# Preservice Teachers' Understanding of Decimals using Standard Algorithm and Alternative Strategies

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The purpose of this study is to explore preservice teachers' (PTs') ability to implement multiple strategies using standard algorithms and alternative strategies and to investigate their conceptual knowledge of decimal operations. There were 37 PTs who participated in this study, all of whom attended a mathematics course in the K-8 Teacher Education Program at a mid-western university. These participants were asked to solve the Decimal Knowledge Test (DKT) in multiple ways. A qualitative approach was used to analyze PTs' responses to the problems on the DKT. The results indicated that using standard algorithms, PTs demonstrated higher ability in solving decimal addition, subtraction, and multiplication involving a whole number. Alternative strategies posed difficulties in solving decimal multiplication problems involving two decimals, and division problems. Similarly, when examining operations, showing smooth or difficult translations between the representations, PTs found it difficult to translate representational modes in operations with multiplication involving two decimals, two division problems. The same results were seen when examining PTs' conceptual understanding based on the Lesh Translation Model. The results provide teacher education programs with opportunities to create various instructional approaches involving translations among multiple representations in mathematics methods and content courses.

Most mathematical problems can be solved using many different solution strategies. The Common Core State Standards for Mathematics (CCSSM, 2010) emphasizes the importance of mathematical proficiency in students' use of different strategies in ways that they should be able to check. Using different solution strategies, they should continually ask themselves, "Does this make sense?" They should understand the approaches other students use in solving complex problems and identify correspondences between different approaches. The National Council of Teachers of Mathematics (NCTM, 2000) indicates that students should develop their "flexibility in exploring mathematical ideas and trying alternative solution paths" (p. 21). NCTM also emphasizes the importance of developing a deep understanding of rational numbers using a variety of representations, particularly, in sixth through eighth grades. Accordingly, Ma (1999) suggests teachers should know the standard algorithms as well as alternative strategies in order to show flexibility in dealing with the nonstandard strategies not included in the textbook.

A fundamental computation theory of reform pedagogy in mathematics has emphasized that students who compare, reflect on, and discuss multiple strategies have advantages in learning mathematics (Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005). Most importantly, students who are learning to use multiple strategies can: (1) obtain higher thinking order skills by using a coherent representation of the learning materials (Große & Renkl, 2006); (2) help students understand each representation and avoid misinterpretations (Große, 2014); (3) enhance autonomy, self-determination, and competence (Große, 2014; Große & Renkl, 2006); (4) facilitate connections between related ideas and different elements of knowledge (Große, 2014; NCTM, 2000; Silver et al., 2005); (5) build up advanced mathematical thinking and knowledge (Becker & Shimada, 1997; Große, 2014; Leikin, 2007); and (6) increase in students' conceptual understanding when students are supported with some prior knowledge (Becker & Shimada, 1997; Große & Renkl, 2006). In addition, the importance of implementing mathematical strategies in different ways has been strongly emphasized at all grade levels (Herman, 2007; Huntley & Davis, 2008; Ma 1999; Tsamir, Tirosh, Tabsch, & Levenson, 2010). In particular, Tsamir et al. (2010) found that even kindergarten children who were able to employ more than one strategy to reach an answer along with those who had little experience with standard mathematical problems were more likely to be open and have a creative approach to mathematics than the older children who are exposed to many years of instruction solving standard one solution problems. According to Lynch and Star (2014a), the use of multiple strategies by students often emphasized the primary school curriculum. For example, the Everyday Mathematics Curriculum of the School Mathematics Project [UCSMP] at the University of Chicago emphasizes that students should be allowed more opportunities to create their own strategies and discuss and justify them with their peers (Carroll, 2000). Accordingly, research on multiple strategies to solving problems shows an impact on students' problem-solving ability in Pre-Algebra and Algebra (Herman, 2007; Huntley & Davis, 2008; Lim, Kim, Stallings, & Son, 2015). Creating multiple strategies also impacted writing and speech while solving mathematical problems (Lim et al., 2015). Although there have been extensive studies about the implementation of multiple strategies, utility of PTs' teaching with multiple strategies is rarely explored.

### *Research Questions*

The current purpose of this study is to explore PTs' ability to implement multiple strategies using standard algorithms (SA) and alternative strategies (AS) and to investigate their conceptual knowledge of decimal operations using multiple strategies of decimal operations. Two specific research questions are listed below.

1. To what extent do PTs use SA and AS, as multiple strategies in solving decimal operations?
2. How do PTs demonstrate their conceptual knowledge of decimal operations using multiple strategies?
   (a) What decimal operations show smooth or difficult translations between the representations?
   (b) To what extent do PTs represent a deep conceptual knowledge of decimal operations?

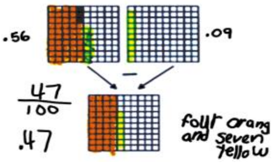
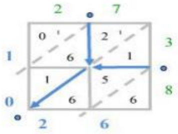
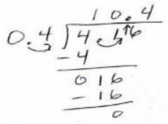
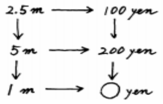
Empirical research done on PTs initial understanding of multiple strategies are very limited. One example is Ma's (1999) study on fractions. She compared the mathematics knowledge of 23 American and 72 Chinese teachers and found that American teachers showed limited knowledge of alternative ways of approaching a specific problem. Thus, mathematics educators can be aware of the weaknesses in the curricula and supplement students' needs to learn mathematics. The importance of this study is twofold. First, this study highlights the need for mathematics educators to identify PTs' initial understanding of multiple strategies of decimal operations, including SA and AS. Second, the results of this study recommend that teacher education (and also professional development) programs should create a variety of instructional approaches or activities for PTs to develop their pedagogical content knowledge in teaching mathematics.

# Theoretical Framework

## *Standard Algorithms and Alternative Strategies in Solving Decimal Operations*

In the past, people focused on SA as a teaching strategy involving teaching numerical steps and rote memorization rather than understanding and explaining the numerical steps. However, currently, the CCSSM specifies that SA can be "developed, discussed, and explained initially using a visual model" (Fuson & Beckmann, 2012, p.16) to develop students' mathematics understanding. In this regard, SA can be used by students to develop a deeper level of conceptual understanding with variations that support the correct use of place value in relation to decimal operations, along with building a strong foundation of procedural fluency. Table 1 presents examples of SA and AS in decimal operations.

Table 1
*Examples of Standard Algorithms and Alternative Strategies*

| Operations | Strategies | Characteristics | Examples |
|---|---|---|---|
| Decimal Addition | Standard Algorithm | Line up the decimal points in a column and add digits in the same place value position | 1<br>1.75<br>+ 0.80<br>2.55<br>(Hatfield, Edwards, Bitter, & Morrow, 2008) |
| Decimal Subtraction | Alternative Strategy | *Pictorial Representation*<br>Draw a 10 × 10 grid and color the decimals |  <br>(Cramer et al., 2015) |
| Decimal Multiplication | Alternative Strategy | *Lattice Method*<br>Draw diagonals, within each square to locate the tens and ones of each partial product coming from the multiplication of two single digits. |  <br>2.7 × 3.8 =10.26<br>(Max et al., 2004) |
| | Alternative Strategy | *Converting into Fractions*<br>Convert decimals to fractions | 5.24 × 1.6 = 8.384<br>$\frac{524}{100} \times \frac{16}{10} = \frac{8384}{1000} = 8.384$<br>(Rathouz, 2011) |
| Decimal division | Standard Algorithm | Move the decimal point. one over to get 41.6 ÷ 4, which is 10.4 | 4.16 ÷ 0.4= 10.4<br> <br>(Hooper, 2015) |
| | Alternative Strategy | *Using proportion*<br>Solve decimal division word problems using proportion | The price of 2.5 meters of ribbon is 100 yen. How much does 1 meter cost? (i.e., 100 ÷ 2.5= 40)<br> <br>(Okazaki & Koyama, 2005) |

Several empirical studies found that when teachers show multiple solutions to a problem, this enhanced the quality of the lessons (Becker & Shimada, 1997; Große, 2014; NCTM, 2000; Silver et al., 2005; Tsamir et al., 2010). In particular, Große (2014) compared the effectiveness of multiple solution methods with uniform solutions using a 2x3-factorial design and found that the group which applied in multiple solutions significantly outperformed the group that implemented uniform solutions.

Also, Becker and Shimada (1997) successfully describe how the use of an open approach has instructional benefits that deepen students' mathematical content knowledge. Open-ended problems refer to problems that are formulated to have multiple solutions. They created lessons using problems with a variety of approaches to the solution. This way the teacher created an environment that was mathematically rich in ways of solving a problem. Accordingly, Ma (1999) emphasized the importance of alternative ways of approaching a specific problem and discussed the ability to judge different approaches. For example, even though it is not always the case, usually it is easier to solve a division problem with decimals by converting it into fractions (e.g., $0.3 \div 0.8 = 3/8$). Son (2016) points out that student-invented strategies involves "a sound understanding of (a) place value, which involves being able to group by tens and to treat the groups as units; (b) properties pertaining to operations such as the associative property of addition, commutative property, and distributive property; (c) the meaning and relationship between addition and subtraction (i.e., subtraction is the inverse operation of addition); and (d) number relationships (i.e., how numbers can be composed and decomposed in different ways)" (p. 107).

### *PTs' Procedural and Conceptual Knowledge in Mathematics Education*

Procedural knowledge can be described as the mastery of computational skills and knowledge of procedures used in identifying mathematical components, algorithms, and definitions. The knowledge of procedures focuses more on applying appropriate computation skills by identifying when and how to use the procedures (Eisenhart et al., 1993; Kilpatrick, Swafford, & Findell, 2001, Rittle-Johnson, Siegler, & Alibali, 2001). Procedural fluency has been used as a tool for analyzing similarities and differences between calculating methods. The methods can be used not only for written paper-and-pencil procedures, mental methods in terms of calculating sums, differences, products, and quotients, but also can be used with calculators, computers, or other manipulatives. Conceptual knowledge is defined as the understanding of the relationships and interconnections of mathematical ideas (CCSSM, 2010; Eisenhart et al., 1993; Hiebert & Lefevre, 1986, Kilpatrick et al., 2001; NCTM, 2014). Hiebert and Lefevre (1986) put more emphasis on knowing "why." For example, students who have conceptual knowledge can understand the logic when solving inverting and multiplying division fraction problems. Students' conceptual knowledge can be effectively developed when representing mathematical ideas in various modes and translating among and within the representations (Cramer, 2003; Lesh & Doerr, 2003).

In school mathematics, researchers agree that procedural and conceptual knowledge are interconnected (Durkin & Rittle-Johnson, 2012; Hiebert & Lefevre, 1986; Kilpatrick et al., 2001, Muir & Livy, 2012; Rittle-Johnson & Siegler, 1998). Kilpatrick et al. (2001) stressed that possessing knowledge of conceptual understanding enables students to learn computation skills in a more effective way, to retain necessary computation skills for a longer period of time, and to reduce susceptible computational errors. In this way, when students are asked to solve higher order thinking problems, even knowledge of procedural fluency can support students' learning process. Students' strong conceptual knowledge could be a stronger predictor for successful problem solving. Considering this, Hieber and Lefevre (1986) pointed out that conceptual knowledge has more of an impact on mathematics learning than procedural knowledge. It is important to note that if students routinely learn the SA with symbols, rules, and procedures in a superficial way, their procedural knowledge can be disengaged from conceptual understanding (Baek, 2006).

### *The Lesh Translation Model in Mathematics Education*

The Lesh Translation Model (LTM) is a theoretical model widely used to explore students' representations of mathematical ideas and their translations among and within the representations (Cramer, 2003; Cramer, Monson, Wyberg, Leavitt, & Whitney, 2009; Moore et al., 2013). LTM identifies five modes of representation: (a) Realistic, (b) Concrete, (c) Pictorial, (d) Language, and (e) Symbolic as shown in Figure 1. The straight arrows between the ovals indicate that students make translations between different representations and the curved arrows on top of each oval indicate that students make transitions within the same representations (i.e., symbolic to symbolic, pictorial to pictorial). For example, if students represent decimal 0.34 using 10 × 10 grid, they are translating from written symbols to the pictorial mode of representation and if students model the same decimal using a 10 × 10 grid and also using a number line, they are translating within the pictorial mode (Cramer et al., 2009). If students represent 0.20 using fraction ⅕, they are translating from written symbols (i.e., symbolic) to the symbolic mode within the same mode of representations. In addition, translating among modes of representations may show students' higher conceptual knowledge. Students can develop their conceptual knowledge through representing mathematical ideas in these five modes of representation and translating among and within the representations (Cramer, 2003; Lesh & Doerr, 2003). From the same perspective, if PTs can translate their mathematical ideas using a variety of modes of representations, then they are more likely to have the conceptual knowledge that will enable the PTs to provide students with effective instruction.

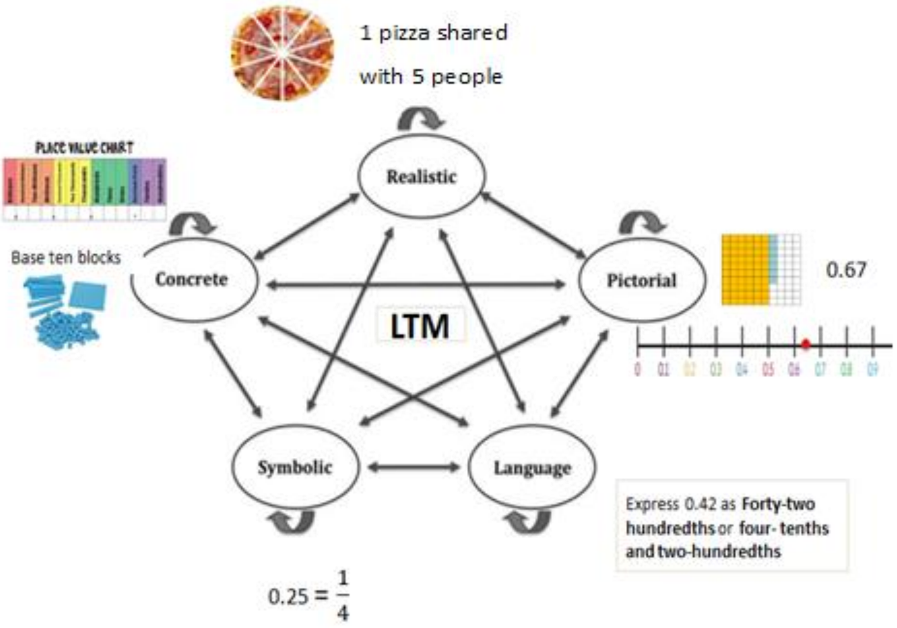


*Figure 1*. The Lesh Translation Model with Concrete Examples

# Methods

## *Sample and Data Sources*

Participants in this study were 37 PTs who attended mathematics course (Course A) in the K-8 Teacher Education Program at a mid-western university. Course A involves the development of rational numbers in the real number system. Of the 37 PTs, ninety two percent were female (34 women, 3 men) and 14 were freshmen, 12 sophomores, 9 juniors, and 2 seniors. In terms of the programs, 20 of them were in Elementary Education (ELED), 8 in Special Education (SPED), and 9 in Early Childhood Education (ECE). Participants in the study were first year PTs and participated in Course A which emphasizes the various representations of mathematical concepts, procedural and conceptual knowledge, as well as teaching primary school mathematics. Course A did not cover decimal operations as part of this course.

Decimal Knowledge Text (DKT) included six decimal operations such as addition (1 item), subtraction (1 item), multiplication (2 items), and division (2 items) (See Appendix A). We divided multiplication and division into two different areas. Studies (e.g., Baroody & Coslick, 1998; Hiebert & Wearne, 1985; Lortie-Forgues & Siegler, 2015) found that students encountered more difficulties in solving the decimal multiplication problem involving two decimal numbers and the division problem involving two decimals, as opposed to the multiplication problem with a whole number and the division problem with a whole number. Test items were orientated to explore PTs' responses to multiple strategies of mathematical concepts used in primary mathematics teaching including a story problem for the given decimal operation. The three professors engaged in an interactive dialogue over the test form questions to make sure they were adequate to observe and measure the multiple strategies of solving decimal operations.

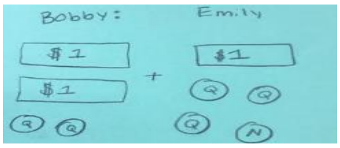
## *Data Collection*

Data were collected through written responses from PTs at the beginning of the first semester. PTs were asked to write their solutions on a sheet of paper developed for this study and to show solutions in more than one way, including story problems. PTs were asked to illustrate multiple ways to solve each decimal operation and their written responses were analyzed.

## *Data Analysis*

The study uses a qualitative content method to analyze PTs' responses to the problems on the DKT in order to examine PTs' use of SA and AS and to explore PTs' conceptual knowledge of decimal operations using the LTM coding. To identify how many SA and AS our PTs had implemented for each operation, PTs' number of SA and AS were counted and then we converted into percentages. Then the three mathematics professors (coders) independently analyzed what modes of representations PTs used based on the LTM coding with percentages. If there were any disagreement, the three coders tried to compromise until the disagreement was finally resolved. Finally, we converted the number of PTs' mode of representations for each decimal operation into percentages. To identify the first research question, we analyzed PTs' multiple strategies for each decimal operation and categorized them with examples. To do this, data sheets for their responses were developed and coded. For example, we categorized the SA when they showed the strategy of carrying out singledigit computations within each unit, but also maintained the place-value of the resulting numbers, and then composed an answer from these individual computations, all other strategies were categorized as the AS. To answer the second question, we used a written test that employed deductive coding system on the basis of the five modes of LTM representations. Two raters analyzed the number of PTs written responses to identify their conceptual understanding by identifying use of AS. PTs responses may fit into two or more categories of modes of LTM representation. We limited the LTM representations to the four that could be done by written responses, and then coded PTs' written responses accordingly as presented in Table 2. For instance, the *Concrete* mode was removed because no PTs used concrete examples (i.e., manipulatives) in their written responses. In order to code PTs' conceptual understanding using multiple strategies, data sheets for their responses were developed to illustrate the coding system (i.e., the transitions within and between the modes). The data sheets were organized by the four modes of the LTM coding, and PTs responses were categorized into the codes accordingly. PTs, representing their strategies in multiple ways and then translating them within and among other representations were considered to have conceptual understanding. Cohen's K coefficient of interrater reliability was 0.95. The raters came to 100% agreement on the discrepancies after further discussion.

Table 2
*Examples of PTs' Coding of Standard Algorithms and Alternative Strategies*

| Coding | Definition with the PTs' Example |
|---|---|
| Realistic | Written responses showing events and objects happening in the real world. For his study, story problems that made connections to real life (i.e., using money in a grocery store) are included in this category.<br>E.g., A student has $3.50 and he lost $1.80 from running around in the park. How much does he leave? |
| Pictorial | Pictorial responses such a hand-drawn picture, tallies, and diagram.<br>E.g.,  ($2.50 +$1.80=$3.50) |
| Language | Written responses that represent use of correct language to explain the mathematical concepts such as money, place value in appropriate words.<br>E.g., When solving *0.25÷5=*,<br>1 quarter=5 nickels; quarter=0.25 of a dollar, nickel=0.05 of a dollar = 5/100. Since 5 is over 100, we must put it in the hundredths value place |
| Symbolic | For this article, any strategies that used to represent *abstract.*<br>using written symbols, numbers, formulas, or any other numerical concepts<br>E.g., $0.05 \times 0.8 = \frac{5}{100} \times \frac{8}{10} = \frac{40}{1000} = \frac{4}{100} = 0.04$ |

An example of both correct and incorrect story problems is listed as below. In the first example, the PT clearly described that a whole pizza is cut into 10 pieces making a connection to the real world. On the other hand, the second example is incorrect because we do not use the expression (i.e., 2.5 pieces of pizza) in our real-life situations. The responses that used ambiguous words or expressions (e.g., 2.5 pieces of pie, 3.5 blocks, etc.) were considered incorrect.

| Correct story problem | Incorrect story problem |
|---|---|
| I have 2 pizzas cut into 10 slices and then 5 slices of another; I want to add another whole pizza and 8 more slices. How much do I have now? | Carson has 2.5 pieces of pizza. Sarah has 1.8 pieces of Pizza. How much pizza do they have if they combine their pieces? |

# Results

## *PTs' Use of Standard Algorithm and Alternative Strategies*

The first research question asked how PTs used SA and AS in regard to decimal operations. As for the SA, we only counted the number of correct, incorrect, and no responses for each operation to identify PTs' current knowledge of SA on the decimal knowledge test. As shown in Figure 2, about 97% of PTs demonstrated a mastery of SA on addition and subtraction operations. The percentages of correct responses to the decimal multiplication problem involving a whole number and involving two decimals were 70.3% and 45.9% respectively, while the decimal division problems involving a whole number and involving two decimals were 45.9% and 56.8%. The two operations that showed the most incorrect responses were the multiplication problem involving two decimals (e.g., 0.05 × 0.8, 32.4%) and division involving a whole number (e.g., 0.25 ÷ 5, 37.8%), respectively. Multiplication and division involving two decimals resulted in the highest percentages of no response (21.6% for each). As for the AS, first, we analyzed the percentages of how many AS our PTs has implemented for each operation.

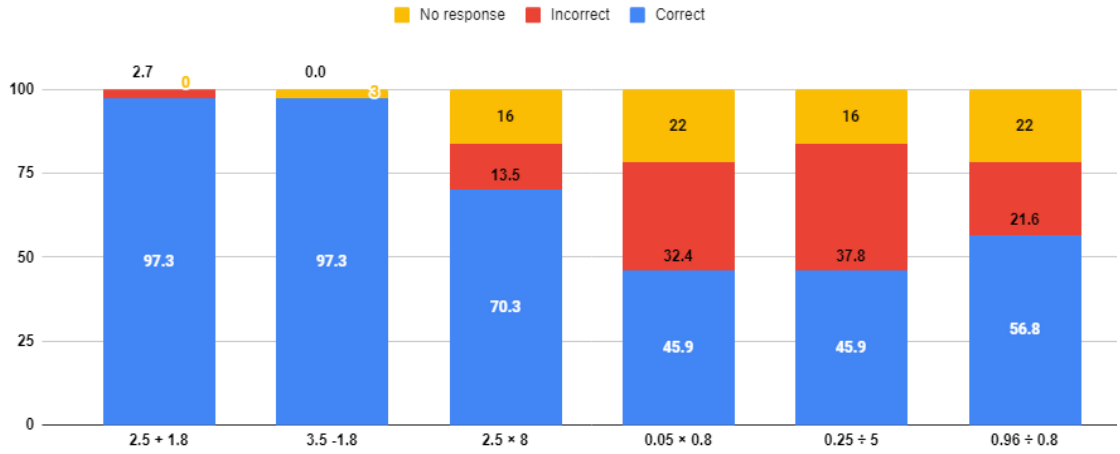


*Figure 2*. Percentages18 of PTs' implementation of SA

As shown in Table 3, the major difference can be seen between the operations of addition, subtraction, and multiplication involving a whole number and the two multiplication and division problems involving two decimals. Obviously, three-fourth of PTs successfully solved the former problems (i.e., 2.5 + 1.8, 3.5 – 1.8, 2.5 × 8) in more than one way; however, more than three-fourth of PTs failed to solved the latter problems (i.e., 0.05 × 0.8, 0.25 ÷ 5, 0.96 ÷ 0.8) in more than one way. No PTs used more than 4 ways. This clearly indicated that our PTs rarely implemented AS when solving decimal multiplication involving two decimals and decimal divisions. As presented in Figure 2 and Table 3, PTs still had difficulties in solving decimal multiplication involving two decimals and both division operations. Obviously, when using SA, slightly more or less than half of PTs have trouble solving those operations; however, for the AS, about four-fifth of PTs failed to provide AS.

Table 3
*Percentages of PTs' Alternative Strategies Use for Each Decimal Operation*

| | 2.5 + 1.8 | 3.5 – 1.8 | 2.5 × 8 | 0.05 × 0.8 | 0.25 ÷ 5 | 0.96 ÷ 0.8 |
|---|---|---|---|---|---|---|
| 0 | 18.92 | 24.32 | 36.84 | *78.38* | *81.08* | *83.78* |
| 1 | 37.84 | 35.14 | 21.05 | 13.51 | 10.81 | 10.81 |
| 2 | 21.62 | 27.03 | 18.42 | 2.70 | 8.11 | 2.70 |
| 3 | 16.22 | 8.11 | 7.89 | 5.41 | 0.00 | 2.70 |
| 4+ | 2.70 | 5.41 | 13.16 | *0.00* | *0.00* | *0.00* |

To identify what strategies were used for our PTs, then, we have categorized the percentages of their AS with examples for each operation (See Appendix B). The most common three AS for addition and subtraction are as follows: *Story problem* (45.95% & 35.14%, respectively), *Composing and decomposing* (32.43% & 29.73), and *Pictorial representation* (27.03% & 18.92). The following strategies, *Story problem* (35.14%), *Lattice method* (25.62%), *Pictorial representation* and *Repeated addition* (18.92% for each), were frequently used in solving multiplication involving a whole number. The strategy, *Whole number thinking* (8.11%), was the most commonly used method in solving multiplication involving two decimals. In solving two division operations, *Converting into fraction* (10.81% for each) was the most often used strategy.

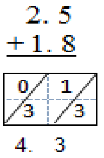
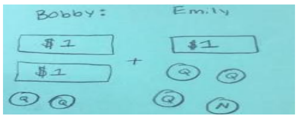
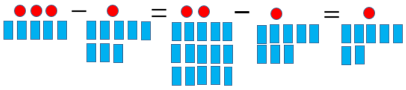
### *PTs' Conceptual Understanding of Decimal Operations using Multiple Strategies*

The second research question asked how PTs demonstrated their conceptual knowledge of decimal operations using multiple strategies. One of the PTs' written work examples is presented in Table 4 to demonstrate her conceptual knowledge by translating within and among the modes of representations. For example, as shown in Table 4, the PT showed 5 different ways of translation in addition, 4 strategies in subtraction, 3 strategies on decimal multiplication involving a whole number, and 2 strategies on decimal division involving a whole number. Specifically, when asking solving addition and subtraction decimals, PT 1 translated from *symbolic* (e.g., 2.5 + 1.8) to *symbolic* (e.g., lattice methods, Decomposition), *realistic* (e.g., Angela goes to a store and she wants to buy two pencils. The price of the two pencils are $2.5 and $1.8. and can you help Angela to get to know how much money she should pay?), and *pictorial* (e.g., number line and money). In contrast, when asking solving multiplication or division decimals, PT 1 demonstrated a very limited knowledge translating within the mode of representation. (e.g., only *symbolic* (e.g., 0.25 ÷ 5)) to *symbolic* (e.g., converting to fraction and decomposing the numerator based on the place value).

*Operations showed smooth or difficult transitions between and among representations.* As indicated in Table 4, If PTs are able to find their solutions based on the LTM modes of representations from *symbolic* with correct procedure to other modes using at least one of can't find the solution with symbolic mode of representation or failed to translate correct symbolic representations to other LTM modes of representations, we consider that they had a difficulty in translating between and among representations.

Table 4

*Examples of the PT's Translations Within and Among Representations*

| # of Strategies | Translations | | PTs' Examples |
|---|---|---|---|
| 5 | Symbolic<br>2.5 + 1.8 | Symbolic | 1.  <br>2. (2.5 + 0.5 + 1.3) =3+ 1.3 = 4.3 |
| | | Realistic | Angela goes to a store and she wants to buy two pencils. The price of the two pencils are $2.5 and $1.8. and Can you help Angela to get to know how much money she should pay? |
| | | Pictorial |   |
| 4 | Symbolic<br>3.5 - 1.8 | Symbolic | 1. $\begin{array}{r} 3.5 \\ -\ 1.8 \\ \hline 2.\overset{\wedge}{3} \end{array} = 2 - 0.3 = 1.7$<br>2. $3\frac{5}{10} - 1\frac{8}{10} = 2\frac{15}{10} - 1\frac{8}{10} = 1\frac{7}{10} = 1.7$ |
| | | Realistic | Billy's mom gave him $3.50 to spend on new toys. He found a race car for $1.80. if he bought the race car how much money would Billy have left? |
| | | Pictorial |  |
| 3 | Symbolic<br>0.05 × 0.8 | Symbolic | 1. 80% of 0.05 = $\frac{80}{x} : \frac{100}{0.05}$<br>$100\,x = 80\ (0.05)$<br>$100\,x = 4;\ \ x = 0.04$<br>2. 5 × 8 = 40, move decimal three ps to the left<br>3. $\begin{array}{c\|c\|c} & 0.025 & 0.025 \\ \hline 0.4 & 0.01 & 0.01 \\ \hline 0.4 & 0.01 & 0.01 \end{array}$<br>0.01+0.01+0.01+0.01= 0.04 |
| 2 | Symbolic<br>0.25 ÷ 5 | Symbolic | 1. $\frac{25}{500} = \frac{1}{20} = 0.05$<br>2. $\frac{0.20+0.05}{5} = \frac{0.20}{5} + \frac{0.05}{5} = 0.04 + 0.01 = 0.05$ |

As shown in Figure 3, PTs' percentages of transitions using LTM coding were identified as follows: addition (78.38%), subtraction (75.68%), multiplication involving a whole number (62.16%), multiplication involving two decimals (24.32%), decimal involving a whole number (18.92), and division involving two decimals (16.22%). On the one hand, PTs showed smooth transitions within and among representations in addition, subtraction and multiplication involving a whole number. On the other hand, PTs made a difficult transition in operations with multiplication involving two decimals, division involving a whole number, and division involving two decimals.

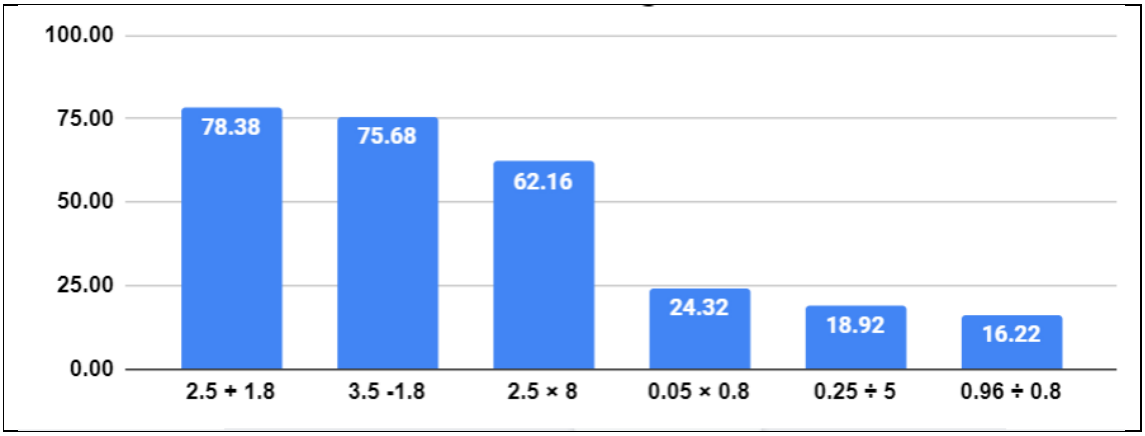


*Figure 3*. The percentages of PTs' translations from symbolic to other modes of representations

*PTs' conceptual knowledge of decimal operations*. PTs' strategies may fit into two or more modes of the LTM. If PTs are able to translate within and among the modes of LTM representations, we consider that they have built their conceptual knowledge by deepening their understanding along with procedural knowledge. To the extent that a PT demonstrated use of more than two different strategies within a representation (curved arrows), this was considered strong conceptual knowledge. The ability to do both – multiple strategies within a representation (curved arrows) and to translate between or among representations (straight arrows) – this combination is considered even stronger conceptual knowledge than the latter. Figure 4 summarizes PTs' conceptual understanding using different modes of LTM representations that were evident for each operation. Three (i.e., pictorial, realistic, and symbolic) of the five modes of the LTM were evident from PTs' AS. The most commonly used representation was symbolic mode.

Approximately half of the PTs solved decimal problems using symbolic mode: addition (54.05%), subtraction (48.65%) and multiplication involving a whole number (51.35%), while one-fifth of PTs (21.62%) showed symbolic mode in solving decimal multiplication involving two decimals. Two decimal divisions showed the least percentages (16.22%, respectively). Next, the percentages showing realistic mode as follows: addition (45.95%), subtraction (35.14%), multiplication involving a whole number (35.14%), and division involving a whole number (2.70%). Pictorial modes were found as follows: addition (29.73%), subtraction (18.92%), multiplication involving a whole number (10.81%), and division involving a whole number (2.70%). Realistic and pictorial modes were not identified in multiplication and division involving two decimals. Only one PT used language mode in solving decimal division with a whole number (2.70%).

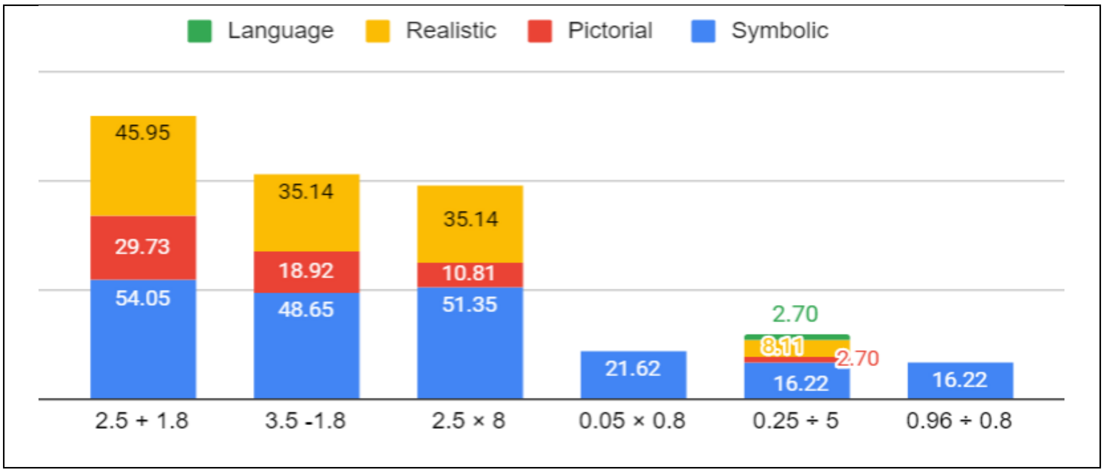


*Figure 4.* Percentages of PTs' conceptual understanding using different modes of LTM

## Conclusion and Discussion

This study investigated PTs' SA and AS to identify their current knowledge of decimal operations and operations that they faced difficulties with and also investigated their conceptual understanding of decimal operations. The results indicated that using SA, PTs demonstrated higher ability in solving decimal addition, subtraction, and multiplication involving a whole number. The AS posed difficulties for PTs in solving decimal multiplication problems involving two decimals, and division problems. In determining PTs' conceptual understanding of decimal operations, they showed smooth transitions in operations with decimal addition, subtraction, multiplication involving a whole number. However, PTs encountered more difficulties in translating representations in operations with multiplication involving two decimals, division involving a whole number, and division involving two decimals. This means that the PTs for this study built a foundation of conceptual knowledge in that they showed smooth transitions in operations with decimal addition, subtraction, multiplication involving a whole number. However, they may have a very limited conceptual understanding in the operations of multiplication involving two decimals and the two types of division problems. This finding can be also supported by the result of our PTs' use of SA and AS.

Several important findings are discussed as follows. First, we can infer that SA is still the dominant strategy used when teaching different ways of solving decimal operations in school mathematics (Fuson & Beckmann, 2012).

Second, SA performance in decimal multiplication and division was influenced by differing numbers of decimal places in the operands not only for primary students but also for our PTs. For example, for the decimal multiplication and division problems between 2.5 × 8 and 0.05 × 0.8 (70.3% and 45.9 % correct) or between 0.25 ÷ 5 and 0.96 ÷ 0.8 (45.9% and 59.8%), more PTs in our study solved multiplication problem involving a whole number than involving two decimals. Interestingly, our result of decimal performance on division is consistent with one of the findings of Hiebert and Wearne's research (1985). In their research, students' correct responses using decimal division involving a whole number (e.g., 0.56 ÷ 7) and division involving two decimals (e.g., 0.24 ÷ 0.03) showed different results. The percentages of correct responses to the decimal division problem involving a whole number for students in grades 7 and 9 was 22% and 57% respectively, while the division problem involving two decimals was 20% and 66%. A possible explanation for this is that the division problem (e.g., 0.24 ÷ 0.03) 9th grade students solved was easily transformed into a whole number such as 24 ÷ 3 by multiplying 100 in both dividend and divisor. This may affect our PTs' problem solving of division involving two decimals (i.e., 0.96 ÷ 0.8).

Third, in analyzing PTs' AS, *Story problem*, *Composing and decomposing*, and *Pictorial representations* were the three major strategies frequently used in solving decimal addition and subtraction (See Appendix B). Ma (1999) emphasized the importance of learning the strategy, *composing* for addition and *decomposing* for subtraction, which helps students understand the concept of place value and that it helps students to make connections to a unit of higher value (i.e., the concept of multiplication). However, for this study, about four-fifth of PTs provided AS in decimal addition and subtraction problems.

Fourth, in solving the problem, decimal multiplication involving a whole number (i.e., 2.5 × 8), *Story problem*, *Lattice method*, *Pictorial representation* and *Repeated addition* were frequently used strategies. For multiplication involving two decimals, *whole number thinking* is the commonly used strategy. In particular, the strategy, *Repeated addition*, was found in operating the multiplication problem involving a whole number, but not found in operating the multiplication problem with two decimals. The PTs for this study can easily make connections to their concept of multiplication as repeated addition.

Fifth, the strategy, *Converting into fractions*, was particularly found when solving decimal division problems rather than any other problems. Thus, this is consistent with the results from Ma's (1999) study that it is easier to solve a division problem involving two decimals by converting it into fractions (e.g., 0.3 ÷ 0.8 = 3/8).

Sixth, this study confirmed that PTs posed large difficulties in operating decimal division problems. The result for this study reflects the findings of other studies that have reported that solving decimal division is difficult when compared to decimal operations in addition, subtraction, and multiplication (Ball, 1990; Hiebert & Wearne, 1985; Muir & Livy, 2012).

Seventh, with regards to what extent PTs represent a deep conceptual knowledge of decimal operations, three (i.e., pictorial, realistic, & symbolic) of the five modes of the LTM were evident from PTs' AS. Obviously, the most commonly used representation was symbolic mode. Realistic and Pictorial modes were not identified in multiplication and division involving two decimals. Only one PT used language mode in solving decimal division with a whole number.

### *Implications for Further Research*

The findings of this research have several major implications for mathematics education. First, the PTs showed the higher correct percentage of division problem involving two decimals (i.e., 0.96 ÷ 0.8) than division involving a whole number (i.e., 0.25 ÷ 5) using SA. This is an interesting finding. Further empirical research on the difference between decimal division with a whole number and those involving two decimals is needed to generalize the results of this study.

Second, PTs used the strategy, *Converting into fractions*, less in solving decimal addition, subtraction, and multiplication problems, while more PTs employed the same strategy when solving division operation problems. These relationships may partly be explained because PTs have more knowledge, experience, and strategies to work with in computing decimal addition, subtraction, and multiplication involving a whole number, but less experience in solving other decimal operations. It would be helpful for mathematics educators to demonstrate the relationship between fractions and decimals that represents the same concept (e.g., 0.3 is equivalent to 3/10) not only in decimal division but also other operations to enhance students' deep understanding of decimals.

Third, this study revealed that only one-fifth of our PTs demonstrated their conceptual understanding of multiplication involving two decimals in the two division problems. The majority of PTs have still encountered difficulties among those three operations. Thus, mathematical educators should provide PTs with more opportunities to translate between and within the five modes of LTM representations when teaching decimal multiplications and decimal divisions. Thus, both preservice and in-service teachers should demonstrate their knowledge of and encourage their students to use different solution strategies because it is the teacher who actually influences students' ability to find multiple strategies in learning mathematics.

Fourth, currently, in many mathematics classrooms, teachers are already using reform or standard-based mathematics curricula that place emphasis on students' ability to apply mathematics in real-life situations. This curricular focus encourages teachers to use collaborative and cooperative learning and student-invented algorithms. Studies (CCSSM, 2010; Clarke, 2005) emphasized the potential dangers of introducing SA too early. However, the adoption of teaching AS first and then SA seems to be not applied in the classrooms as teachers themselves will likely not have been taught AS. More research should be conducted on this matter.

Finally, as mentioned in the literature review (Bingolbali, 2011; Leikin, 2007; Ma 1999), it is not easy for teachers to engage students in solving problems in different ways. Mathematics educators including teacher education should provide an opportunity for pre- and in-service teachers to develop a deep understanding of decimal operations through mathematics content, methods courses and professional development programs. If they don't fully understand their students' difficulties, they can't take appropriate action to develop their student's decimal understanding and skills. These programs can be specifically designed to help teachers implement strategies for solving problems in multiple ways, and to identify effective ways to evaluate different approaches students can apply in solving problems (NCTM 2000, Rittle-Johnson & Star, 2007).

The current research has several limitations. First, this study focused only on the topic of decimal operations. Further research is needed to investigate other mathematics content such as other rational numbers like fractions, percentages, integers, etc. Second, the decimal knowledge test for multiple strategies consists of six items. Although we used these six specific items for this study, other decimal numbers may produce different results. For our study, we used decimal multiplication and division items involving two decimals less than 1 (e.g., 0.05 × 0.8 or 0.96 ÷ 0.8). The results of using decimal multiplication and division problems more than 1 may be different. Thus, it cannot be generalized to other ethnic groups. Fourth, it is difficult to generalize the result of the study because of the small sample size. Only 37 PTs were used for this study. A large sample size would be more representative of populations. Lastly, to identify our PTs' understanding of decimals, a majority of PTs demonstrated 3 major modes of representations (i.e., Realistic, Symbolic, and Pictorial) from the five LTM coding on our DKT. The data from PTs' interviews will help in identifying the connections between procedural knowledge and conceptual understanding.

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# Appendix A

## *Decimal Knowledge Test*

1. What are the different ways to get the solution? Show as many different strategies as you can to find the answer for 2.5 + 1.8. You may use real-world situation or story problems to explain the meaning of this decimal addition.

2. What are the different ways to get the solution?  Show as many different strategies as you can to find the answer for 3.5 – 1.8. You may use real-world situation or story problems to explain the meaning of this decimal subtraction.

3. What are the different ways to get the solution? Show as many different strategies as you can to find the answer for 2.5 × 8. You may use real-world situation or story problems to explain the meaning of this decimal multiplication involving a whole number.

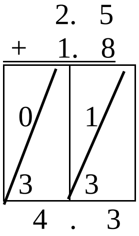
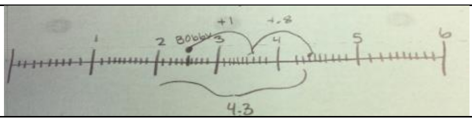
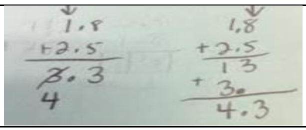
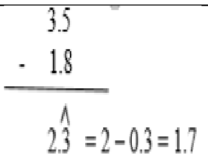
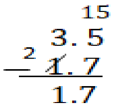
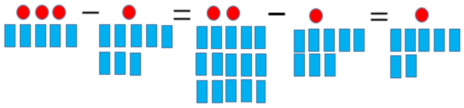
4. What are the different ways to get the solution? Show as many different strategies as you can to find the answer for 0.05 × 0.8. You may use real-world situation or story problems to explain the meaning of this decimal multiplication involving two decimals.

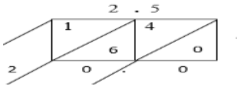
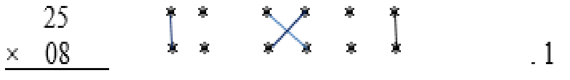
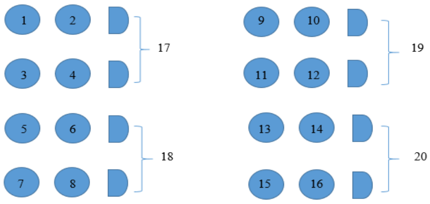
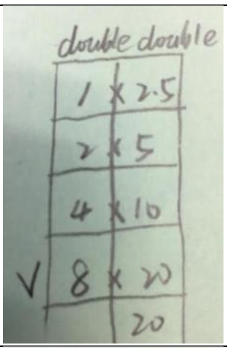
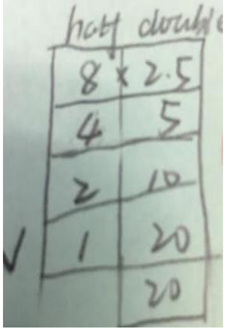
5. What are the different ways to get the solution? Show as many different strategies as you can to find the answer for 0.25 ÷ 5. You may use real-world situation or story problems to explain the meaning of this decimal division involving a whole number.

6. What are the different ways to get the solution? Show as many different strategies as you can to find the answer for 0.96 ÷ 0.8. You may use real-world situation or story problems to explain the meaning of this decimal division involving two decimals.

# Appendix B

*Categories of Alternative Strategies with Examples*

| Alternative Strategies | | | |
|---|---|---|---|
| | Category | Example | Percentages |
| 2.5 + 1.8 | *Composing and decomposing* | 1.5 + 1.5 + 1+ 0.3 = 4.3 | *32.43* |
| | Lattice method |  <br>Add tenths digits to get 1.3 and add ones digits to get 3.0. Then add along the diagonals. Place decimal pt. by lining up. | 10.81 |
| | *Pictorial representation* |  | *27.03* |
| | Whole number thinking | 25 + 18 = 43, then place the decimal pt. to make 4.3 | 10.81 |
| | Combine left and right/ right to left method |  | 5.41 |
| | *Story problem* | Emily went to the beach and drove 2.5 miles. She stopped at a gas station for gas then drove another 1.8 miles to the beach. How many miles was her total drive? | *45.95* |
| 3.5 -1.8 | *Composing and decomposing* | 2.0 + 1.5 – 1.0 + 0.8 = 1.0 + 0.7 = 1.7 | *29.73* |
| | Cap method |  | 13.51 |
| | Newzeland method |  | 13.51 |
| | *Story problem* | You have $3. 50. You want to buy a candy bar that costs $1. 80. How much money will you have left assuming all taxes are included? | *35.14* |
| | Converting into fraction | $3\frac{5}{10} - 1\frac{8}{10} = 2\frac{15}{10} - 1\frac{8}{10} = 1\frac{7}{10} = 1.7$ | 5.41 |
| | Whole number thinking | 35 -18 =17, then place the decimal pt. to make 1.7 | 8.11 |
| | *Pictorial representation* |  | **18.92** |
| 2.5 × 8 | Composing and decomposing | (2.5 × 2) × 4 = 5 × 4 = 20 | 13.51 |

| | | | |
|---|---|---|---|
| | *Lettice method* |  | *21.62* |
| | Mental computation (Vedic method) |  | 2.70 |
| | German method | 2 \| .5<br>8 \| 16 \| 4<br>16+4=20 | 5.41 |
| | Combine left to right/ right to left method | 2.5<br>× 8<br>1 6<br>+ 4<br>2 0 | 10.81 |
| | *Pictorial representation* |  | *18.92* |
| | *Story problem* | There is a room that is 8 squares long and 2.5 squares wide. How many total squares are in the room? | *35.14* |
| | *Repeated addition* | $\overbrace{2.5+2.5+2.5+\dots+2.5}^{\times 8}=20$ | *18.92* |
| | Egyptian multiplication |  Take one number (i.e., 2.5) and multiply it by 2. This is done repeatedly until you get the other number (i.e., 8). Take the corresponding numbers(s) (i.e., only 20) and add them together, which is 20. | 2.70 |
| | Russian multiplication |  Place each number at the top of one of two side-by-side columns. One column's entries are doubled, and the others are divided by two. Check the odd multiplier until you have 1 or 0 and discard others. Add all of matched multiplicand (i.e., 20). | 2.70 |

| | | | |
|---|---|---|---|
| | Brad's method | $\begin{array}{r} 25 \\ \times\ 80 \\ \hline 160 \\ 00 \\ 40 \\ 0 \\ \hline 200 \end{array}$<br>Then move one decimal pt. to the left to get 20. | 2.70 |
| 0.05 × 0.8 | *Whole number thinking* | 5 × 8 = 40, then move 3 decimal pts to the left | *8.11* |
| | German method | $\begin{array}{c\|c\|c} & 0.025 & 0.025 \\ \hline 0.4 & 0.01 & 0.01 \\ \hline 0.4 & 0.01 & 0.01 \end{array}$<br>0.01+0.01+0.01+0.01= 0.04 | 5.41 |
| | Converting into fraction | $\frac{5}{100} \times \frac{8}{10} = \frac{40}{1000} = \frac{4}{100} = 0.04$ | 5.41 |
| | Using percents | 80% of 0.05 = $\frac{80}{x} : \frac{100}{0.05}$<br>100 $x$ = 80(0.05); 100 $x$ = 4; $x$ = 0.04 | 2.70 |
| 0.25 ÷ 5 | *Converting into fraction* | $\frac{25}{500} = \frac{1}{20} = 0.05$ | *10.81* |
| | Story problem | I have a quarter which is worth 0.25 of a dollar, what is a quarter broken up into 5 equal parts? | 8.11 |
| | Composing and decomposing | $\frac{0.20+0.05}{5} = \frac{0.20}{5} + \frac{0.05}{5} = 0.04 + 0.01 = 0.05$ | 5.41 |
| | Pictorial representations | <br>0.25 ÷ 5 = 0.05 | 2.7 |
| 0.96 ÷ 0.8 | *Converting into fraction* | $\frac{96}{100} \div \frac{8}{10} = \frac{96}{100} \times \frac{10}{8} = \frac{960}{800} = \frac{6}{5} = 1\frac{1}{5}$ | *10.81* |
| | Reducing fraction | $\frac{96}{\cancel{100}_{10}} \times \frac{\cancel{10}^{1}}{8} = \frac{96}{80} = 1\frac{16}{80} = 1\frac{2}{10} = 1.2$ | 2.7 |
| | Scaffold method | $\begin{array}{r} 0.2 \\ 1 \\ 0.8\,\overline{)0.96} \\ \underline{0.8} \\ 0.16 \\ \underline{0.16} \\ 0 \end{array}\Big]$<br>Line up the digits in the quotient with the digits in the dividend. Then add the two partial quotients to obtain the final quotient (i.e., 0.2 +1 = 1.2) | 2.7 |