Conceptions of Zero Among Students With and Without Learning Disabilities[1]

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Zero has been philosophically and symbolically interpreted differently from other numbers throughout history. While it is foundational to all mathematics, in early grades, it is denounced as representing "nothing." In this study, the understanding of zero among students with learning disabilities (LD) and no learning disabilities (NLD) were investigated, and conceptual similarities and differences were examined. This research is a multiple-case study based on qualitative design. The study participants were six students, three with LD and three with NLD, aged between 10 and 12 years. Data were collected through clinical interviews that focused on zero as a number, the meaning of zero as "nothing," zero as the cardinality of an empty set, and zero's use as a symbol (place value, placeholder, digit). According to the findings obtained through the content analysis method, no patterns demonstrated clear distinctions between students with LD and NLD in interpreting zero as a number or the meanings of cardinality. In other words, LD and NLD students had differences and similarities in their zero understandings but were not sharp enough patterns to identify the groups. The students' explanations of their conceptualizations of zero provided some notable results. Significant misconceptions within all students' understandings of zero suggest the need for primary school mathematics teachers to pay more attention to the meaningful understanding of zero.

Introduction

Historically, zero has both inspired and confused people. Zero is ubiquitous beginning in primary education and should be understood as being a number, filling the gaps in number systems, and serving different purposes in operations. Zero is an essential reference for counting and a very abstract concept (Haylock & Cockburn, 2008). It has paved the way for significant discoveries in mathematics in a historical process emerging from the need to assign a value to "nothing." Resistance to the idea of zero is described as a developmental limitation that took eons for societies to confront, accept, and represent (Sarama & Clements, 2009). It can be difficult for students to express a non-existing object with a number or, in other words, to consider zero as a number and a symbol (National Research Council (NRC), 2011; Russell & Chernoff, 2011). Students with learning disabilities (LD) may not form a link between number magnitude and the symbolic system used to express it (Rousselle & Noel, 2007). In this context, the conceptions of students with LD regarding zero as a symbol expressing number magnitude and as a number indicating "nothing" are issues of concern. The related literature confirms that students' knowledge of the concept of zero is weak (Grouws & Reys, 1975; Reys, 1974; Reys & Grouws, 1975; Tsamir, Sheffer, & Tirosh, 2000; Seidelmann, 2004; Catterall, 2006; Erdoğan, 2019).

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Learning about and discussing the concept of zero makes it possible to observe students' reasoning processes (Wellman & Miller, 1986). Accordingly, examining the concept of zero among students with learning disabilities (LD) compared to students with no learning disabilities (NLD) can provide deeper insights into the mathematical reasoning of these respective students. While there are several previous studies examining knowledge and usage of numbers among students with LD (e.g., Geary, Bailey, & Hoard, 2009; Lambert & Tan, 2017), no study was found examining the knowledge and understanding of zero among students with LD in terms of the specific number as a concept with comparisons of understandings of zero between groups of students with LD and NLD. Addressing this gap in the literature, the present study aims to examine the understanding of zero among students with LD and NLD, as zero is a concept philosophically and symbolically interpreted differently from other numbers throughout history. The similarities and differences in the understanding of zero among students with LD and NLD are discussed herein.

Significance of the Study

The current study investigates the similarities and differences between the understanding of students with LD and NLD in terms of the meaning of zero as a number and its use as a symbol. The place of zero is essential in many concepts and key mathematical ideas, such as an equation to equal zero, the zero function, core functions, the degree of a zero polynomial, a series converging to zero, or a zero vector. Understanding zero and its use as a numerical counterpart support understanding many concepts and algorithms essential in primary education.

Counting is a universal skill generally considered easily acquired at an early age. However, sophistication is required to match a number with a set, count physical objects, investigate the one-to-one correspondence of cardinality, or answer the question of "how many?" These require complex ways of thinking and abstractly seeing relationships between the whole and pieces (e.g., reaching a point by partitioning a line or a part of a line and reaching zero with a dimensionless point) (Smith, 2012) and offer a basis for the study of the concept of zero.

The concept of numbers depends on the development of counting skills, one of the starting points in learning and comprehending arithmetic. A number reflects quantity, or what it means to be "one" or "two." Counting is further divided into rote counting and rational counting. Rote counting, based on recall, involves counting from 1 to 10, as done by children at the ages of 2 or 3. Rational counting involves labeling objects in a group with numerical names (Smith, 2012).

Although zero is an important concept in counting, zero remains abstract in the context of counting objects. For example, although a young child must touch objects to learn to count, zero is rarely encountered (Sarama & Clements, 2009). Expressing a non-existing object with a number, or considering zero as a number, can be difficult for students (NRC, 2001; Gullberg, 1997).

Wellman and Miller (1986) opine that thinking about zero produces deep reasoning, especially among primary students. Thus, it is essential to investigate primary students' thinking and understanding regarding the concept of zero. The present study may offer helpful information to other researchers on students' (with LD and NLD) understanding of zero.

The concept of zero has been investigated in the contexts of transitions from natural numbers to integers (Gallardo & Hernández, 2006), division operations (Crespo & Nicol, 2006; Grouws & Reys, 1975; Reys & Grouws, 1975; Tsamir & Sheffer, 2000), odd or even numbers (Levenson, Tsamir, & Tirosh, 2007), empty sets (Merritt & Brannon, 2013), and transitions from arithmetic to algebra (Gallardo & Hernández, 2005). Most studies that report zero as a difficult concept for students to grasp are relatively old (e.g., Gullberg, 1997; Inhelder & Piaget, 1969; Reys & Grouws, 1975; Wheeler & Feghali, 1983). More recent studies examining zero in depth with all its meanings and uses are quite limited (e.g., Catteral, 2006). Seeing students' different interpretations of zero can guide educators in their instructional design regarding zero. Additional modern studies regarding students' conceptions of zero through multiple meanings and uses remain valuable.

The present study investigates the differences and similarities in the understanding of students with LD and NLD regarding the concept of zero. It provides insights into learning disabilities and the characteristics of LD students by examining students' thinking and reasoning about this special concept. Lambert and Tan (2017) previously noted that exploring the mathematical world from the perspectives of different and diverse learners provides an understanding of the relationship between knowing and concretizing mathematics.

Theoretical Framework

Learning disabilities. In the literature, specific learning disabilities are generally defined as behavioral situations with a biological basis, with no formal definition currently being agreed upon (Mazzocco, 2007). Individuals with LD form a heterogeneous group and may experience difficulties in reading, mathematics, or verbal language (Pierangelo & Giuliani, 2006; Namkung & Peng, 2018). Although the cause of LD is not known, the existence of possible causes rather than a single reason is emphasized (Namkung & Peng, 2018). Andersson and Östergren (2012) state that there is not a single central cause for LD and that multiple cognitive deficits can lead to difficulties in learning. The performance of a student with LD in specific academic skills is 1 to 2.5 years below the average, or -1 to -2.5 standard deviation intervals (American Psychiatric Association (APA), 2013). Similarly to the description from the Ministry of National Education of Turkey (Ministry of National Education (MoNE), 2015), the U.S. Individuals with Disabilities Education Act (IDEA) describes LD as follows:

> "Specific learning disability" means a disorder in one or more of the basic psychological processes involved in understanding or in using language, spoken or written, that may manifest itself in an imperfect ability to listen, think, speak, read, write, spell, or to do mathematical calculations… (IDEA, 2004)"

Students with LD may have difficulty perceiving numbers or using number facts or calculations, mathematical reasoning (APA, 2013), and number-related processes and procedures (Andersson & Östergren, 2012). It is also stated that these students may have difficulty reading written numbers, sorting numbers or pieces of information, and understanding cardinality (Munro, 2003). Students with LD cannot grasp symbols that are number representations and cannot match number symbols and number quantity representations. For this reason, these students have difficulties in all activities requiring the manipulation of these symbols, such as addition, subtraction, comparisons of the quantities of two numbers written with symbols, or showing the place of a number in a number line (Mejias et al., 2012). However, students with NLD and aged 8-11 could see zero as a number because the symbol 0 was seen on a number line with other known number symbols (Catteral, 2006).

Zero as a number. Historically, zero was defined as a number centuries after it had begun being used as a placeholder (Seife, 2000). Zero is a complex mathematical concept due in part to its connection to the word "nothing" (Blake & Verhille, 1985). The absence of quantity is a source of students' confusion about zero's numerosity (Blake & Verhille, 1985; Gialamas & McCann, 1991; Seidelmann, 2004). Since number determines quantity, students' understanding of zero may contain misconceptions such as "zero is not a number" (Catteral, 2006; Sen & Agarwal, 2015). Arithmetic or algebra involving zero is also problematic for many primary school students, preservice teachers, and teachers (Grouws & Reys, 1975; Reys, 1974; Reys & Grouws, 1975; Seidelmann, 2004; Wheeler & Feghali, 1983). Some teachers (e.g., Russell & Chernoff, 2011), preservice teachers (e.g., Wheeler & Feghali, 1983), and students (e.g., Seidelmann, 2004; Catteral, 2006) struggle to identify zero as a number.

According to the Peano axioms for the axiomatic thinking system of mathematics, "zero" is defined as a number that is not a successor to any other number. When primary school children learn about natural numbers and basic arithmetic operations, they do mathematics following this systematic structure, even if they are unaware. Zero, which is included in this structure, has the feature of being its starting point. Due to the complexity born from language and descriptions, the concept of zero must be approached carefully in mathematics education. Gialamas and McCann (1991) stated that a dichotomy in usage exists only for zero, as zero is both a placeholder and a number.

Zero as a symbol. Students also struggle with zero as a symbol, particularly as a place value. The concept of a number representing "nothing" or "none" was first signified with a dot by the Hindus, and later the current symbol "0" was used (Sharma, 1993). Children's difficulties in accepting zero as a place value parallel historical development (Wellman & Miller, 1986). The symbol for zero was used as a placeholder to show when a digit in a numeral was "empty." As Sharma noted, "It is not difficult for children to represent 'nothing' by zero, but it is much more difficult to use zero in a place value system" (Sharma, 1983, p. 6).

The present study examines how students with LD and NLD understand "zero" in terms of all meanings of the number, including the cardinality of an empty set and as a symbol that entails the usage of digits, a place value, and a placeholder. With that purpose, the following research question was addressed: "What are the similarities and differences in the conceptions of zero as a number and a symbol among students with LD and NLD?"

Method

Research Design

This qualitative study investigated and compared understandings of the concept of zero among students with LD and NLD without any intervention. According to Stake (1995), multiple-case studies can be utilized to understand the differences and similarities between cases. In this context, the current study is a multiple-case study, and the considered cases are students with LD and NLD. These students' understandings of the concept of zero are the units of analysis.

Study Group

The participants of this study were determined by criterion sampling among the purposeful sampling methods. Six students volunteered to participate with approval from their parents. These volunteers included three students with LD (coded as LD1, LD2, and LD3) and three with NLD (coded as NLD1, NLD2, and NLD3). Two were female students (LD3 and NLD1), and the others were male. LD2, NLD1, NLD2, and NL3 were 10 years old while LD1 was 11 and LD3 was 12 years old. To deeply investigate the concept of zero with all its meanings, participants were chosen to be in at least the 4th grade, or at least the age of 10, because such participants could be expected to have encountered zero and operations with zero and to have knowledge about zero and natural numbers according to the national curriculum in Turkey up to the 4th grade (MoNE, 2015). Because students with LD are roughly two years behind their peers in educational achievements (APA, 2013; Judge & Watson, 2011), students with LD were selected within the age range of 10-12 years. The relevant counseling and research centers confirmed these students to have LD. According to information from their families and teachers, the students with NLD had similar achievement levels among themselves, were not low achievers, were not diagnosed with LD, and were not known to be under any other risks.

Research Process

Data collection. Data were collected in individual clinical interviews conducted in sessions lasting 40-50 minutes at most with each student. Interviews were audio and video recorded with the approval of the students and their families. The necessary permission and approval to conduct the study were obtained from the MoNE. Worksheets and field notes were also utilized for document-based data. Clinical interviews were conducted in environments where students would feel comfortable, such as their homes or school. A quiet but interactive environment was maintained during the interviews. Pilot studies were carried out with two students, one with LD (female, 12 years old) and one with NLD (male, ten years old), selected by the same sampling methods and the same conditions applied for the students participating in the main study, such as attending the same schools and being in the same age range.

Opinions were obtained from an expert in mathematics education who studies student thinking, particularly concerning numbers. As a result of these pilot studies and expert opinions, the interview questions were revised in terms of comprehensibility, questions that were inappropriate for the students' levels were removed, and some necessary questions were added. For example, the question "How many sheep are there in this zoo?" asked with a picture of a zoo to evaluate the understanding of the cardinality of sets was changed to "How many crocodiles are there in this zoo?" as one of the students found the idea of sheep in a zoo meaningless. This step was important as it is emphasized in the literature that tasks or questions prepared to reveal students' mathematical thinking should be presented in a meaningful and realistic context (Hunting, 1997).

Clinical Interview Questions. Clinical interview questions aiming to clarify students' thoughts on and understandings of the concept of zero were prepared according to the 4th-grade level and are given in Appendix 1. With these questions, the mathematical meaning and usage of zero were first addressed. Zero's meaning of "nothing" and its meaning as a number were approached philosophically and in the context of the problem being considered in the interviews (Leeb-Lundberg, 1977; Wheeler & Feghali, 1983; Wheeler, 1987; Crespo

& Nicol, 2006; Wilcox, 2008). At the same time, the interviews also addressed the use of zero in chunking numbers, its feature of being the cardinality of an empty set, and its use as a symbol (place value, placeholder, digit).

The "nothing" meaning of zero was examined in the context of the non-existence of a thing, absence, and the relation of zero to the state of absence. Comics appropriate for the students' levels and explicit verbal problems were employed to highlight their relations.

The meaning of zero as a number was indirectly considered in the "Number Hunting Game." The students were asked to find a single-digit number in a table filled with numbers, including zero. Furthermore, whether the students used zero in chunking integers was determined in the context of a problem about chunking numbers (see Wheeler & Feghali, 1983). It was also directly asked whether zero is a number.

Zero as a counting number was examined via questions adapted from the study of Wheeler and Feghali (1983), such as "Can you count backward from 10, 5, and 2? Can you count backward from 1?" and "Do you use zero in counting the items in your pencil case?"

Zero as the cardinality of an empty set was examined in the context of the number of apples in an empty basket and the amount of money in an empty wallet.

For *the use of zero as a symbol (place value, placeholder, digit)*, the meaning of zero was investigated in numbers such as 401, 041, 410, 10 TL, 10, 1.04, 004, 04, 40, and so on.

Data Analysis

Since it enables comparisons of similarities and differences between multiple cases, the case-oriented approach proposed by Miles et al. (1994) was applied during data analysis. In this context, data analyses were conducted for each student. Each case was analyzed according to the content analysis method. Categories were obtained by coding students' expressions, behaviors, and writings to understand zero as a number and symbol. The transcriptions and documents of tasks were analyzed to identify repeated patterns. For example, the answer of "there is zero" to the question "How many apples are there in an empty basket?" was coded as "directly assigning zero to nothing." The answer of "there are no apples in this basket [...] none, zero" was coded as "reaching zero from nothing." When it was seen that similar codes were obtained with similar answers and understandings, two categories were accordingly labeled as "Assigning nothing to zero" and "Assigning zero to nothing."

When a new understanding was detected unsuitable for existing categories and codes, a new category was created, giving it a name to express that understanding. For example, it was noticed that a student's interpretation of the concept of zero as space and space as a black hole did not fall into any existing category, and it was therefore included among the categories as a different interpretation of "Regarding zero as space." The categories and codes of these discrete cases were then compared.

Data analysis was conducted separately by each researcher, and it was found that the correlation coefficient between the researchers was 83%. Elaborating on the categories and reaching a consensus on them, the insights of the students with LD and NLD regarding zero were presented under the two main headings of the meaning of zero as a number and the use of zero as a symbol. Many subcategories were established under these two main headings. While some subcategories were only identified among the students with LD, such as focusing on the position of zero, others were only obtained from students with NLD, such as relative zero/absolute zero. Others were obtained from both groups of students.

Findings and Discussion

The findings presented here include the students' understanding of the meaning of zero as a number and its use as a symbol. As a result of the analysis, in addition to the categories and subcategories determined in line with the literature and mathematical facts (see the section above on "Clinical Interview Questions"), the place of zero in numbers and its feature of being an identity element in number categories were obtained as new categories. The insights obtained within each category are presented first in figures, then relevant explanations are given.

Meaning of Zero as a Number

As a result of the analysis of the meaning of zero as a number, the categories of "nothing" and "cardinality" were determined regarding zero as a number and various interpretations.

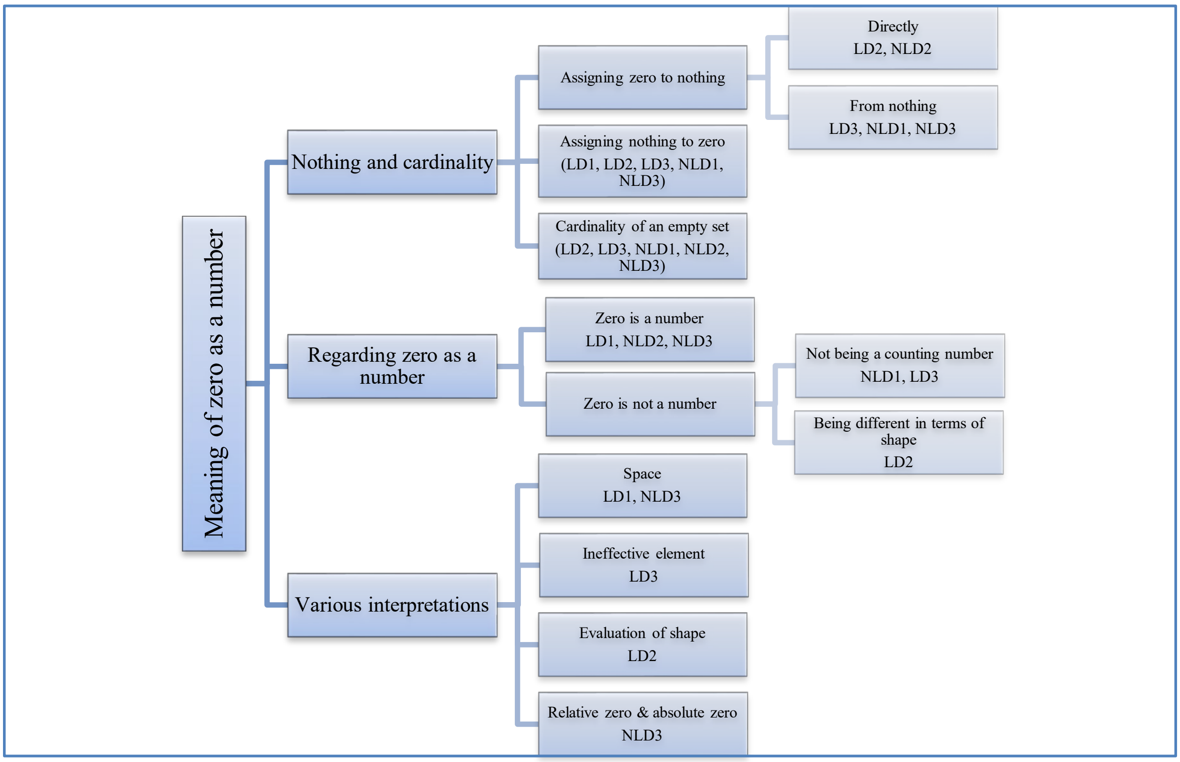

Figure 1. Students' understanding of the meaning of zero as a number

Nothing and Cardinality. In clinical interviews, the cardinality meaning of zero (i.e., no element in the set) and assigning zero to the one not existing in a set in the context of numerosity were questioned independently. However, *nothing and cardinality* meanings were addressed in the analysis, with zero findings assigned to nothing in both contexts. For this meaning, three different categories were determined: assigning zero to nothing, assigning nothing to zero, and zero as the cardinality of an empty set.

Assigning zero to nothing, expressing an element that does not exist in the set with zero (in the context of a quantity): Assigning zero to nothing means relating zero to the non-existence of something, absence, and the state of being absent, except for LD1, all participating students assigned zero to nothing. Students with LD and NLD were heterogeneously distributed in this regard. LD2 and NLD2 directly assigned zero to the state of absence without referring to nothing, while LD3, NLD1, and NLD3 reached zero from nothing. For example, LD2, who directly assigned zero to nothing, replied to the question "How many apples are there in an empty basket?" with "There are zero," while NLD3, who reached zero from nothing, said, "There are no apples in this basket [...] none, zero." NLD2 did not use the word "none" except in operations with zero. It can be thought that this is because NLD2 directly assigned zero to nothing.

LD1 had difficulty assigning a number to non-existing things and could not assign zero to nothing. For example, an empty transparent cup was shown to LD1, and it was confirmed that the cup was empty, and then we put an eraser inside the cup:

Researcher 1 (R1): Now the eraser is inside the cup, isn't it? Now the eraser is removed.
LD1: Now it's empty, there's no eraser in it.
R1: Now, how can we answer the question "How many erasers are there in the cup?"
LD1: None.
R1: How else can we express this?
LD1: Empty.
Researcher 2 (R2): Can we say there are no erasers?
LD1: I don't know.
R1: Does that make sense? How do we express "none" in mathematics?
LD1: [Silence.]
R1: What do you think of when we say "none"?
LD1: I think of nothing.
[We then wrote on a piece of paper "There are no erasers in the cup" and "There is/are ... eraser/s in the cup."]
R2: We want these two sentences to express the same thing. What do you think we should write in the blank?
LD1: [Silence.]
R1: This sentence [pointing to the second sentence] will mean that there is no eraser in the cup. What should we write here?
LD1: … "there is".
R2: What should we write in the blank?
LD1: … "many"
R1: But there are no erasers. Do "many" erasers exist?
LD1: [Silence.]

As can be seen, LD1 reached nothing and absence, but he did not relate the concepts of nothing and absence to the concept of zero. In the question about chunking the numbers, he split the number 3 into "nothing" and "3." This overlaps with the fact that LD1 did not assign zero to nothing.

Assigning nothing to zero: All students except NLD2 assigned nothing to zero. For example, when asked what they think of when we said "zero," NLD1 replied "I have no money," and LD3 replied, "There are zero boards in the classroom; I mean, there are no boards left. An empty basket. Nothing is written on the board." Thus, they both assigned nothing to zero. The desired understanding is that students do not assign nothing to zero in terms of mathematical situations; instead, they assign zero to nothing. However, NLD2 did not assign nothing to zero in the questions asked in the context of daily life.

Assigning zero to the cardinality of an empty set: As in the context of quantity, all students except LD1 assigned zero to the cardinality of an empty set. For example, in the context of cardinalities of sets, NLD3 replied, "There are zero apples, I mean, there are no apples" to the question, "How many apples are there in an empty basket?" LD1's reply was "Empty."

Regarding zero as a number: While LD1, NLD2, and NLD3 said that zero is a number, LD2, LD3, and NLD1 said zero is not a number. The reasons why these students did not consider zero a number differed.

As seen in Figure 1, one of these reasons was *not regarding zero as a number as zero is not a counting number* (LD3 and NLD1). While all students knew that zero was not a counting number, LD3 and NLD1 overgeneralized this notion. The reason for not regarding zero as a number was related to counting:

R1: Why isn't zero a number?
LD3: Because we don't start counting from zero.
R1: When do we count a number?
LD3: We start counting from 1 like 1, 2, 3…
R1: What is a number, in your opinion?
LD3: A number, a digit. I mean something like 1, 2, 3.
R1: Can you write down the digits?
LD3: [She wrote] 1, 2, 3, 4, 5, 6, 7, 8, 9.

LD3 confused the cardinal numbers or labeling of things with the counting of numbers. While expressing the idea of counting numbers, she thought that she counted the numbers. Therefore, she considered zero not a number since she did not count zero while counting numbers. LD3 and NLD1 knew that zero was not a counting number, but they generalized this to the fact that zero was not a number. Similarly, LD3 did not count zero while counting backward from 10, 5, 2, and 1. This is consistent with the fact that she did not regard zero as a number (Wheeler & Feghali, 1983). The conversation between the researchers and NLD1 was as follows:

NLD1: Zero is not a number as I do not count it like 1, 2, 3.
R2: When I say a single-digit number, don't you show zero?
NLD1: It may be a number, but I don't think of it as a single-digit number because I don't use it when I'm counting.

In this case, there may be two understandings. NLD1 may think that zero is not a number as there is no labeling with zero while counting things, or she may confuse the counting of numbers with cardinal numbers as LD3 does. This becomes clear later in her interview. Such an understanding among students may be common. NLD3 referred to this misconception and decided that zero was a number by distinguishing between these two situations:

R1: Do you think that zero is a number?
NLD3: Well… We sometimes count zero, I mean, we use it when we count numbers, for things, but normally we say 1, 2, 3… [He thought for a while]. I think it's a number.

NLD3 explained that zero is used when counting numbers, but zero is not used when counting things. Consequently, he consciously concluded that zero was a number by differentiating between the counting of numbers and counting numbers. Despite not being counting numbers, the existence of numbers such as -1, $a + ib$, or $a + ib + jc + kd$ has been accepted throughout history. Similarly, NLD3 could distinguish between zero as a number and zero not being a counting number.

Another situation regarding zero involved describing it as *different from other numbers in terms of its shape*. LD2 thought zero was different from other numbers as it had a round figure. He thought that numbers like 1, 2, and 3 "indicate something," but zero "does not indicate anything" as it is round. He did not see zero as a number because he evaluated it physically as a figure and focused on its use as a symbol. While LD2 may have overgeneralized ideas, he suggested some perceptual-based reasons. This supports findings that students with LD tend to use perceptual knowledge rather than digital logic (e.g., Munro, 2003).

When examining students' explanations regarding *zero as a number*, LD1, NLD2, and NLD3 said zero was a number. NLD2 defined a number as "the unit, the statement of the number of a thing by seeing." NLD2 first mentioned seeing. Considering that objects are things we can see, it can be said that NLD2 refers to objects with this expression. The expression "the number of" refers to counting. The same student also referred to assigning names to the objects he counted with the expression "the unit of." Therefore, he defined the concept of numbers by including all elements in his own language. Even if he did not use the word "statement" as a term, he was aware that a symbol is required to express numbers when the notion of the discourse includes symbols (Gee, 1999). NLD2, for example, was aware of the meaning of the number two and that 2 is used as a symbol to indicate the meaning it refers to. NLD2 knew symbols created to refer to numbers are just representations of those numbers:

R2: So, if you think again, according to your definition, is zero a number?
NLD2: Yes, it's a number. There is *say* [Turkish: to count] in the word's root, so I'm counting. As *sayı* [number] comes from the root of *saymak* [to count], we are counting them. If we count zero, then it's a number…

NLD2 explained the idea of zero being a number by considering that the concept of numbers comes from the root of the word for counting in Turkish, as seen in the quotation above. Since NLD2 used zero while counting backward from 10, his explanation that the word "number" comes from the root of "counting" in Turkish is in line with using zero in counting numbers.

Various Interpretations of Zero

Regarding zero as space: The students' perceptions of space varied. For instance, LD1 and NLD3 regarded space as a space vacuum or black hole. LD1 interpreted the concept of zero as space and space as a black hole. LD1's interpretation of a black hole as a thing that swallows everything may reflect a perception regarding the feature of zero being the absorbing element in multiplication.

Regarding zero as an identity element: During the clinical interviews, LD3 defined zero as an identity element. She stated that zero was not a number as it was an identity element. LD3 did not qualify zero as an identity element in place values, multiplication, or division. However, she viewed zero as ineffective in the number "04" in its place among other numbers and in addition and subtraction.

For natural numbers, she wrote "1, 2, 3, 4, 5, 6, 7, 8, 9":

R2: These are natural numbers?
LD3: Yes.
R2: Then what is zero?
LD3: Zero is an identity element.
R1: An element of what?

LD3: [Showing 04] An element of 4.
R2: Say, for instance, 401.
LD3: It's not an element here because it has a meaning here.
R1: What meaning?
LD3: Well, I can't explain that. It increases one to ten?

In cases where zero was a placeholder as a symbol, she interpreted it as ineffective since it did not add value to the number. To provide evidence that zero is an identity element and not a number, she gave the example that it did not have meaning in the number "04." She was also aware that this was different from the meaning of zero in the number "401."

Evaluation of shape: LD2 evaluated zero in terms of its shape by indicating that zero was not a number as it had a round shape different from other numbers. Furthermore, when he was asked to give examples of zero from daily life, he gave an example using the pronoun "it," like the findings of the study of Reys and Grouws (1975).

Relative zero and absolute zero: When NLD3 was asked to give an example of the use of zero in daily life, he replied: "For a mindless person, we can say she or he has an IQ of 0." He assigned the meaning of zero as a number in its daily use to the value of a test that measures intelligence. It can be thought that NLD3 was intuitively aware of the difference between relative zero and absolute zero (it is a similar example of relative zero to note that water freezes at 0 °C, which equals 32 °F). The same student was then asked: "Should we think that a person with an IQ of 0 has no mind?" He said: "No, he has a little mind, but his IQ is 0." When asked to give another example like this, he did not answer, and then he was asked: "Is it a similar situation when the temperature outside is zero degrees?" He said: "I don't know, as if the weather may be a little hot." He was then asked: "Well, how many elephants are in our room?" He said: "There are none at all, zero, but the other is different." Accordingly, it can be said that he could make the right distinction in the examples intuitively.

Meaning as a Symbol

Students' perceptions and understandings about the meaning of zero as a symbol were examined in terms of zero being a placeholder, zero's place value, and zero as a digit. In addition to these categories that have already been considered in the literature regarding the meaning of zero as a symbol, the categories of focusing on the place of zero in a number and zero as an identity element in a number are newly introduced to the literature because of these interviews.

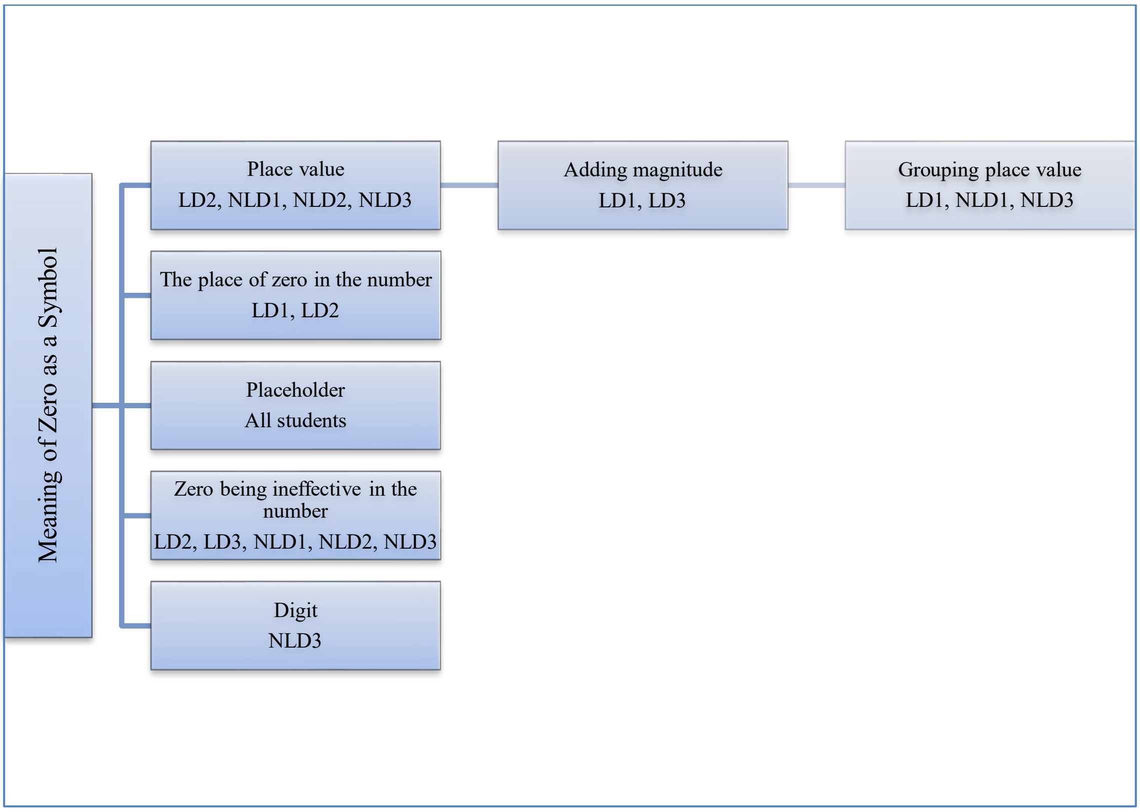

Figure 2. Students' Understanding of zero as a symbol

Interpretation of zero as a place value. Place value is the value that digits take according to their location in a number. Other than LD1 and LD2, the students were aware of the place value meaning of zero. They knew a number would change according to its position in terms of place value. While LD1 and LD3 said that zero adds value to the numbers, NLD1, NLD2, and NLD3 analyzed the numbers and explained the place value of zero. LD3 explained that for 10, it increased from one to ten, and when LD1 was asked whether 30 or 03 was a higher number, he said "30" and gave the following explanation:

> LD1: Because this [pointing to 03], 3 must be here [to the left of 0]. Then this is also big, as it is on this side, it is zero three. As this is big, it is thirty.
> R1: For example, 400?
> LD1: That's an even bigger number. That's 400, it can go to millions to be a bigger number. And for being a small number, the zeros will come before the four.

As can be seen, LD1 was aware of occasions in which zero added a magnitude to a number, but LD1 stated that both 02 and 20 were two-digit numbers and that the difference between them lay in the position of zero being on the right or the left. This indicates that LD1 could not understand the place value and interpreted its use as a symbol to express the place value of zero. In addition to the fact that students with NLD were aware that zero added magnitude to numbers, they focused on finding the place value of zero in a number by analyzing it. For example, while NLD1 examined the meanings of zero in given numbers, she explained: "The zero in 410 is in the units digit, and zero in 401 is in the tens digit. When there is no zero, they both become 41. But it is there. Since zero is there and I know where it is, the numbers are different."

Focusing on the position of zero in a number: LD1 and LD2 did not consider the given numbers as a concept and did not mention the place value meanings of the digits in numbers. They were only concerned with the shapes of the numbers or expressions and the order of digits in the numbers. While LD2 considered the numbers as a whole, LD1 emphasized the digits in the numbers. For example, when LD1 was asked about the difference between 410 and 401, he explained: "If 1 is not near 4 and on the other side, then 1 is near 4." Another remarkable point is that he considered other numbers rather than zero. Although he was asked about the meaning of the zero, he commented on the positions of other numbers by taking the zero as the reference point. Thus, it could be understood that LD1 was aware that 410 was a higher number. When LD2 was asked about the differences in the zeros in the numbers 410, 401, and 041, the following conversation ensued:

LD2: In the second [number], zero is between 4 and 1. In the third, it is in front of 4 and 1.
R1: What has the zero changed?
LD2: [Showing 410 and 401:] It changed this and that.
R2: How has it changed?
LD2: Zero is in the middle, it is four hundred and one.
R2: What if it hadn't been there?
LD2: Forty-one. Since there is a zero, it is four hundred and one.
R1: Why is that so?
LD2: It was forty-one without zero, so if there is zero, it is four hundred and one.
R1: We have zero here [041] as well, but it is not 401. Why not?
LD2: It would have been forty-one again if it wasn't zero, but since zero is in the front, it is zero forty-one.

As is seen above, LD2 was aware that zero changed the number as a whole, but he did not know the place value meaning of zero.

Knowing the placeholder meaning of zero: To make students feel the need to assign a symbol to the gap, they were asked to interpret the comic in Figure 3. By evaluating it concretely, the students indicated that it was normal for the man in the comic to be confused, stating that two zero numbers could fit into the blank. Therefore, the students were aware that zero could be used as placeholder.

R1: Why do you think he's confused?
LD1: Because the middle is blank.
R1: Is he right to be confused?
LD1: Yes. Then it's a thousand and two.
R2: Why?
LD1: Because two zeros fit there.

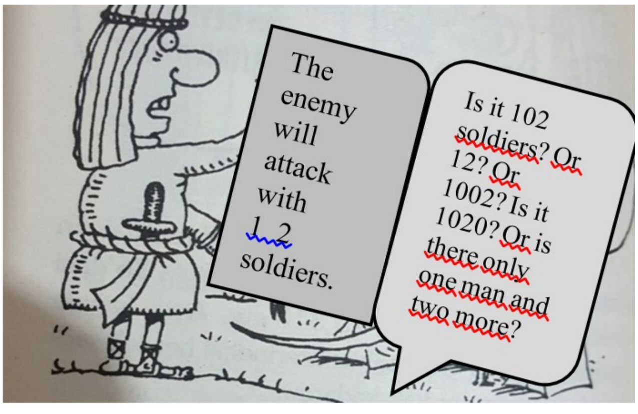

Figure 3. The comic used to investigate the placeholder meaning of zero (Poskitt, 2005, p. 38)

Zero as an identity element in a number: In the clinical interviews, students were asked to examine numbers such as 004, 04, and 041 prepared by the researchers. The aim here was to elaborate on students' understanding of the meaning of zero as a symbol. While LD2 had the misconception that numbers like 04 had a place value rather than placeholder meaning (07 is a two-digit number, 007 is a three-digit number), LD1, LD3, and all NLD students were aware that 04 was 4, a single-digit number. LD3 explained that zero was an *identity element*, NLD1 viewed the zero in 04 as an *ornament*, NLD2 indicated that zero was included to "show off," and NLD3 indicated that it was a number whether there was a zero or not. In the Number Hunting Game, NLD3 gave the example of 05 for a single-digit number with a zero in it. He said: "It's actually 5, but no need to write…." With this answer, NLD3 showed that he thought the 0 had no meaning in 05 and thus approached 05 as a single-digit number with 0 in it.

Zero as a digit: Except for NLD3, none of the students could justify the idea that zero was a digit. For example, LD3 confused natural numbers with digits. For natural numbers, she wrote the numbers from 1 to 9 and described the concept of numbers as "a number is a digit."

When LD2 was asked to compare 07, 7, and 70 in terms of their sizes, it was seen that he did not consider these quantities as a whole, and he separately compared the place values of the digits in the numbers. Therefore, although LD2 said that zero was a digit, it is thought that he did not sufficiently understand the concept of digits. This is because he considered the symbols expressing the places in numbers as numbers, not as digits. This could be understood by his comparison of numbers.

Similarities and Differences in Students' Understandings of the Concept of Zero

The similarities and differences in the understanding of zero among students with LD and NLD are summarized in Table 1. Table 1 provides individual students' understanding and the comparisons of those understandings. Similar understandings are shaded with the same colors so that similarities and differences in Table 1 can be observed more clearly. When the similarities in students' understandings are examined, it can be said that the only understanding similar for all students was *the placeholder meaning of zero in symbolic usage*. However, it can be added that no meaningful difference was found between the students with LD and NLD in terms of interpreting zero as a number or the meanings of cardinality and nothing. In understanding zero as a number, students with LD and NLD showed similar understandings.

The understanding of the students with LD and NLD differed little from each other in terms of assigning zero to nothing. Only LD1 had difficulty in assigning nothing to zero. It was seen that LD3 and NLD1 made similar explanations. LD1 and LD2 had different perspectives, such as evaluating the shape of zero or not assigning zero to nothing. Unlike the other students, LD2 evaluated zero's shape, LD1 did not assign zero to nothing, and NLD3 did not assign nothing to zero. In partitioning the number 3 in the context of natural numbers, it was seen that NLD2 did not consider zero while LD2 did. Unlike other students, NLD3 referred to relative zero and absolute zero. The fact that LD1 did not realize that zero was the cardinality of an empty set may be related to his inability to assign zero to nothing. It is noted in the literature that students with LD may have difficulty understanding cardinality (Munro, 2003).

Table 1
Differences in the Students' Understanding of the Concept of Zero

		Zero as a number	Zero as a symbol
LD	LD1	He did not assign zero to nothing.	He decided according to its place in the number.
		He did not recognize its meaning as the cardinality of an empty set.	Zero adds magnitude to a number.
		Zero is a number.	
	LD2	As zero is different in terms of its shape, it is not a number.	He decided according to its place in the number.
			He did not recognize its place value.
	LD3	As zero cannot be counted among numbers, it is not a number.	Zero adds magnitude to a number.
NLD	NLD1	As zero cannot be counted among numbers, it is not a number.	-
	NLD2	He did not assign nothing to zero	-
		Zero is a number.	
	NLD3	Relative zero, absolute zero	He recognized its use as a digit.
		Zero is a number.	

As a result of the findings in the context of zero's meaning as a symbol, it was seen that the understanding of the students with LD and NLD varied. Only NLD3 among all students was aware of using zero as a digit. While LD2 evaluated that zero was holding a place in terms of its shape, other students evaluated zero in terms of adding magnitude to a number and as a digit holding a place. Despite not using appropriate mathematical language, LD3 was aware of the meaning of place value, similar to the students with NLD. In addition, LD3 assigned the feature of being ineffective to the zero in the number "07", which has meaning as a placeholder. Due to the uncertainty in the expressions of LD1, it is impossible to assess whether he was aware of this meaning. It can, however, be said that LD1's and LD2's understanding of zero as a symbol was more immature than their understanding of zero as a number. This may be because the former understanding is primarily knowledge-based and algorithmic (number analysis). For example, while students with LD could compare 70 to 7 in size, they made different interpretations for 07. At the same time, students commented about the place of zero in numbers when they were directed to evaluate zero as a symbol. Hence, the categories of *the place of zero in numbers* and *the feature of being the identity element in numbers* were added to the currently existing categories.

Conclusion and Recommendations

The current study has examined the similarities and differences between the understanding of students with LD and NLD regarding the concept of zero. Interestingly, although the students had similar understandings of zero, their justifications differed. Even when the justifications were the same, the interpretations of the results were different, falling into different categories. Accordingly, the zero concepts revealed a broad spectrum of understanding for different learner profiles, not following a standard pattern.

As a result of this study of the understandings of students with LD and NLD regarding the concept of zero, new categories have been added to the literature regarding the different meanings and uses of zero. As a contribution to the literature, different interpretations of zero, such as *space*, *identity element*, *shape*, and *relative zero and absolute zero* have emerged regarding the meaning of zero as a number. Furthermore, it was seen that students' philosophical approaches to the meaning of "zero as nothing" can be differentiated into the "assignment of nothing to zero" and "zero to nothing." It was concluded that zero not being a counting number may affect its comprehension as a number. In this context, it is recommended that more time be allocated to the concept of zero and more focus be placed on these differences conceptually in mathematics teaching.

It has been shown in this work that it is important to differentiate between assigning nothing to zero and zero to nothing. While the former causes problems in learning and interpreting the different uses of zero as a symbol, the latter is a communication tool used in the mathematical expression of verbal situations. For students to use zero correctly and conceptually, they must assign zero to nothing and assign nothing to zero in appropriate mathematical contexts. In daily life, zero is assigned to objects that do not exist. In this context, students are generally expected to attribute zero to nothing, while the opposite is a fallacy that adversely affects their learning (Russell & Chernoff, 2011; Cockburn & Little, 2008).

Among the students participating in the present study, only one student with LD had difficulty assigning a number to something that did not exist and could not assign zero to nothing. This shares similarity with humanity's historical struggles to develop the notion of zero (e.g., Blake & Verhille, 1985). In addition, it is seen that the assignment of zero to nothing among students could also reflect differing philosophical approaches in the interpretation of zero. For teachers to encourage students' understanding of zero as a number, it may be helpful to emphasize various meanings of zero rather than "nothing" (Russell & Chernoff, 2011). The handling of zero as a vacuum of space or a black hole by students with both NLD and LD is also a new finding for the literature, as well as an example of the fact that students with LD and NLD may create similar mental images when they use their mathematical knowledge together with their imaginations.

No sharp distinction was observed in this study between students with LD and NLD in conceptual examinations of zero as a number. No patterns in student responses revealed clear distinctions between students with LD and NLD in terms of interpreting zero as a number or the meanings of cardinality and nothing. While LD and NLD students had differences and similarities in their understanding of zero, these distinctions were insufficient to distinguish the groups significantly. The reason for this may have been the conceptual examination of the concept of zero in this study.

However, the study's findings regarding the symbolic meaning of zero demonstrate that the understanding of students with LD and NLD was very dissimilar. This is consistent with the fact that two of the three participating students with LD were less developed in their understanding of the use of zero as a symbol than in their understanding of the meaning of zero as a number. This may be due to weaknesses in visual and symbolic perception among students with LD, a finding that has been previously noted in the literature (Andersson, 2010). Examining the difficulties experienced by students with LD in arithmetic in this context may be helpful for the further advancement of the literature in this field. Considering the statement of one participant with LD that since zero is different from other numbers in terms of its shape, it is not a number, our study supports previous findings that students with LD may make decisions based on perceptual knowledge rather than numerical logic (Munro, 2003).

In conclusion, since zero develops deep reasoning among students, especially at the primary school level (Wellman & Miller, 1986), it is remarkable that students with LD have an understanding similar to those with NLD regarding zero as a number. In this context, it can be suggested that more scientific research on number conceptions should be carried out to further understand the mathematical thinking and learning processes of students with mathematical LD. Various researchers have noted that students with LD particularly have difficulties with arithmetic and numbers (e.g., Geary, 1990; Mejias et al., 2012; Namkung & Peng, 2018).

As another important result of this study, we conclude that the concept of zero generated a different range of concepts without specific patterns across different learner profiles, once again revealing the complex nature of the concept of zero. Considering this finding, further scientific studies to observe how students' understandings of zero develop hierarchically over time or how the learning trajectories of this concept change will be beneficial for both enriching the mathematics education literature and guiding mathematics teachers.

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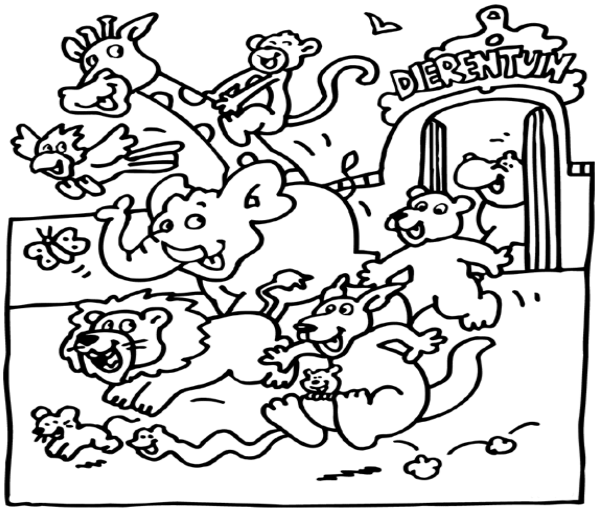
Appendix 1.

Interview questions on the concept of zero

1. This is a zoo. How many alligators are there in this zoo? (*A question about nothingness*)



2. How many apples are in an empty basket? [Prompt questions: If the student says "Nothing", he or she is asked "What do we say for nothing?" If there is no answer, questions such as "How many beds are in this room?" are repeated.] (*Cardinality of the empty set*)

3. What is "nothingness" represented by?

4. How much money do I have if I have no money left in my wallet? (*Cardinality of the empty set*)

5. What comes to mind when we say "zero"? What do you think zero is? Give an example. (*Questioning the concept*)

6. I have zero [with hands open]. Can you talk about this? What would you say about this? (*Questioning the concept*)

7. How do we represent zero? (*Notation*)

8. If you were the first to find zero, how would you show it? What symbol would you use? (*Notation*)

9. Is zero a number? (*What do you think a number is?*)

10. Do we use zero when counting the pencils in our pencil case? (*Can you count with zero? Is the student aware that zero is not a counting number?*)

11. Number Hunting Game (MoNe, 2015): Let's find the numbers with zero in this game. Find and write at least 10 different numbers. (*The case of distinguishing the number and symbol meanings of zero; there is also the use of the symbol of zero here and the student can only choose zero; if the student does not say zero, he or she will be asked to find a two-digit number*)

Number Hunting Game

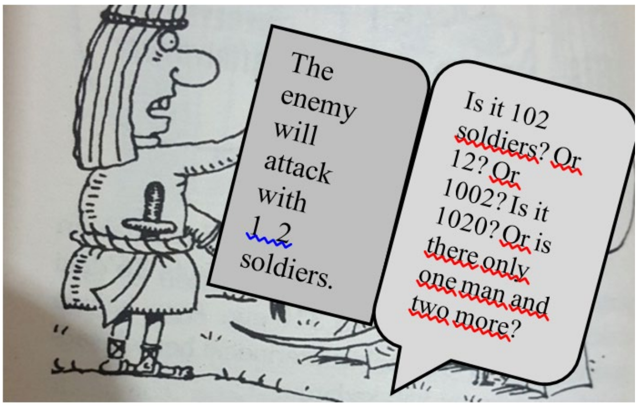
3	2	2	7	9	8	5	6	5	5	6	4
1	8	3	7	5	4	8	6	5	2	6	8
6	5	6	0	5	5	1	1	5	9	9	0
0	5	8	4	7	9	9	1	2	8	6	6
3	3	5	0	3	5	2	3	2	0	6	4
3	6	9	9	3	8	6	7	3	7	3	2
9	5	4	4	5	6	9	8	0	4	4	8
9	7	3	5	9	2	0	0	0	8	0	8
1	0	5	7	0	7	0	7	2	9	5	5
5	7	3	6	3	2	1	2	6	0	0	1
5	2	6	6	4	0	0	7	9	2	2	4
1	9	2	3	5	6	5	0	6	2	9	1
9	8	8	3	7	1	5	3	8	8	5	6
8	1	7	1	0	7	2	6	6	6	9	4
8	1	2	6	5	7	4	0	5	3	8	7
3	9	3	1	0	0	6	8	4	3	3	5
5	8	5	3	1	0	7	4	7	4	4	5
5	8	7	6	3	9	3	5	4	5	3	5
9	0	3	7	1	5	0	2	1	5	7	8
6	2	6	3	2	2	3	6	5	8	5	1
5	0	6	5	1	5	6	2	4	2	9	2
7	9	6	8	5	7	1	8	5	1	5	8

12. Examine the quantities given below. What would you say about the meaning of zero in each of these examples? [Prompt questions: "How many digits are in the following

numbers? How many zeros can we put at the end of 1.0390?"] (*Use of symbols, digits, values, placeholders*)

40 | 10 *TL* | 410 | 401 | 041 | 04 | 004 | 1.04

13. Interpret the picture below. (*Use of symbols*)



14. Two friends, Ali and Ahmet, go fishing. A total of 3 fish are caught. How many do you think each friend caught? (*Understanding of zero as a number*, adapted from the study of Wheeler and Feghali (1983))

15. There are no apples in a basket. If I divide the apples into 4 groups, how many apples will there be in each group? (*Number meaning*)