly demonstrate the student's limited mathematical understanding. The student continues to demonstrate limited comprehension of concepts, such as degrees being an invariable measurement regarding the lengths of the respective rays depicting the angle and shows an inability to distinguish between valid and misleading information. The student uses imprecise language concerning mathematical concepts ("The lines look close to the same"), provides few and terse responses, and, in at least one case, responds only with a confused look.

The student exhibits additional examples of comprehensible input, comprehensible output, and interlanguage. The student's language is more social (informal) than academic (formal), and the activity remains teacher-centric. The student's more extended responses may evidence emergence from the silent period, albeit these more extended responses seem only to be the result of strong questioning skills from the teacher. And yet, the student again demonstrates no examples of confirmation and comprehension checks or requests for clarification; these still reside singularly in the teacher's actions.

In summary, through these three combined transcript fragments, the student seems to be fully in the receiving mathematics stage. We see no indication from this exchange that the learner might be close to transitioning into the next stage, replicating mathematics.

Scenario 2: On Angle Measure Versus Rotation

In the following transcripts, four students work together through concepts related to measuring angles with a teacher. As before, the lengthy transcript is portioned into smaller sections with some analysis following each portion.

Teacher: In Euclidean geometry, what is the greatest and least angle we can measure?
Student 1: There isn't a limit. We can go to $-\infty$ and to $+\infty$.
Teacher: How do you know? What says that you can do this?
Student 2: If you go around and around counter-clockwise, you can keep going around and getting higher and higher degrees. And the same for clockwise and getting smaller and smaller degrees.
Student 1: You don't mean smaller degrees. You mean negative degrees of greater absolute value.
Student 2: Right. That still gets us to $-\infty$ and to $+\infty$.
Teacher: What if I said that is incorrect?
Student 3: How?
Teacher: I am stating categorically that it is incorrect. I want you to figure out why.
Student 4: It's got to be about the negative angles. Maybe it goes to positive infinity but in either a positive or negative direction.
Student 3: [Asking the teacher] Is that right?

Student 2 immediately uses imprecise mathematics language describing rotations creating "higher degrees" and "smaller degrees." However, Student 1 corrects this notion by distinguishing between small angular measurements and negative angles, and Students 1 and 2 come to agree that $-\infty \leq$ angle measure $\leq \infty$. Whether or not leading to correct understanding, the students communicate, listen to each other, and learn from each other. While mathematical understanding is growing, mathematical conceptual understanding is limited, and concepts are held disjointedly.

In this brief portion of a transcript, there is evidence of comprehensible input (being able to discuss $\pm\infty$ without fully understanding such), comprehensible output (having a grasp of angular rotations, but not being able to adequately articulate their ideas regarding such), and interlanguage (mention angles as positive and negative rather than angle measures as positive or negative). These students are well beyond the silent period and freely share their ideas. Additionally, we see confirmation checks, comprehension checks, and requests for clarification as students interact, question themselves and others, seek to learn from other students' communications, and refine both individual and corporate ideas and communication. The group's language is becoming more formal and academic (moving from "big" and "small" to $+\infty$ and $-\infty$) as they wrestle with more cognitively demanding notions. The learning environment is more student-centric, as students communicate with, and learn from, their peers.

Student 1: What if we say that the angle is always positive, but it can be either in the clockwise or counterclockwise direction?
Student 2: In trig, we look at positive and negative angles. We can look at the sine of -300 degrees.
Student 3: Maybe [the teacher] is wrong.
Student 4: I don't know. Let's list what we definitely know? We know that an angle can be zero or greater.
Student 1: Is there an upper limit?
Student 4: I don't think so. So, we have for sure [writing] $0° \leq x° \leq \infty°$. Everyone ok with this so far?
Student 2: Yup. But it still kinda bothers me that we have negative angles in trig.
Student 1: But [the teacher] said that was wrong.
Student 3: [To the teacher] Is [writing] $0° \leq x° \leq \infty°$ correct?
Teacher: No.
Student 3: Why?
Teacher: [Shrugs and silently indicates that he is not answering and that the question belongs to the students to analyze]
Student 1: Ok. So, it's probably because we have ∞. Maybe we can just go up to 360°, and everything bigger is really just a multiple of 360° plus some angle.
Student 2: Then we want [writing] $0° \leq x° \leq 360°$. [To the teacher] Is that right?
Teacher: No.

The students focus on familiar heuristics; they begin with $-\infty \leq$ angle measure $\leq \infty$ and modify it as little as possible, and only incrementally. Their mathematical understanding and communication are improving; however, they continue to use imprecise mathematical verbiage and demonstrate limited conceptual understanding. They are making connections outside of geometry to trigonometry and the notion of upper limits. These students continue to evidence instances of comprehensible input, comprehensible output, and interlanguage, confirmation checks, comprehension checks, and requests for clarification. Additionally, they now have moved toward valuing writing in their learning process, as Student 4 suggests listing all they know about the topic, and three students write out inequalities and ranges for the angle measures.

Student 2: I found this [from their textbook]: "In Euclidean geometry, the measure of an angle is $0° < x° < 180°$."
Student 3: That can't be right. We use zero angles all the time.
Student 4: And we know straight angles. 0 and 180 must be ok.
Student 1: And we have reflex angles which we know are [writing] $180° < x° < 360°$.
Student 2: I think that the book is wrong.
Teacher: The book is correct. Figure out what is going on.
Student 3: Ok… If the book is correct, then there is no such thing as 0° angles, straight angles, or reflex angles.
Student 1: Then why do we use them.
Student 4: If the definition is correct, then these angles are wrong.
Student 2: I guess that we just have to accept that.
Student 1: But we have been using them.
Student 2: But we weren't supposed to.

In this portion of the transcript, the students read contextualized mathematics from a textbook. As they discuss the textbook definition and their deliberations, we see evidence of talking and writing about mathematical experiences. They also begin to make connections to zero, straight, and reflex angles and check their comprehension regarding such. We continue to see the students learning through communicating with each other. In doing so, they continue to evidence confirmation checks, comprehension checks, and requests for clarification. The observation of fewer instances of comprehensible input and comprehensible output may be because the students are growing in understanding and are better able to articulate their ideas. The learning environment is student-centric with minimal teacher interaction. Students are considering increasingly complex ideas and are doing so more through speaking and writing.

Student 1: What about a quadrilateral like this? [Draws Figure 5] We have an internal angle at D, which is a reflex angle. I wonder if this is a valid quadrilateral if D is not a valid angle in Euclidean geometry?
Student 4: If we have to accept this definition, then we may need to modify our definition of a quadrilateral – or any polygon with internal reflex angles.

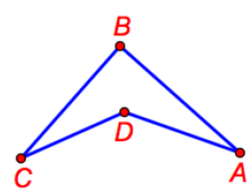

Figure 5

Student 1: How?
Student 4: Well, we could say that a "Euclidean quadrilateral" has internal Euclidean angles.
Student 1: Then what would we do with this quadrilateral?
Student 4: I wouldn't consider it a quadrilateral. I would consider it a composite of two triangles. [Draws segment BD to create Figure 6] See. Now no angles are greater than 180°.
Student 2: Ok. That's an ok workaround. But I'm still struggling with the fact that we use both negative and larger positive angles in trig.
Teacher: Is there a possibility that "angle" means something different in trig?

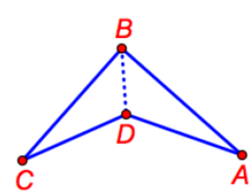

Figure 6

Student 3: The definition for angle in the book said that it was a definition for angle in Euclidean geometry. It did not say that that it was the definition for angle in trig.

Teacher: In trigonometry, we often use the word “angle” when we really mean “angle of rotation.” This is a significant difference.

Student 3: And rotations can go round and round and not stop. Or they can go in positive or negative directions. So, $-\infty° \leq x° \leq \infty°$ works for angles of rotation. Cool.

Student 4: Really cool.

In this final portion of the dialogue, students wrestle with the new information and realize that they have made mistakes in their previous deliberations. The students attempt to reconcile what they have read in the textbook with their previous discussions and demonstrate growing mathematical conceptual understanding as they attempt to explain the angles in the quadrilateral they are investigating. While the students are learning through communication with others and gaining precision in communication, they still struggle with novel topics and integrating mathematical ideas (wrestling with the notion of the angles on the nonconvex quadrilateral). After the teacher provides the students with a hint about how they might need to think about terminology, they begin to understand concepts in different contexts and see mathematical concepts and applications discretely. We can recognize that these students rely on each other and not the teacher for learning. They also focus on writing appropriately for the investigation (drawing a geometric figure).

In their exchanges, we encounter examples of students who are principally operating in the stage of replicating mathematics and show some evidence of moving past this stage into negotiating meaning. This is investigated in more detail in the discussions below.

Discussion and Implications

Recalling that two questions—one formal and one informal—drove this research project, we now consider one at a time through this discussion.

The Formal Question

> How does the MAF speak to the work, reasoning, and articulations of the 5th and the 10th grade students in this study as they consider geometry and angles?

The transcripts analyzed in this study provide evidence of student behaviors and communications consistent with stages in the MAF (Bossé et al., 2018a; Bossé et al., 2018b). For instance, the 5th grade student in Scenario 1 exhibited many of the characteristics associated with the stage receiving mathematics, including: listening to explanations of examples; demonstrating limited comprehension; providing few responses to questions; an inability to distinguish between valid and misleading information; recognition of simple computations and solutions; and use of imprecise language about mathematical concepts. As previously considered, many of these characteristics evolved from notions from the literature regarding cognitive development, language acquisition, and mathematics learning (Biggs & Collis, 1982; Burger & Shaughnessy, 1986; Christiansen, 1997; Clements et al., 2001; Cummins, 1979, 1984, 1991; Dienes, 1960, 1971; Dienes & Golding, 1971; Dulay & Burt, 1973; Fuys, 1985; Fuys et al. 1988; Krashen, 1977, 1982; Lightbrown & Spada, 1999; Pica, 1996; Selinker, 1992; Swain, 1985; Swain & Lapkin, 1995).

The 10th grade students in Scenario 2 seem to pose a more difficult case to consider concerning the MAF. Consistent with the MAF stage of replicating mathematics, the earlier transcript portions recognized that the students: comprehend contextualized information; respond to simple questions; talk and write about mathematical experiences; understand mathematical concepts disjointedly; use imprecise mathematical communication; demonstrate limited mathematical conceptual understanding; focus on familiar heuristics; attempt to replicate what is observed; and read simple contextualized mathematics. However, latter transcript portions seem to indicate that the students were more in the stage of negotiating meaning, as they: possess limited mathematical repertoire; follow simple ideas but struggle to track novel ideas; grow proficiency in communicating ideas, but struggle with precise mathematical intricacies; practice correctly communicating mathematics; use multiple, albeit disconnected, representations; and see mathematical concepts and applications discretely. These are consistent with numerous learning theories (Biggs & Collis, 1982; Christiansen, 1997; Cummins, 1979, 1984, 1991; Dienes, 1960, 1971; Dienes & Golding, 1971; Garfinkel, 1967; Krashen, 1977, 1982; Krashen & Terrell, 1983; Pica, 1996, 1997; Selinker, 1972, 1992; Van Hiele, 1986). Negotiating meaning, as evidenced in these transcripts, shows components from both personal (applying mathematics to what they know; engaging independently in mathematical investigations; applying mathematical concepts to their interests; becoming more involved in textbook readings and class notes; and an inability to create novel mathematics) and interpersonal (discussing mathematical ideas to learn from others; practicing with and using more formal mathematical language; beginning to understand concepts in different contexts; experimenting with ideas provided by others) forms. Therefore, it is apparent that students can belong simultaneously in more than one stage of the MAF. Additionally, it may be that students can move from one to a subsequent stage of the MAF during a learning experience.

In addition to students in this study being recognized as being in one or more stages in the MAF, comparative evidence shows that the student in Scenario 1 differs from the students in Scenario 2 in several dimensions. The learning environment for Student 1 is teacher-centric, employing informal (social) language, considering cognitively undemanding tasks, and focused on listening to the teacher. The students in Scenario 2 are increasingly in a student-centric learning experience, employing formal (academic) language, considering more cognitively demanding mathematics, and, while focused on listening to one another, use reading, speaking, and writing to communicate ideas (Bossé et al., 2018a; Bossé et al., 2019; Bossé et al., 2018b; Cummins, 1979, 1984, 1991; Krashen & Terrell, 1983; Selinker, 1972, 1992). Thus, many dimensions are in flux.

As also expressed in Bossé et al. (2018a), Bossé et al. (2019), and Bossé et al. (2018b), understanding the MAF and how students progress through the MAF provides a new window into students' mathematical errors and erroneous communications. As linguistic errors (comprehensible output and interlanguage) are recognized as natural components of the learning process, so too can mathematical errors be recognized as natural in mathematics learning. With this perspective, teachers can welcome student errors as windows into learning.

The Informal Question

> How do the work, reasoning, and articulations of these student participants speak to the MAF? Might the participants' work provide an additional layer of validation to the MAF and demonstrate that the MAF may be used from increasingly broader audiences considering a broader swath of mathematical concepts?

While the lens of the MAF allows students to be recognized as being in a particular stage, research currently remains lacking as to how to help a student transition from one stage to the next. Quite possibly, to do so, more investigations need to be made connecting the MAF to theories that already propose either intermediate stage ideas (e.g., the SOLO Taxonomy, Biggs & Collis, 1982) or phases that connect stages (e.g., Van Hiele, 1986). Techniques which help students transition through would be invaluable to education.

It must be wondered how teachers' understanding of the MAF may assist them in providing stage-appropriate learning opportunities to students. The MAF may indicate that not all learning experiences are appropriate for all students. Instructional differentiation may far exceed simply modifying content. It may entail modifying the learning environment, the investigations and activities considered, and greater focus on listening, reading, speaking, and writing.

Altogether the MAF provides another window into student learning, student understanding, and possibly teaching. It is hoped that others would follow up with further research into student learning.

Limitations

Further studies are needed to determine how these stages might be expressed in other areas of mathematics and how the additional stage of producing mathematics could be observed and analyzed in classroom settings. Bossé et al. (2018a) [applying the framework to fractions and decimals learning] provide such an analysis for the learning of fractions that also examines students who are moving from replicating mathematics to negotiating meaning. Another path for future research would be to identify examples illustrating how students move from each stage in this framework to the next and whether these transitional stages have identifiable characteristics.

References

Bossé, M. J., Bayaga, A., Fountain, C. A., Lynch-Davis, K., Preston, R., & Adu-Gyamfi, K. (2018a). Fraction learners: Assessing understanding through language acquisition. *International Electronic Journal of Elementary Education, 11*(2), 113-124.

Bossé, M. J., Bayaga, A., Fountain, C. A., Slate Young, E., & DeMarte, A. (2019). Mathematics learning through the lens of language acquisition. *IEJEE, 12*(1), 103-113.

Bossé, M. J., Ringler, M., Bayaga, A., Fountain, C.A., & Slate Young, E. (2018b). Acquiring math: Connecting math learning and second language acquisition. *International Journal of Mathematics Teaching and Learning, 19*, 223-252.

Bailey, N., Madden, C., & Krashen, S. (1974). Is there a 'natural sequence' in adult second language learning? *Language Learning, 24*, 235-243.

Biggs, J. B., & Collis, K. F. (1982). *Evaluating the quality of learning - The SOLO taxonomy*. Academic Press.

Bogdan, R. C., & Biklen, S. K. (2003). *Qualitative research for education: An introduction to theories and methods* (4th ed.). Allyn and Bacon.

Bruner, J. S. (1966). *Toward a theory of instruction*. Norton.

Bruner, J. S. (1979). *On knowing*. Belknap Printing.

Burger, W. F., & Shaughnessy, J. M. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education, 17*, 31–48.

Christiansen, I. (1997). When negotiation of meaning is also negotiation of task: Analysis of the communication in an applied mathematics high school course. *Educational Studies in Mathematics 34*(1), 1-25.

Clements, D. H., Battista, M. T., & Sarama, J. (2001). Logo and geometry. *Journal for Research in Mathematics Education*, Monograph Series, 10.
Creswell, W. J. (2003). *Research design: Qualitative, quantitative and mixed methods approaches* (2nd ed.). Sage Publications.
Cummins, J. (1979). Cognitive/academic language proficiency, linguistic interdependence, the optimum age question and some other matters. *Working Papers on Bilingualism*, 19, 121-129.
Cummins, J. (1984). *Bilingualism and special education: Issues in assessment and pedagogy*. Multilingual Matters.
Cummins, J. (1986). Empowering minority students: A framework for intervention. *Harvard Educational Review*, *56*, 18-36.
Cummins, J. (1991). Language development and academic learning. In L. Malavé & G. Duquette (Eds.) *Language, culture and cognition*. (pp. 161-175). Multilingual Matters.
Dienes, Z. P. (1960). *Building up mathematics*. Hutchinson Education Ltd.
Dienes, Z. P. (1971). An example of the passage from the concrete to the manipulation of formal systems. *Educational Studies in Mathematics, 3*(3/4), 337-352.
Dienes, Z. P. & Golding, E. W. (1971). *Approach to modern mathematics*. Herder and Herder Co.
Dulay, H., & Burt, M. (1973). Should we teach children syntax? *Language Learning*, *23*, 245-258.
Dulay, H., & Burt, M. (1974). Natural sequences in child second language acquisition. *Language Learning, 24*, 37-53.
Fuys, D. (1985). Van Hiele levels of thinking in geometry. *Education and Urban Society*, *17*(4), 447-462.
Fuys, D., Geddes, D., & Tischler, R. (1988). The van Hiele model of thinking in geometry among adolescents. *Journal for Research in Mathematics Education*. Monograph 3.
Garfinkel, H. (1967). *Studies in ethnomethodology*. Prentice-Hall.
Krashen, S. (1977). Some issues relating to the monitor model. In H. D. Brown, C. Yorio & R. Crymes (Eds.) *Teaching and learning English as a second language: Trends in research and practice* (pp. 144-158). TESOL.
Krashen, S. D. (1982). *Principles and practice in second language acquisition*. Oxford: Pergamon.
Krashen, S. D., & Terrell, T. (1983). *The natural approach: Language acquisition in the classroom*. Prentice-Hall Europe.
Lightbrown, P. M., & Spada, N. (1999). *How languages are learned* [2nd ed.]. Oxford University Press.
Miles, M. B. & Huberman, M. N. (1994). *Qualitative data analysis: an expanded sourcebook.* Sage.
Piaget, J. (1972). *The psychology of the child*. Basic Books.
Pica, T. (1987). Second language acquisition, social interaction, and the classroom, *Applied Linguistics, 8*, 3–21.
Pica, T. (1996). The essential role of negotiation in the communicative classroom. *Japan Association of Language Teaching Journal*, *18*(2), 241-268
Selinker, L. (1972). Interlanguage. *International Review of Applied Linguistics*, 10, 209-231.
Selinker, L. (1992). *Rediscovering interlanguage*. Longman.

Swain, M. (1985). Communicative competence: some roles of comprehensible input and comprehensible output in its development. In S. Gass, S. and C. Madden, (Eds.), *Input in second language acquisition*. Newbury House.

Swain, M. & Lapkin, S. (1995). Problems in output and the cognitive processes they generate: a step towards second language learning. *Applied Linguistics*, *16*(3), 371-391.

Van Hiele, P. M. (1986). *Structure and insight: A theory of mathematics education*. Academic Press.

Vygotsky, L. S. (1978). *Mind in society.* Harvard University Press.

Wodak, R. (2009). *Aspects of critical discourse analysis*. Sage Publications.

Wodak, R., & Meyer, M. (2009). *Methods of critical discourse analysis*. Longman.