Promoting Prospective Mathematics Teachers' Understanding of Derivative across Different Real-life Contexts

Mahmut Kertil
Marmara University
mkertil@marmara.edu.tr

Hande Gülbagci Dede
Marmara University
handegulbagci@gmail.com

This study investigates prospective secondary school mathematics teachers' developing conceptions about the meaning of derivative (as rate of change) while working on a series of tasks involving different non-motion real-life contexts. The study was conducted by employing a case study method. The participants of the study were 25 prospective secondary school mathematics teachers enrolled in mathematics teaching methods course. A series of contextual tasks involving different real-life situations about derivative was designed and implemented in the form of problem-based learning as a regular part of the course. Pre- and post-questionnaires, written group solutions to the tasks, individual reflection papers, and video-recorded online classroom discussions were the data sources. A constant comparative analysis method was employed in analysing the data. The results showed that prospective teachers have difficulties in interpreting the meaning of derivative in different real-life contexts. While they were frequently using "amount of change" and "slope" interpretations, they started to use "rate of change" interpretation. However, the classroom discussions showed that most of the prospective teachers were using the "rate of change" expression as a memorized term without engaging in its contextual meaning. We made some inferences about the teaching and learning of derivative and the professional development needs of teachers.

Rate of change, as one of the central ideas in the secondary mathematics curriculum, is the most inclusive interpretation of derivative which is necessary for conceptual understanding of many calculus concepts (Bezuidenhout, 1998; Byerley & Thompson, 2017; Thompson, 1994a, 1994b; Herbert & Pierce, 2011). A coordinated understanding of many big ideas involving variable, function, covariation, ratio, and rate is required for a coherent understanding of the rate of change (Byerley, 2019). However, the rate of change interpretation of derivative is generally ignored or not extensively addressed with the required quantitative meanings in the curricular documents and textbooks (Berry & Nyman, 2003; Bingolbali, 2008; Byerley & Thompson, 2017; Teuscher & Reys, 2012). Many researchers have reported that students or mathematics teachers have various conceptual difficulties in understanding derivative and rate of change (e.g., Bezuidenhout, 1998; Byerley & Thompson, 2017; Bingölbali, 2008; Feudel, 2018; Herbert & Pierce, 2012; Mkhatshwa & Doerr, 2018; Teuscher & Reys, 2012; Thompson, 1994a; 1994b).

Derivative and rate of change concepts have a large spectrum of application areas in a variety of non-kinematic science and engineering contexts (Jones, 2017). However, overdependence on motion context (i.e., distance-time or velocity-time graphs) is observable in curricular materials and in-class practices of teachers in introducing the rate of change (Berry & Nyman, 2003; Herbert & Pierce, 2008; Jones, 2017; Wilhelm & Confrey, 2003). Therefore, students have difficulties in transferring their ways of reasoning about the rate of change across different (non-motion) contexts (Bektaşlı & Çakmaklı, 2011; Doorman & Gravemeijer, 2009; Herbert & Pierce, 2012; Jones, 2017; Wilhelm & Confrey, 2003; Zandieh, 2000; Zandieh & Knapp, 2006). Thompson (1994b) and Jones (2017) emphasize the use of contexts involving simultaneously changing non-temporal quantities, in addition to kinematic concepts, to support students' understanding of the rate of change. In the study by Wilhelm and Confrey (2003), Algebra-I students had difficulties in projecting their understanding of the rate of change in motion context to Bank Account context and the researchers voiced the necessity of using multiple contexts in teaching. Similarly, the studies by Herbert and Peirce (2011, 2012) also showed the inconsistencies in students' understanding of the rate of change across different mathematical representations and different real-life contexts.

The aforementioned research studies commonly indicate the need for devising new tasks contextualized in multiple areas for advancing students' coherent and robust understanding of rate of change. To do that, there is a need for more research studies proposing effective teaching practices supported by well-designed sample activities contextualized in novel situations. Overreliance on the motion context while teaching derivative and the resulting student difficulties may also be related to teachers' ways of knowing (Byerley & Thompson, 2017). Prospective and in-service teachers can be supported in realizing the meaning of derivative and rate of change, specifically in non-motion contexts. However, there is little research on describing and supporting prospective or in-service teachers' understanding of derivative (e.g., Byerley & Thompson, 2017). At this point, because it is a critical factor for students' ways of forming mathematical conceptualizations, teachers' understanding of derivative and supporting their professional knowledge seems important (Byerley & Thompson, 2017; Teuscher & Reys, 2012). This seems important in terms of improving teachers' knowledge of derivative which will go along with an increase in students' understanding of this concept in the future. Therefore, as little research has attended, planning and devising research on supporting mathematics teachers' understanding of derivative across different contexts is worth a particular focus of attention. In this study, we investigate prospective mathematics teachers' (PST) developing conceptions of derivative and rate of change across a series of tasks involving non-kinematic contexts. The research questions guiding this study are as follows:

- How did prospective mathematics teachers' understanding of derivative appear while solving tasks from different real-life contexts?
- How did engaging in solving a series of tasks contribute to prospective mathematics teachers' understanding of derivative and rate of change across different contexts?

Review of Literature

A wide range of research put into words students' understandings of rate of change from various aspects including (i) overdependence on the motion context and difficulties in interpreting the rate of change in non-kinematic contexts (e.g., Doorman & Gravemeijer, 2009; Herbert & Pierce, 2008; Jones, 2017; Wilhelm & Confrey, 2003), (ii) conceiving it as an amount of change (e.g., Rowland & Javanoski, 2004; Mkhatshwa & Doerr, 2018), (iii) difficulties in forming connections among different interpretations such slope and difference quotient (Byerley & Thompson, 2017; Herbert & Pierce, 2008; Zandieh, 2000), and (iv) difficulties arising from a lack of quantitative and covariational reasoning (Thompson, 1994a, 1994b; Thompson & Carlson, 2017).

A collection of recent papers investigated calculus students' understanding of derivative in non-kinematic contexts such as related rate or cost, revenue, and profit (Feudel, 2016, 2017, 2018; Jones, 2017; Mkhatshwa, 2018, 2020; Mkhatshwa & Doerr, 2016, 2018). The study by Mkhatshwa (2018) evidenced that business calculus students did not see marginal change as a rate in the form of a difference quotient, rather they saw it as an amount of change (difference in only one quantity). In the following study, Mkhatshwa and Doerr (2018) reported that business calculus students had difficulties in discriminating and interpreting the quantities of the total cost, total profit, marginal cost, and marginal profit. Similar results were reported in the studies by Feudel (2017, 2018) investigating calculus students' understanding of derivative in the context of economics. According to Mkhatshwa (2018), as the denominator is one, equality of the numerical values for the difference quotient (rate) and the difference (amount) quantities is the main source of this difficulty. This is also pointed out by Cooney, Beckman, and Lloyd (2010), in a way that thinking with one-unit increment in the independent variable provides students with a great amount of information about the rate of change just by looking at the change in the dependent variable. Jones (2017) reported that, while working on non-kinematic contexts, calculus students (i) forced to use covariational reasoning, (ii) saw the derivative value as an "amount" quantity, not as a "rate", and (iii) invoked "time" as an independent variable although it is not explicitly stated. These researchers commonly emphasized the need for teaching derivative and rate of change within diverse contexts (Jones & Watson, 2018; Zandieh, 2000).

Research also shows that student difficulties with the concept of rate of change can be related to their lack of quantitative and covariational reasoning (e.g., Kertil et al., 2019; Carlson et al., 2002; Thompson & Carlson, 2017; Cooney et al., 2010; Castillo-Garsow, 2012; Confrey & Smith, 1994; Byerley & Thompson, 2017; Thompson, 1994a, 1994b). Students' conceptions of quantities, variation in one quantity, and covariation between quantities, and being able to see all these in a unit of structure is critical for a robust conception of rate (Thompson, 1994b; Thompson & Carlson, 2017). Conceiving rate of change as an amount of change in one quantity, as many students and teachers demonstrated, is seen as the result of lack of covariational reasoning (Bezuidenhout, 1998; Thompson & Carlson, 2017; Cooney et al., 2010; Zandieh, 2000). Thompson (1994b) offered the use of non-temporal contexts in developing students' conceptions of the relationship between quantities. As supportive of this argument, Jones (2017) recently evidenced how using tasks involving non-kinematic contexts forced students to attend to each quantity and so to use covariational reasoning while thinking on the rate of change.

Teachers have similar difficulties as students regarding the concept of derivative. However, compared to the number of studies conducted with secondary or undergraduate level students, there are relatively few studies focusing on describing or supporting mathematics teachers' knowledge of derivative and rate of change (e.g., Akkoç et al, 2008; Kertil, 2014, Byerley & Thompson, 2017; Duran & Kaplan, 2016; Gökçek & Açıkyıldız, 2016; Rodríguez-Nieto et al., 2021; Yoon et al., 2015). Akkoç et al. (2008) investigated the developments in one prospective teachers' content knowledge during a technology-enriched workshop and they reported an enrichment in her conception of a derivative as an instantaneous rate of change in a real-life context. In a recent study on 251 in-service high school mathematics teachers, Byerley and Thompson (2017) showed that the majority of teachers' understanding of slope and rate of change was formulaic and procedural. Only a small percent of them could demonstrate an understanding of rate of change as a new quantity obtained from the multiplicative comparison between two changes or relative sizes. Byerley and Thompson (2017) strongly emphasized the need for professional development support for in-service or pre-service mathematics teachers regarding the topic of rate of change.

Conceptual Framework

Zandieh (2000) introduced an analytical framework for describing a robust conception of derivative. According to this two-dimensional framework, the horizontal dimension involved contexts (or representations) involving slope of a tangent line, rate of change, speed or velocity, and symbolic (limit of the different quotient). The vertical dimension involves ratio, limit, and function layers for analysing a students' understanding at each representation level. In this study, we specifically focused on the "rate of change" interpretation at the horizontal dimension, but we also considered PSTs' transitions and fluency between different representations. We used Thompson's (1994b) theory of quantitative reasoning in describing PSTs' understanding of derivative as a rate of change. Quantitative reasoning theory discriminates between the basic terms of quantity, quantification, quantitative operation, numerical operation. According to Thompson (1994b, 2011), quantities are attributes of objects conceived in situations and quantification involves conceptualizing an object and measuring its quality. Quantitative operations are mental operations by which a new quantity is conceived while numerical operations are used to evaluate those newly constructed quantities (Thompson, 2011).

The concept of derivative involves different but connected interpretations such as rate (of change), slope, and difference quotient. From the quantitative reasoning perspective, rate of change is an intensive quantity obtained as a result of the multiplicative comparison of changes in two simultaneously changing quantities (Thompson, 1994b). Rate of change is an abstract quantity that can be conceived first as "fastness," and can be conceptualized as the multiplicative comparison of two extensive quantities (distance and time) and measured by using the numerical operation of division. According to Thompson and Carlson (2017), a robust conception of rate of change requires conceptualizations of ratio, quotient, covariation, quantity, and proportionality. Additionally, covariational reasoning has been emphasized as being critical for a mature understanding of rate of change (e.g., Kertil et al., 2019; Byerley & Thompson, 2017; Thompson & Carlson, 2017). Saldanha and Thompson (1998) defined covariational reasoning as "holding in mind a sustained image of two quantities' values simultaneously" (p.299). Rate of change is a new quantity obtained as the result of the multiplicative comparison of two quantities (relative size), and so the three quantities form a quantitative structure (quantity-1, quantity-2, result of multiplicative comparison) (Thompson, 2011). Without covariational reasoning, students can see derivative as an amount of change in only one quantity (e.g., Byerley & Thompson, 2017). In the current study, we benefited from quantitative reasoning theory in describing PSTs conceptions of derivative across different real-life contexts. Quantitative reasoning theory emphasizes understanding and transferring of mathematical concepts across different contexts as it emerges with one's comprehension of situations in terms of the quantities involved and the relationships between them. Although rate of change is an interpretation of derivative, in this study, we used "derivative" and "rate of change" concepts interchangeably as we asked about the contextual meaning of derivative across the tasks.

Methods

In this study, the case study method was employed which includes the in-depth analysis of a group of PSTs within its phenomenological bounds (Yin, 2011). PSTs' ways of reasoning on a series of contextual tasks about derivative and rate of change during their individual written solutions or in group works were the unit of analysis.

Participants

The participants in this study were 25 senior years PSTs enrolled in a secondary mathematics teacher preparation program at a public university in Turkey. The study was conducted as a regular part of the mathematics teaching methods-II course which was carried out online because of the restrictions of the COVID-19 pandemic. Regarding the content and pedagogical content knowledge background of the PSTs, all of them completed the fundamental mathematics courses such as calculus-1, calculus-2, and linear algebra as well as a mathematics teaching methods-I course.

Selection and Intervention of the Tasks

In the study, four tasks about derivative were designed by researchers. All the real-life contexts used in the tasks were selected from non-motion contexts. The details of the tasks are provided in Table 1. The first task was selected from an economic context, and it was about the relationship between cost and production. In this task PSTs are provided with a verbal expression, a particular value of cost as a function of the amount of production, and a derivative value at a particular point. The second task was about the world population between 1998-2010 years provided with tabular data. The third task was about the price of a car as a function of mileage (see Figure 1). PSTs were provided with tabular data and asked to find an approximate derivative value at a point. The fourth task was about the filling bottle context adapted from the study of Kertil and Küpcü (2021). In this task, PSTs were provided a picture of a conical bottle with its dimensions, and they were asked to interpret the volume as a function of the radius of cross-sections and volume as a function of height. In each task, PSTs were asked to interpret the meaning of derivative in the given context. Additionally, they were asked to interpret the long-term behaviour of the function and its derivative.

Table 1
The Detail of the Tasks

Task	Context of the task	Variables of function	The figure or representation provided	Duration of classroom discussion
Cost of Product	Cost as a function of a product	The length of the produced rope & Total cost	Verbal (with some symbolic expressions)	2 hours
Population	World population over the years	Year & Population	Table	2 hours
Price of a Car	The price of the car as a function of mileage	Price of the car & Mileage	Table	2 hours
Filling Bottle	Filling a conical bottle with a constant flow rate	Height & Radius Radius of cross-sections & Volume	Picture	2 hours

All the tasks, except the population, were selected from non-temporal contexts. An overreliance of students and PSTs on algebraic formulas and algebraic procedures about derivative concept has been reported in the literature (e.g., Berry & Nyman, 2003; Gökçek & Açıkyıldız, 2015). Therefore, we did not provide any formula for the functions in any of the tasks to see PSTs' ways of reasoning about derivative in different contexts without algebraic formulas. But PSTs were free to obtain an algebraic formula by using the givens in the task context as appeared in The Price of a Car, Population, and Filling Bottle tasks.

The tasks were implemented sequentially in the form of group and classroom discussions. As the course was instructed online because of the COVID-19 pandemic situation, all the tasks were designed in the Desmos web-based activity builder platform. Each task was assigned to PSTs via a classroom code constructed in the Desmos two-days before the class-hour. PSTs were asked to solve the tasks with their group mates. All of the 8 groups were asked to work on the tasks altogether by using an online platform (e.g., Zoom, Skype). They submitted their group answers via the Desmos platform before the class-hour. The instructor (one of the authors of this paper) analysed the answers before the class, and she specified the sample solutions to share with the class during the classroom discussion. These solutions were selected carrying out some properties as uniqueness, involving a different way of reasoning, or representing a common way of reasoning. During the classroom discussions, PSTs had an opportunity to present and explain their ways of reasoning. Classroom discussions were carried out (and recorded) on Zoom online teaching platform and lasted three weeks (about 8 class-hours).

The Price of a Car

The value of a car gradually decreases because of damage, mileage, and some other reasons. If we assume to keep all other reasons constant, the price of a car (*V*) depending on its mileage (*d*) can be modelled with a function of *P(d)=V*. In the table below, tabular data is provided showing the value of a new car as a function of its mileage.

d (*1000)	V (*1000 TL)
20	192
50	180
150	148
300	109
600	60

According to the table,

a) What does the expression $P(300) = 109$ mean in the given context?

b) What does the symbolic expression $\frac{P(600)-P(300)}{600-300}$ mean in the given context?

c) Discuss the meaning of $P'(300) = 0.21$, and what is its unit?

d) What is the approximate value of $P'(600) =$?

Figure 1. Price of a car (Task 3)

Data Sources and Analysis

In this study, we benefited from multiple data sources that are (i) pre and post questionnaires, (ii) written group reports, (iii) video-recorded whole class discussions, and (iv) reflection papers. In the beginning, a questionnaire including two open-ended questions was administered to see PSTs' background knowledge about the definition and contextual meaning of derivative. The questions were "What is derivative?" and "Provide four different contexts in which derivative was used and explain". The same questionnaire was administered after the implementation of the tasks. The second data source was the written group reports of PSTs about each task. Another source of data was video-recorded classroom discussions by which we could reach the details of PSTs' ways of reasoning. Finally, after the implementation of all tasks, PSTs were asked to write a reflection paper (individually) for evaluating the discussions on the tasks and their learning outcomes.

Table 2
Categories of PSTs' ways of Interpreting Derivative and Rate of Change in Different Real-life Contexts

Ways of Interpreting Derivative	Explanation and Examples $V'(h)$
Amount of change (AoC)	Considering derivative as the amount of change in the dependent variable e.g., $\frac{dV}{dh}$ *here shows the change in volume with respect to height.*
Rate of change (Smooth-Relative size) (RoC)	Explaining derivative as a rate of change and considering the rate of change as a smooth continuous covariation and as a relative size e.g., *Volume and height change together, and any sized change in height result in a change in volume as "derivative value" times as large.*
Rate of change (Chunky) (RoC-C)	Explaining derivative as a change in the dependent variable per unit change in the independent variable, thinking with 1-unit increments e.g., $\frac{dV}{dh}$ *means a change in volume per unit change in height.*
Difference quotient with limiting (DQ-L)	Explaining derivative with the $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ e.g., $\frac{dV}{dh}$ *means the limit of the rate of change in volume to the change in height as the change in height approaches zero.*
Slope of tangent line or gradient (Slope)	Explaining derivative as the slope of the tangent line. e.g., $\frac{dV}{dh}$ *means the slope tangent from a point x=h.*
Other or irrelevant	The explanations cannot be labelled under either of the above categories.

A constant comparison method was used in analysing the data (Strauss & Corbin, 1998). The data analysis was started with the questionnaire administered one week before the intervention. In the pre- and post-questionnaires, based on PSTs written explanations, their understanding of derivative was analysed according to the coding scheme in Table 2. Examples of real-life contexts provided by PSTs were categorized (e.g., motion, economics, etc.) and analysed according to the attributes of the variables (e.g., temporal, or non-temporal). The written group solutions and whole-class classroom discussions for each of the tasks were also analysed according to the coding schema in Table 2. The written reports were analysed by focusing on the meanings of verbal expressions or other mathematical representations. The analysis of reflection papers informed us about PSTs' self-evaluation of their development at the subject matter knowledge level.

Findings

In this section, we first started by presenting the descriptive data about PSTs' explanations about the meaning of derivative and the real-life examples that they provided in pre- and post-questionnaires. Later, we followed with reporting PSTs' ways of reasoning that they demonstrated during the solution of each task and the whole class discussions, findings obtained from their lesson plans and reflection papers.

Table 3
PSTs' Explanations for the Meaning of Derivative in Pre and Post Questionnaire

Explanations	Pre-test f (%)	Post-test f (%)
Amount of change	11 (37%)	10 (19%)
Rate of change	7 (23%)	23 (44%)
Rate of change (Chunky)	1 (3%)	-
Difference quotient with limiting	3 (10%)	8 (15%)
Slope of a tangent line	8 (27%)	9 (17%)
Other	-	2 (4%)
Total (f)	30 (100%)	52 (100%)

PSTs emphasized more than one meaning of derivative in their explanations in pre and post questionnaires. Table 3 shows that "amount of change" was the most frequent conception of derivative that PSTs demonstrated in the pre-questionnaire. The percentage of explanations in which derivative was considered as the rate of change was only 23% before the intervention. After the intervention, while the "amount of change" conception decreased to 19%, the "rate of change" conception increased to 44%. PSTs provided and continued to provide "difference quotient" and "slope of a tangent line" interpretations for the contextual meaning of derivative before and after the intervention.

In the pre- and post-questionnaires, PSTs were asked to provide four different well-defined real-life examples for derivative. The real-life contexts provided by PSTs in questionnaires were analysed and presented in the table below. We also analysed if they were involving temporal or non-temporal variables and the descriptive data is provided in Table 4.

Table 4
PSTs' Preference of Real-life Contexts for Explaining Derivative in Pre-post Questionnaires

Contexts	PSTs' examples of derivatives in pre and post questionnaires							
	Temporal		Non-temporal		N/A		Total	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
	f	f	f	f	f	f	f (%)	f (%)
Motion	20	18	-	-	-	-	20 (31%)	18 (17%)
Length-Area-Volume	-	3	-	3	-	1	-	7 (7%)
Economics	-	8	1	6	3	5	4 (6%)	19 (18%)
Temperature-Heat	6	8	-	1	-	1	6 (9%)	10 (10%)
Filling water	1	9	-	4	1	-	2 (3%)	13 (13%)
Population	5	13	-	1	-	1	5 (8%)	15 (15%)
Optimization	-	-	11	6	-	-	11 (17%)	6 (6%)
Other	5	7	7	4	5	4	17 (26%)	15 (15%)
Total	37	66	19	25	9	12	65	103

Note. Context-free examples are not counted.

As seen in Table 4, PSTs provided 65 real-life examples in the pre-questionnaire while they provided 103 in the post-questionnaire. While 31% percent of the examples were from the motion context in the pre-questionnaire, it decreased to 17% percentage in the post-questionnaire. The number and diversity of real-life contexts provided by PSTs were considerably increased in the post-questionnaire. The frequency of temporal and non-temporal examples was both increased. However, the percentage of examples involving non-temporal contexts slightly decreased. It was also observable that PSTs provided similar real-life examples as those used during the intervention such as filling water, economics, and population.

Table 5
PSTs' Conceptions of Derivative across the Tasks

	Cost of Production		Population		Price of a Car		Filling Bottle	
	f	%	f	%	f	%	f	%
AoC	9	52.9%	11	24.4%	8	33.3%	3	6.3%
RoC	5	29.4%	20	44.4%	12	40.0%	45	93.7%
RoC-C	2	11.8%	9	20.0%	1	4.2%	-	-
DQ-L	1	5.8 %	-	-	-	-	-	-
Slope	-	-	2	4.4%	1	4.2%	-	-
Irrelevant	-	-	3	6.7%	2	8.3%	-	-

Table 5 shows PSTs ways of reasoning about derivative across the tasks. Although PSTs' explanations for the contextual meaning of derivative change according to the context, the use of "rate of change" increased in their verbal explanations. As an overall picture, the descriptive data indicates that the intervention involving a series of four tasks from different contexts about the meaning of derivative seems to have a limited impact on PSTs' making sense of derivative in non-temporal real-life contexts. The diversity in their definitions of derivative increased. In the following parts, we will report data about PSTs' ways of thinking during the solution of each task supported with the instances from written solutions, in-class discussions, and reflection papers.

Task 1. The Cost of Product

The first task was about the cost of producing rope (*M*) as a function of the amount of production (r). In this task, one value of the cost function and one value of its derivative at a point were provided (i.e., $M(2000) = 800$ and $M'(2000) = 0.35$). PSTs were asked to describe the contextual meaning of derivative at a point, the long-term behaviour of the function, and the meaning and long-term behaviour of the derivative function.

As summarized in Table 5, almost all the groups (except G5) explained the meaning of derivative as "the amount of change in the cost function". Five groups provided explanations involving "amount of change" and "rate of change" at the same time. G6's explanations were consistently involving "amount of change" while G5 consistently used rate of change or rate of change (chunky) reasoning. During the in-class discussions about the meaning of $M'(r)$, PSTs from different groups explained their reasoning as follows:

Researcher: What does derivative mean in this context? How did you explain it?

PST5 (G3): We benefited from the definition of the derivative. We said the rate of change, speed of change… The rate of change in cost with respect to rope production in meters… (*RoC*)

PST25 (G2): We tried to think with the definition of speed. Speed is the distance travelled per unit of time. Thinking, in the same way, $M^{'}(r)$ means a change in cost per unit of product (in meters) (*RoC-C*).

PST19 (G6): We only changed the words in the definition of the derivative. And we expressed the instantaneous change in the cost. **(***AoC***)**

As can be seen, PSTs from different groups provided different explanations for the meaning of derivative. With the expression "change in cost per unit of the product", G2's chunky way of reasoning was observable. But they first visited the meaning of speed in the motion context, and then they adopted the definition of speed in Physics as "*change in distance per unit of time*" to the cost-product context. G6 explained the meaning of derivative as the instantaneous change in the cost function. In the following parts of the discussion, PSTs continued to discuss the meaning of each expression and we asked them to clarify the contextual meaning of $M'(r)$. In their group report, G8 explained the expression $M'(2000) = 0.35$ as a percentage of change (35%) in the cost at the given meter of rope production. An excerpt from the classroom discussion on G8's answer is below:

Researcher: What does the expression $M^{'}(2000) = 0.35$ mean? Different from the other groups, Group 8 stated that "the increase in cost is 35 % when the rope is 2000 meters". What does that mean?

PST1 (G8): I don't know exactly, but it was meaningful for us at that moment. Was it 0.35%, I get confused now?

Researcher: PST11, what's your opinion?

PST11 (G8): I don't think it is 0.35%, the instantaneous rate of change is 35%, I guess.

Researcher: What does the percentage mean here?

PST11 (G8): We just reported 0.35 as 35%, I mean we just converted the decimal into a percentage. We mean the amount of change, the cost increased by 0.35 Turkish Liras (₺). Yes, the percentage expression can be wrong…

PST1 (G8): I think there is no difference between saying the cost increases by 0.35 or it increases by 35%

PST11 (G8): But here the increase in the cost is 0.35₺. There is a huge difference when we say it increases by 35%. Here, the cost increases by 0.35₺ per unit production. But if we say it increases by 35%, it means the cost will be 135 when it was 100.

Researcher: Ok, PST15 what do you think? You said, “the rate of change in cost at the 2000th meter of production.”

PST15 (G4): I am not sure about percentage interpretation. But $M^{'}(2000) = 0.35$ means the rate of change at 2000.

Researcher: Ok PST15, what does this mean exactly? Is this value indicating a change in Liras or something else? What does this value exactly tell us?

PST15 (G4): Hmmm, 0.35... Can we say a percentage change? I am not sure.

PST9 (G4): We still have difficulty interpreting 0.35. Will we add this to the current cost value (800 ₺) or do we interpret it as a percentage of change? We are not sure.

As seen in the episode above, PSTs had difficulty in interpreting the meaning of derivative ($M'(2000) = 0.35$) even though they correctly labelled its unit. About half of the groups explained this value as the “amount of change” in the cost function and about 25% of the groups provided an expression “rate of change in cost”. However, even the groups indicating the verbal expression of “rate of change” had difficulty in interpreting the meaning of it in the context. One different answer provided by G8 was indicating the percentage of change in the cost. PST11 and her group members had difficulty explaining when the researcher asked the meaning of “35% increase in the cost”. PST11 realized that this expression means the cost changed from 100 units to 135 and this could not be true in this context, and she returned to the amount of change conception. One interesting situation here was the groups who provided the true expression “rate of change in cost” also indicated their difficulty in interpreting what that means in reality. So, we realized here that PSTs might have difficulties in interpreting the rate of change as smooth continuous covariation or relative size even if they use the true verbal expression.

Researcher: Can we interpret 0.35 as “the change in the cost will be 35% of the change in production”, or the change in cost will be 0.35 times as large as the change in production, because it is a rate. If the production changes one meter, the cost will change 0.35 multiple of it, or if the production change with 0.2 meters, then the cost will change 0,070₺. [Silence about 3 seconds]

PST19 (G6): Yes exactly, it is reasonable. I am clued in now. We could not think this way.

The researcher provided an explanation involving relative size. Although some of the PSTs realized the idea of relative size through the explanation of the researcher, the difficulties continued. The following excerpt shows how PSTs tried to interpret the meaning of the instantaneous rate of change.

PST21 (G1): In fact, I could not understand if this is an increase in cost or not, but we reported it as a rate of change in cost at the 2000^{th} meter of production.

Researcher: Again, what does 0.35 stand for here? We try to make it meaningful.

PST8 (G8): It is really difficult, I got so confused.

PST2 (G7): Let’s think of this situation on a graph. It should be a line or curved graph, the meter of production is the independent and cost is the dependent variable. If we plot the point 2000, the numerical value 0.35 is the slope of the tangent line at that point.

Researcher: Ok, exactly. What does this slope tell us in this context?

PST2 (G7): It tells us if the function is increasing or decreasing. It also tells us the cost is increasing at the 2000th meter of production.

Researcher: Can we guess the cost at the 2001st meter of production by the slope?

PST2 (G7): Yes, we can. We intuitively know the graph is increasing. The cost will increase by the increase in production.

Researcher: Then, can you tell me the cost at the 2001st meter of production?

PST2 (G7): I can't say, because we don't know the algebraic formula.

Researcher: We know the total cost for producing 2000 meters of rope is 800₺. We also know the instantaneous rate of change in cost (0.35 $₺/rope$). Can we guess the total cost for producing 2001 meters of rope?

PST17 (G7): We did it, we thought of the cost as a linear function with a slope of 0.35. The cost will be 800.35 for producing the 2001st meter of rope. But this will be a rough guess…

PST8 (G8): This is not true. We can compute a numerical value, but this will not be the exact value. Because we don't know the algebraic function.

PST8 (G8): I agree, we cannot compute. We don't know the function.

The above episode shows although they used the verbal expression of "rate of change in cost". PST21 and her group members had difficulty in interpreting the meaning of it. Other groups had similar difficulties. This result shows that using the "rate of change in cost" verbal expression does not always show one's understanding of its contextual meaning. Moreover, the idea of marginal cost (change in total cost coming from producing one additional unit) did not seem to make sense to PSTs. Although the instructor tried to direct them, PSTs did not interpret the derivative in the cost-product context as the economists' way of understanding (marginal cost). The researcher asked what the approximate cost value at the 201st meter of rope production would be. Most of the PSTs stated that they could not find the cost value without the algebraic function. The researcher insistently asked PSTs about the approximate value of cost at the 201st meter of production and he clearly stated, "could it be 800.35₺", this idea did not make sense to them. PST2 and PST8 indicated that the exact value of the cost function could not be found without knowing the algebraic function. The interesting point here was that although some of PSTs conceived 0.35 as an amount change in cost at 200th meter of production, they did not interpret it as the cost added by producing one additional meter of production. By the way, during and after the implementation process of this task, PSTs were not provided any technical information about the marginal cost meaning of derivative. The reported data here demonstrate PSTs' ways of interpreting derivative in an economic context.

Another outstanding result observed in this task was PSTs' insufficient knowledge about the cost-production context. Although most of the groups correctly interpreted the dependent and independent variables with their corresponding units in the given situation, their lack of contextual knowledge was observed in their interpretations about the long-term behaviour of the cost-production function. While seven groups reported that the cost-production was an increasing function, only one (G8) of the groups stated that nothing can be said about the long-term behaviour of the cost-production function. Only one group (G6) could report the cost function was increasing at a decreasing rate (in their terms, "Cost increases decreasingly"). Other groups either thought as if there was a linear relationship between cost and production (G3, G5) or they just indicated "the cost increases" (G1, G2, G4, G7). The discussions on the behaviour of the cost-product function took a long time. Most of the PSTs indicated that the long-term behaviour of the cost-product function cannot be known without its algebraic equation. At the end of the activity, PST14 indicated his view as follows:

PST14 (G6): … we could not reach a common understanding about the long-term behaviour of the function. Nothing changed after the discussion, I think still everybody thinks whatever they think before. The question is open to different ways of thinking and every way of thinking has some reasonable aspects…

Researcher: What would it be in reality?

PST14 (G6): In fact, I believe we cannot add our personal interpretations to mathematical questions… We only look at what we are given in the question context. Therefore, even if "the cost function increases at a decreasing rate" seems a reasonable way of thinking that some of our friends indicated, we were not provided any information in the task that we can reach this conclusion.

In the episode above, PST14 indicated the impossibility of making such a conclusion about the long-term behaviour of the cost-product function as no extra information was provided in the problem context. PST14's expression of "we cannot add our own interpretations to mathematical problems" gives important clues about his beliefs about mathematics and the nature of mathematical problems. PSTs' lack of general knowledge about cost-production context resulted in some inconsistent interpretations about the comparison of derivative at different points and the long-term behaviour of the derivative function.

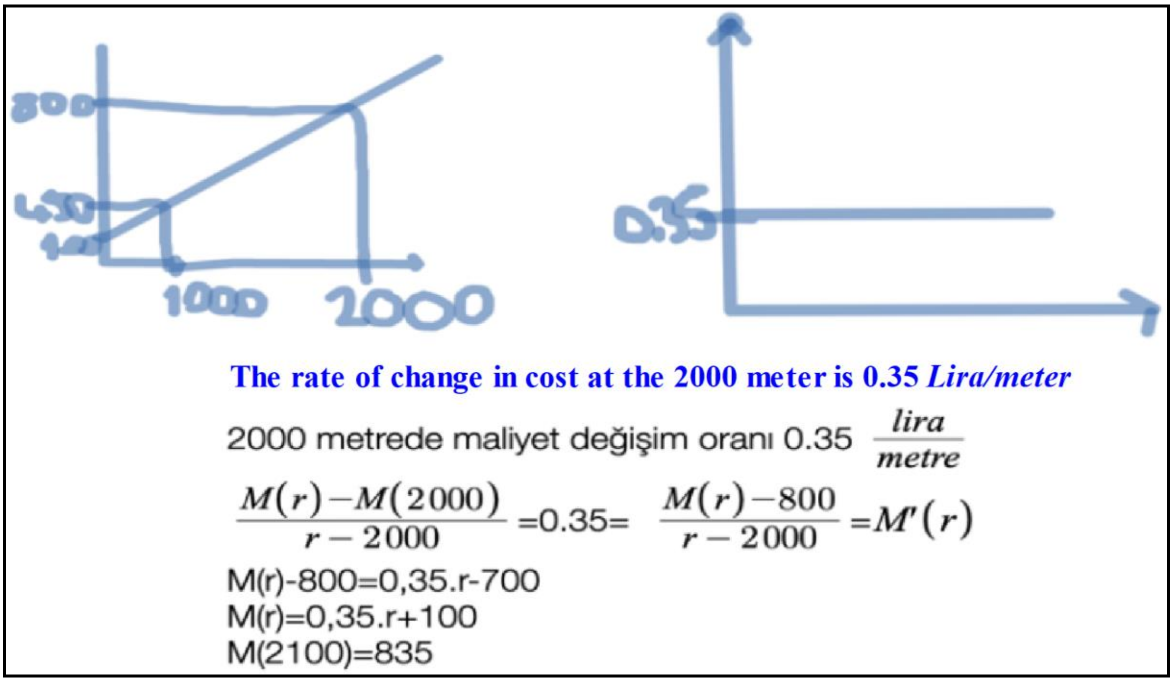

Figure 2. G5's way of reasoning about the cost function and its derivative

In Figure 2, G5's linear way of reasoning is observable. They considered the cost-production function as a linearly increasing function and therefore they assumed a constant rate of change. Based on this way of reasoning, G5 calculated the approximate value of $M(2100)$ as seen in the figure. About the question, "if the expression $M'(5000) = -0.1$ is possible or not", they indicated the impossibility of getting this value again depending on their conception of the constant rate of change. G1's way of reasoning about the long-term behaviour of cost-production function was "cost increases at an increasing rate". Therefore, they concluded that the numerical value of $M'(2000)$ was less than the numerical value of $M'(3000)$. Regarding the numerical expression $M'(5000) = -0.1$, members of G1 indicated that "In this situation, the revenue is less than the cost and so the company will lose money." G6 was the only group consistently indicating "cost increases at a decreasing rate". These results show PSTs' lack of general knowledge about an economic context not only in terms of the derivative but also of the function concept.

Task 2. Population

The second task was about the world population between the years 1998-2010. A tabular data was provided with unequal year intervals and PSTs were asked to interpret an algebraic expression involving an average rate of change and the meaning of derivative at particular years. As represented in Table 5, 24% of PSTs explained the meaning of derivative as an "amount of population change." About 44% of them provided an explanation considering derivative as the rate of change in population with respect to time. About 20% of PSTs used a chunky way of thinking on the rate of change like "yearly population change". Only G2 and G5 consistently used rate of change or rate of change (chunky) conceptions in all sub-questions of this task. Compared to the previous task, the increase in the frequency of usage of "rate of change" expression was observable.

In one of the sub-questions in the task, PSTs were asked to compute the average rate of change in population between the year intervals of 1999 and 2010 and the meaning of this value in the given context. The following excerpt from a classroom discussion shows PSTs' interpretations of the numerical value that they obtained:

> Researcher: What does this value mean in the given context?
>
> PST17 (G7): For us, it indicates the change in population between the 1999-2010-year interval. It is a rate, and it also shows the increase in population in that year interval.
>
> PST22 (G5): We found the same value (74.8), but we interpreted this value as yearly population change or rate of change in population with respect to years. I mean, it shows the yearly population change on average.
>
> PST5 (G3): We also used "yearly population change".
>
> Researcher: So, what does 74.8 mean in this context?
>
> PST17 (G7): We stated that this value shows the increase in population in 11 years, but we did not mean this increase happened every year.
>
> PST24 (G5): I think, this value shows the yearly population increase as an average value of 11 years. I mean, the increase in population may be 73 million for one year, it may be 75 for the other year. This is an average value.
>
> Researcher: What can be the better verbal expression for this value?
>
> PST5 (G3): I think, the yearly change or per year can be true.
>
> Researcher: What about the expression "rate of change in population with respect to year" provided by G2?
>
> PST17 (G7): I think, "Average rate of change" seems better, doesn't it?
>
> PST6 (G2): We can add an "average" expression, yeah…
>
> PST8 (G8): This expression (rate of change in population with respect to year) is not wrong. It is a different way of stating the yearly population change.

As seen in the episode from the classroom discussion, PSTs tried to make clear the meaning of the "average rate of change" in the population context. PSTs interpreted 74.8 as yearly population change. Yearly population change (chunky way of reasoning) seemed to be the dominant idea that PSTs used in the population context. PST24 tried to explain the importance of adding the "average" expression. Although PSTs agreed upon the "average rate of change in population with respect to years", they preferred to use yearly population change.

The most difficult part of this task for PSTs was interpreting the derivative at a point. They had difficulty in interpreting the meaning of derivative at a particular year. They were asked to find the approximate value of $P'(2000)$ and to provide an explanation of its meaning. Because tabular data was provided, four of the groups obtained an algebraic formula by using MS Excel (G1, G6, G7, G8). By using the algebraic formula, they computed the derivative value of $P'(2000)$. Three groups (G2, G4, G5) used the tabular data for calculating the value of $P'(2000)$ by using the left-hand side or right-hand side by narrowing down the interval. However, PSTs could not provide a satisfactory explanation about the meaning of derivative even though they indicated the technical expression like "it's the instantaneous rate of change in population with respect to time". Some of the explanations provided PSTs during the classroom discussion are provided below:

> Researcher: How did you compute the approximate value of $P'(2000)$?
>
> PST2 (G7): Because the derivative at 2000 is asked, it is impossible to narrow down the interval by using tabular data. Therefore, we obtained a formula by using MS Excel.
>
> PST25 (G2): We also obtained a linear graph by using MS Excel, but we did not use it. We preferred approximating from the left and right sides by using the given data.
>
> PST9 (G4): We computed the slope approximating the year from the left and right sides, and we took the average of these two values.
>
> Researcher: You found different values, it is ok. What does this value mean?
>
> PST2 (G7): It is the rate of change in population or yearly population change. But it is difficult to think of this value as an instantaneous rate because 1 year is too long an interval.
>
> PST8 (G8): Rate of change, the instantaneous rate of change, or average rate of change, I wonder if these expressions have differences for this situation. We can accept 1 year as a big interval, and so this can be accepted as an average rate of change.
>
> PST5 (G3): Derivative at a point is an instantaneous rate of change, not average.

Some of the groups benefited from MS Excel to obtain an algebraic linear function. By using the algebraic function, they procedurally computed the derivative value at 2000. Some other groups used the left and right-hand side approximation for finding the derivative. However, PSTs had difficulty in interpreting the meaning of derivative in 2000. Although they used (instantaneous or average) rate of change expressions, they did not seem to understand this value as relative size as also observed in the previous task. The first difficulty here was about scaling of the time variable (year, month, week, day, hour, minute, second…) that requires thinking about the meaning of 1 hour or minute, in terms of a year. And secondly, PSTs' lack of relative size conception of derivative was also observed here. They could not provide a satisfactory explanation about the meaning of derivative in terms of the population and year variables, for example, "population change is a multiple of (multiplied by the derivative value) change in year."

Task 3. Price of a Car

The third task was about the price of a car as a function of its mileage. In this task, tabular data was provided. All groups demonstrated a clear understanding of the situation in terms of the functional relationship between the price and mileage variables as appeared in their answers to the sub-questions about the variables and their units.

As seen in Table 5, 33 % of the answers were involving the meaning of derivative as an "amount of price change." About 40% of them provided an explanation considering derivative as the rate of change in the price of a car with respect to mileage. Only 4 % of PSTs used a chunky way of thinking about the rate of change like "change in price per unit mileage". G1, G5, and G7 consistently used rate of change or rate of change (chunky) conceptions. In this task, the frequency of using the "rate of change" conception relatively decreased, and the "amount of change" conception increased when compared with the population task. One of the sub-questions in the task was asking about the meaning of $V'(300) = -0.21$. As also observed in previous tasks, PSTs had difficulties in explaining the meaning of derivative in price-mileage context even if they used the true terminology of the rate of change. Some of the answers from group reports are provided below.

G1: This expression [$V'(300) = -0.21$] shows the instantaneous rate of change in the price of a car, and it is -0.21 ₺/km. This shows a decrease in the price of the car (Written report of G1).

G5: The value of the car decreased at a rate of 0.21 ₺/km when its mileage reached to 300.000 kilometres. It also means "rate of change in the price of the car at 300.000 kilometres." (Written report of G5).

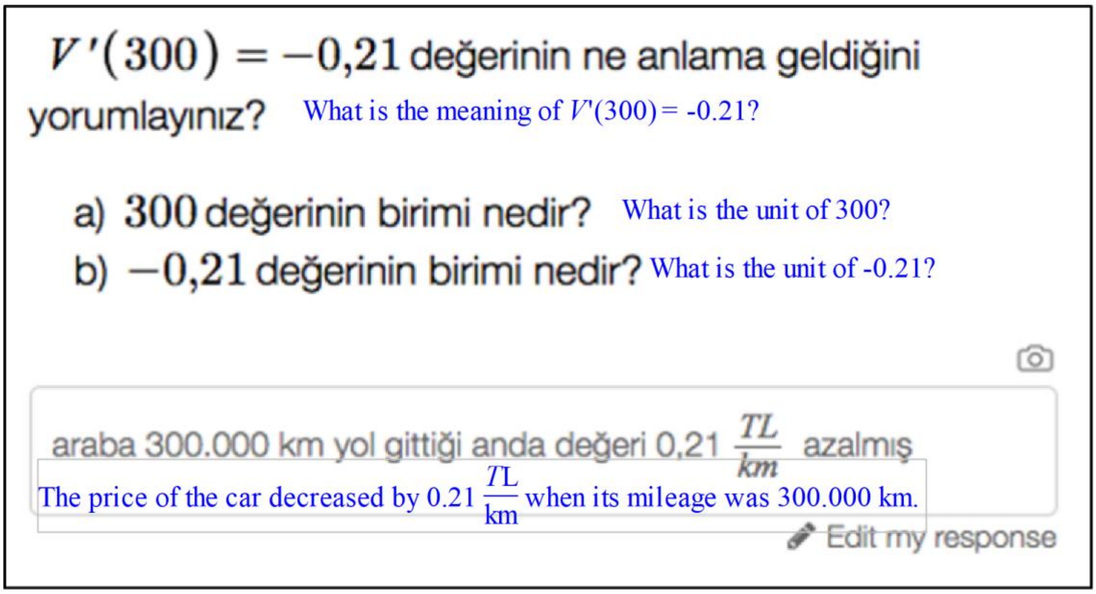

Figure 3. A screenshot from G2's group report in Desmos

When the verbal expressions analysed in the group reports, G1 and G5 used rate of change, but G2 and G6 used an amount of change in interpreting the derivative value (-0.21). G2's answer involves an amount of change conception of derivative as they interpreted -0.21 as the amount of decrease in the price of the car. During the classroom discussion, we asked about the contextual meaning of derivative.

Researcher: What does $V'(300) = -0.21$ mean in this context?

PST1 (G8): As the derivative is negative here, the function is decreasing. We can say, the rate of decrease in the price of the car is 0.21 at the 300000th meter of mileage.

Researcher: Ok, what does this rate of change value tell us? For example, if the mileage increased by 2 meters, what can we say about the new price?

PST4 (G3): We cannot know the exact value as we don't know the algebraic function. We can only say, the price will decrease as the sign is negative.

PST15 (G4): I agree, if this value was positive, we would say an increase in the price.

PST2 (G7): It shows the instantaneous rate of change in the price of a car when its mileage is at 300 thousand kilometres. Or it shows the slope of the tangent line drawn at the given point.

As seen in the episode above, PSTs can explain the meaning of derivative by using verbal expressions such as rate of change (rate of decrease) in price. They can also figure it out as the slope of the tangent line. However, as also observed in previous tasks, PSTs had difficulties in interpreting and using the derivative value as a relative size between changes in two quantities. In other words, they are not familiar with using a linear approximation of $\frac{\Delta V}{\Delta d} \cong -0.21 \rightarrow \Delta V \cong -0.21 \times \Delta d$ for guessing the value of a function around a point very near to the reference point.

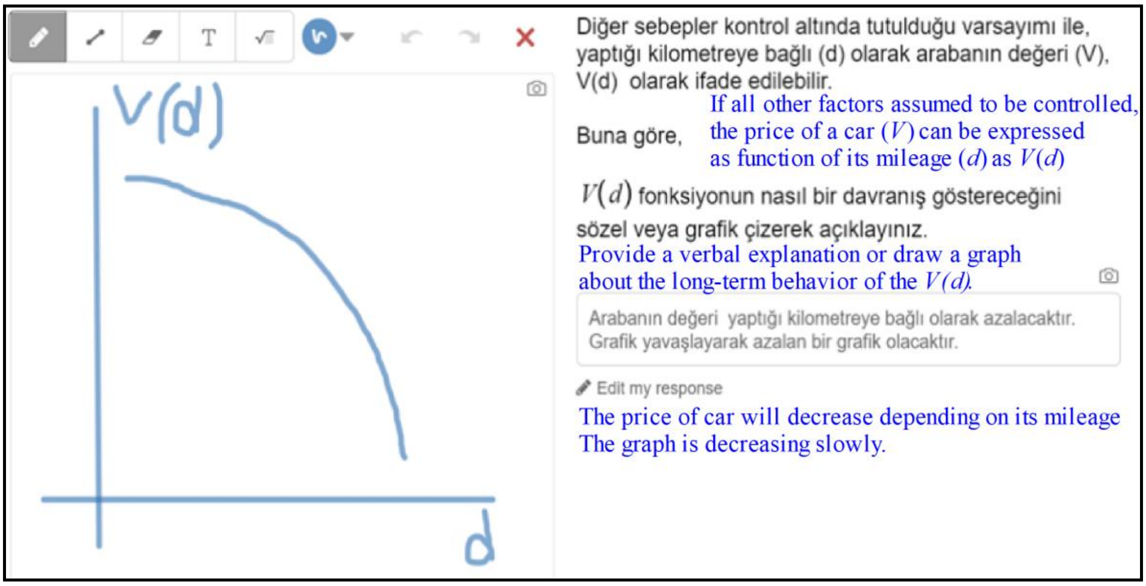

Figure 4. A screenshot of G7's graph

Moreover, again PSTs' insufficient general knowledge about the price-mileage context was observable. Even though all groups seemed to be aware of the long-term behaviour of price-mileage function was decreasing, some of PSTs had difficulties in specifying the character of the decreasing function. As seen in Figure 4, G7 drew a concave-down decreasing graph for the price-mileage function.

Task 4. The Filling Bottle

Filling Bottle task was asking about the relationships between height, volume, and radius of cross-section variables as the bottle was being filled with a constant flow rate. Specifically, PSTs were asked about the (i) radius of cross-sections versus the height of water, (ii) height versus volume, and (iii) radius of cross-sections versus volume. As the measures of the conical shape were provided, PSTs could create an algebraic model for the volume as a function of height or volume as a function of the radius of cross-sections. They could also create rough graphs for each of the related variables.

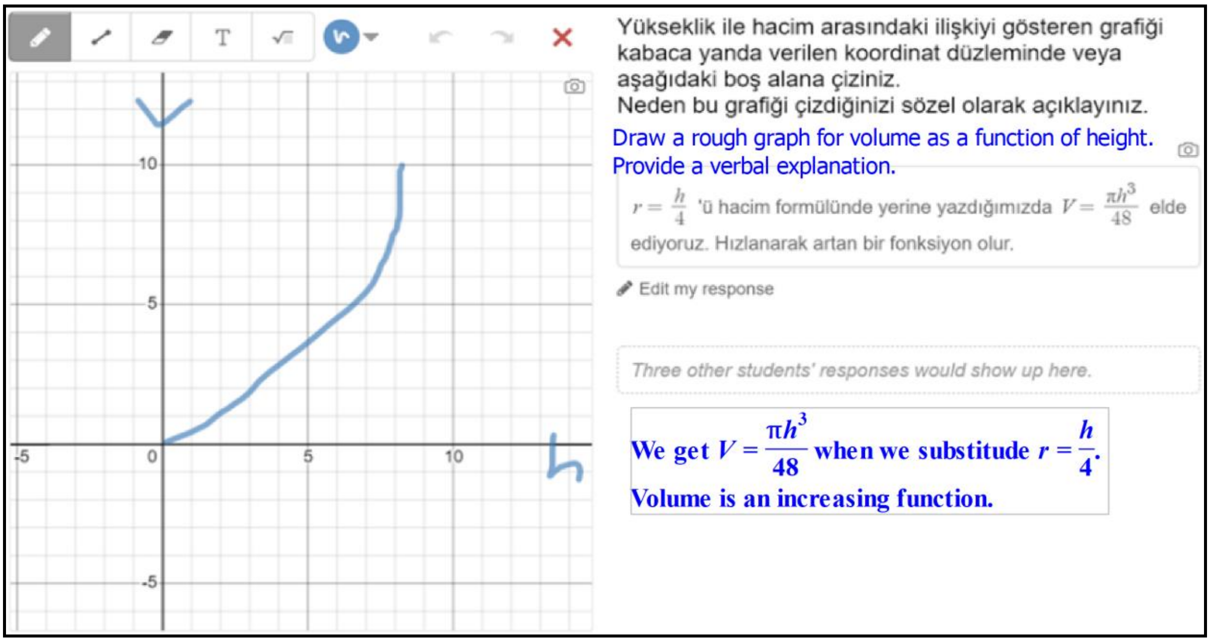

Figure 5. A screenshot of G5's drawing of volume as a function of height in Desmos

In the task, PSTs were also asked about the meaning of derivatives in volume-height (and volume-radius of cross-sections) context. We asked them to explain the meaning of $V'(h)$ (the derivative of volume-height function) and $V'(3)$ in the given context. We also asked the meaning of $V'(r)$ which is the derivative of the volume-radius function. Their answers to these questions in the group reports were analysed and presented in Table 5.

As seen in Table 5, slope, difference quotient and rate of change (chunky) conceptions of derivative did not appear in group reports. Most of the groups (93.7%) consistently used the rate of change verbal expression while explaining the symbolic expressions $V'(h)$, $V'(3)$, and dV/dr. They frequently provided explanations such as "instantaneous rate of change in volume as a function of height", "rate of change in volume with respect to the radius of cross-sections". This shows PSTs started to use the true verbal expression of rate of change while explaining derivative in different contexts.

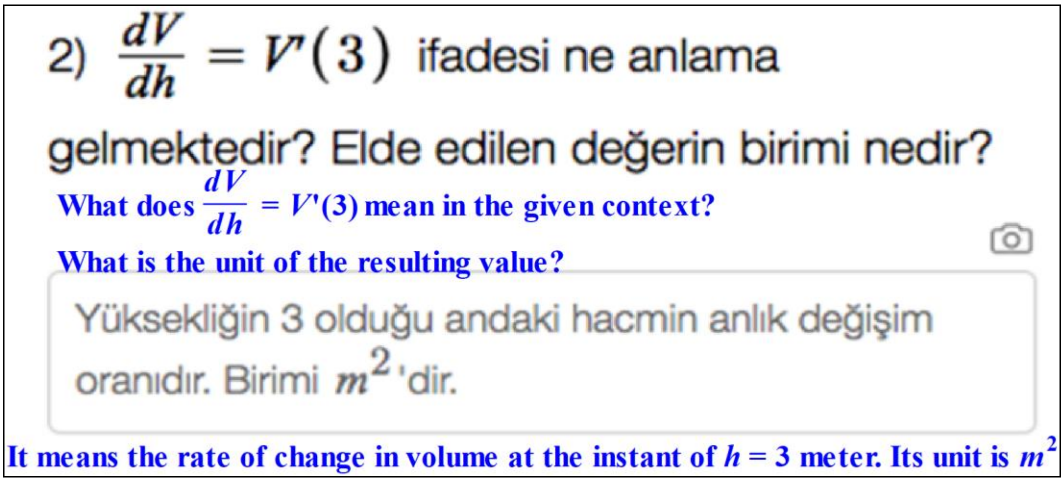

Figure 6. G4's explanation for the contextual meaning of derivative at a point

However, again PSTs had difficulties with the contextual meaning of rate of change and derivative. Just by looking at the units of the variables ($\frac{m^3}{m} = m^2$), some of the groups interpreted $V'(h)$ as a measure of an area. In the group reports, only G1 and G8 could realize that $V'(h)$ should produce a measure of an area, but they had difficulty in interpreting where this area was. For example, G1 used the explanation that "the derivative of volume gives an area", but they could not specify where this area was. G8 was the only group that can interpret $V'(h)$ as the area of cross-sections, but they had difficulty in explaining $V'(r)$ (according to them this can be the surface area of a sphere with radius r). A small episode from the classroom discussion on this issue is provided below.

> Researcher: You explained $V'(h)$ as the rate of change in volume as a function of height, and its unit is m^2. What does that mean?
>
> PST8 (G8): Can we think of $V'(h)$ as a surface area.
>
> Researcher: You stated it as the rate of change in your verbal explanations. What does dV/dh mean, if it is an area?
>
> PST4 (G3): It seems to me as if it is a projection from 3-dimension to a 2-dimension. The projection of three-dimension can give a two-dimensional cross-section, it is an area…
>
> PST17 (G7): As the bottle is filled up with water, the cross-sectional area is gradually changing. Can it be related to the area of cross-sections?
>
> PSt5 (G3): Yes, the sum of the cross-sectional areas will give the volume.
>
> Researcher: What do you think? The rate of change in volume as a function of height gives us the area of cross-sectional circles…
>
> PST2 (G7): We can prove it algebraically. The unit of the derivative is m^2 and this unit shows an area. If we write down the volume formula ($V = \pi . r^2 . h$) and if we divide it by height, we will get the area of the cross-section at a given height.
>
> PST17 (G7): Exactly, it gives us a cross-section area without any height because the delta h value approaches 0.
>
> Researcher: Could you realize at first glance that the rate of change in volume as a function of height gives us a cross-sectional area?
>
> PST4 (G3): Actually, we did not think in that way. We only computed by using the volume-height units, but we did not think about the meaning of m^2.

In the episode above, PSTs tried to interpret the meaning of $V'(h)$.Although almost all of the groups provided a verbal explanation involving the rate of change, they had difficulty in interpreting its meaning in the given context. Based upon the unit of the $V'(h)$, they realized it could be related to an area, but they could not easily specify where the area was. PST17 (G7) indicated that it could be related to the cross-sectional areas as the cross-sections were gradually changing. And then, PST4 (G3) thought volume as the sum of overlapping cross-sectional areas. Then PST2 (G7) was also convinced, and he argued to prove this fact algebraically. Up to this discussion, even though PSTs used rate of change terms in their verbal explanations, they did not think about the physical meaning of derivative in the filling-water context. This was also indicated by PST4 (G3) as "we did not consider derivative here as an area of cross-sections."

PSTs' Self-Reflections about Their Learning Gains

After applying all the tasks, we asked PSTs to write a reflection paper to evaluate what they learned about derivative in this process. Most of the PSTs (16 out of 25) indicated that they deepened their understanding of derivative. They indicated that their understanding of derivative was procedural, and they have a more robust and multi-faced understanding of derivative after engaging in solving these tasks. Excerpts from three PSTs' reflection papers are presented below.

> PST21 (G1): I was thinking as if I knew, but during solving these tasks I realized that I did not know the relationship between derivative and rate of change exactly. I understood the meanings of average and instantaneous rate of change in a better way through the group and classroom discussions.

> PST23 (G5): While solving these tasks, I realized that my knowledge of derivative was very superficial. The group discussions and classroom discussions provided me a great opportunity to widen my knowledge of derivative. I believe I understood the meaning of derivative in different contexts.

> PST18 (G2): Unfortunately, our conception of the derivative is that it is the slope of the tangent line as most high school students have. We had difficulty in interpreting derivative with its different meanings other than slope. These activities helped us to broaden our understanding of the meaning of derivative…

As can be seen in the excerpts from PST21, PST23, and PST18's reflection papers, most of the PSTs think that solving these tasks provided them to foster their understanding of derivative with its different meanings in various contexts. These tasks created an opportunity for PSTs to discuss and think about the meaning of derivative. Moreover, PSTs also had an opportunity to observe derivative in non-motion contexts. PST7 (from G6) indicated in her reflection paper that "*Only the slope on the velocity-time graph was coming into my mind for exemplifying the derivative before, but now I can provide different examples*". Other PSTs also stated that they were familiar with the examples only from the motion context, but after engaging in solving these tasks PTSs became aware of the derivative examples from different contexts.

Discussion and Conclusions

We investigated PSTs' understanding and ways of interpreting derivative in different real-life contexts. We also analysed if there were developments in PSTs' conceptions of derivative across different contexts as they engaged in working on a series of tasks employing group and in-class discussions. The results showed that PSTs had difficulties in interpreting the meaning of derivative in different non-motion contexts. While their explanations of derivative were frequently involving the amount of change at the beginning, they started to use "rate of change" in their verbal explanations. They got familiar with the "rate of change" expression while interpreting the contextual meaning of derivative. Also, the diversity of the real-life contexts that they provided as examples about derivative increased by the post-questionnaire. However, it is difficult to say PSTs could be able to conceptualize the meaning of derivative as a new quantity in different contexts. As also observed in the study of Byerley and Thomson (2017), PSTs seem to express "rate of change" without involving in-depth reasoning about its contextual meaning as a measure of relative size between the smoothly covarying variables.

The findings of the current study are in line with the literature reporting student difficulties in interpreting the meaning of derivative in non-motion contexts (e.g., Doorman & Gravemeijer, 2009; Herbert & Pierce, 2008; Jones, 2017; Mkhatshwa & Doerr, 2018; Wilhelm & Confrey, 2003). In questionnaires, PSTs examples of contexts for the concept of derivative were dominantly from the motion context, but a relatively small increase in the diversity of contextual examples was observed by the post-questionnaire. They also visited the meaning of derivative as velocity in the physics context to make a meaningful interpretation about the derivative of the cost-product function. Moreover, PSTs did not interpret the derivative in the cost-product context as the economists' way of understanding (marginal cost). The interesting point here was that although most PSTs conceived derivative as an amount change in cost (at the 200^{th} meter of production), they did not interpret it as the cost added by producing one additional meter of production. PSTs had similar difficulties in interpreting derivative in population, price-mileage, and volume-height contexts.

From the perspective of quantitative reasoning, different real-life contexts require different ways of conceiving the variables and so of the derivative (Jones & Watson, 2018; Thompson & Carlson, 2017). Even if PSTs started to use the "rate of change" in their verbal expressions, the need for supporting them in conceiving derivative as a new quantity with its meaning in the given context appeared. PSTs had difficulties in conceiving the derivative as a new quantity and attending to its meaning even though they used the "rate of change" expression in a memorized way. For example, in the volume-height context, derivative indicates the rate of change in volume as a function of height as also indicated by most of the PSTs, but physically it also means the area of cross-sections. In the population context, PSTs explained the rate of change as "yearly population change" and they did not conceive it as a rate as also observed in the study by Kertil and others (2017).

From the mathematical point of view, the marginal cost interpretation of derivative has some problematic aspects in terms of underestimating the infinite small differentiation (Feudel, 2016, 2017, 2018; Mkhatshwa & Doerr, 2018). It emphasizes a chunky way of reasoning (change in cost by one additional unit of product) and may result in conceptualizing derivative as an amount of change as also observed in this study (Mkhatshwa, 2018; Mkhatshwa & Doerr, 2018). However, the idea of marginal cost (profit or revenue) or "per unit change" can be used as a steppingstone for conceptualizing derivative as a relative size between two smoothly covarying quantities if it can be supported with the use of the linear approximation method. The covariational reasoning literature reports that this chunky way of reasoning may be robust enough to make true mathematical inferences about the concavity of function graph (Johnson, 2012; Thompson & Carlson, 2017) and it can easily be shifted to a relative-size conception of the rate of change (Kertil et al., 2019). In other words, we can ask students to guess in the change of a dependent variable as a result of different (small) amounts of change in the independent variable (other than 1 unit) provided with a derivative value. We realized that PSTs need more practice for finding an approximate value of a function at a point (without an algebraic formula) based on its derivative value as we tried to do during in-class discussions. These practices seem to have the potential to provide students or PSTs with the meaning of derivative and rate of change.

Although it was not a focused issue at the beginning of the study, in the progress we also observed PSTs' lack of context knowledge concerning the mathematical functions. They drew, for example, linear or concave-up increasing graphs for cost-production context. And similarly, they drew a linearly increasing graph for population context and a linearly decreasing graph for price-mileage context. The graphs and PSTs arguments about why they preferred to draw them indicated their lack of context knowledge not only in terms of derivative but also for functions. As also addressed in the PISA competency framework (OECD, 2018), knowledge about which functions can be used to model what kind of real-life situations is an important indicator of students' or teachers' mathematical modeling competencies or their level of mathematical literacy (Henning & Keune, 2007; Maaß, 2006; Niss & Jensen, 2011).

Lastly, this study can be seen as an effort of searching for effective professional support for PSTs in developing conceptions of derivative across different real-life contexts. The tasks used in this study were selected from different real-life contexts, but we only focused on the verbal (or rate of change) interpretation of derivative (Jones & Watson, 2018; Zandieh, 2000). We strongly suggest future intervention studies aiming at developing teachers' content knowledge of derivative covering its different representations with ratio, limit, and function layers. The results of this study may also contribute to the awareness of the mathematics education community about the commonly lacking aspects of teachers' content knowledge resulting from the long-term effect of their way of teaching and learning school mathematics. PSTs need long-term experiences and practices for internalizing contextual meanings of mathematical concepts. Therefore, the content of high school (or undergraduate) algebra or calculus courses, curricular materials, and textbooks in terms of the tasks, problems, and real-life contexts they have been presented with needs substantial reconsideration.

References

Akkoç, H., Bingolbali, E., & Özmantar, F. (2008). Investigating technological pedagogical content knowledge: A case of derivative at a point. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda, A. (Eds.), *Proceedings of the Joint Meeting of PME 32 and PME-NA XXX (Vol. 2, pp. 17-24). México: Cinvestav-UMSNH*

Bektaşlı, B., & Çakmakçı, G. (2011). Consistency of students' ideas about the concept of rate across different contexts. *Education and Science, 36* (162), 273-287.

Berry, J. S., & Nyman, M. A. (2003). Promoting students' graphical understanding of the calculus. *The Journal of Mathematical Behavior*, *22*(4), 481–497. https://doi.org/10.1016/j.jmathb.2003.09.006

Bezuidenhout, J. (1998). First year university students' understanding of rate of change. *International Journal of Mathematical Education in Science and Technology*, *29* (3), 389-399. https://doi.org/10.1080/0020739980290309

Bingolbali, E. (2008). Türev kavramına ilişkin öğrenme zorlukları ve kavramsal anlama için öneriler. M. F. Özmantar, E. Bingolbali, & H. Akkoç (Eds.), *Matematiksel kavram yanılgıları ve çözüm önerileri* (pp. 223–255). Ankara: Pegem A.

Byerley, C. (2019). Calculus students' fraction and measure schemes and implications for teaching rate of change functions conceptually. *The Journal of Mathematical Behavior*, *55*, 100694. https://doi.org/10.1016/j.jmathb.2019.03.001

Byerley, C., & Thompson, P. (2017). Secondary mathematics teachers' meanings for measure, slope, and rate of change. *Journal of Mathematical Behavior*, *48*, 168-193. https://doi.org/10.1016/j.jmathb.2017.09.003

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33(*5), 352–378. https://doi.org/10. 2307/4149958

Castillo-Garsow, C. (2012). Continuous quantitative reasoning. In R. Mayes & L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context (WISDOMemonograph)* (Vol. 2). Laramie, WY: University of Wyoming.

Cooney, T. J., Beckman, S., & Lloyd, G. M. (2010). *Developing essential understanding of functions for teaching mathematics in grades 9-12*. Reston, VA: National Council of Teachers of Mathematics.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26*, 135–164. https://doi.org/10.2307/749228

Doorman, L. M., & Gravemeijer, K. P. E. (2009). Emergent modeling: discrete graphs to support the understanding of change and velocity. *ZDM*, *41*(1-2), 199-211. https://doi.org/10.1007/s11858-008-0130-z

Duran, M., & Kaplan, A. (2016). Lise matematik öğretmenlerinin türevin tanımına ve türev-süreklilik ilişkisine yönelik pedagojik alan bilgileri [High school teachers' pedagogical content knowledge on definition of derivative, and relationship between derivative and continuity]. *Erzincan University Journal of Education Faculty*, *18*(2), 795-831. https://doi.org/10.17556/jef.68600

Feudel, F. (2016). Relevant knowledge concerning the derivative concept for students of economics – A normative point of view and students' perspectives. In E. Nardi, C. Winsløw, & T. Hausberger (Eds.), *Proceedings of the first conference of International Network for Didactic Research in University Mathematics s (pp. 181–190)*. Montpellier, France: University of Montpellier and INDRUM.

Feudel, F. (2017). Students' interpretation of the derivative in an economic context. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the 10th congress of the european society for research in mathematics education*. Dublin, Ireland: DCU Institute of Education & ERME.

Feudel, F. (2018). C'(x) = C(x+1)-C(x)? - Students' connections between the derivative and its economic interpretation in the context of marginal cost. In V. Durand-Guerrier, R. Hochmuth, S. Goodchild, & N. M. Hogstad (Eds.) *Proceedings of INDRUM 2018 second conference of the International Network For Didactic Research In University Mathematics* (pp. 95-104). Univer Kristiansand, Norway: University of Agder and INDRUM

Gökçek, T., & Açıkyıldız, G. (2016). Matematik öğretmeni adaylarının türev kavramıyla ilgili yaptıkları hatalar [Preservice mathematics teachers' errors related to derivative]. *Turkish Journal of Computer and Mathematics Education*, *7*(1), 112. https://doi.org/10.16949/turcomat.14647

Henning, H., & Keune, M. (2007). Levels of modelling competencies. In W. Blum, P. L. Galbraith, H. Henn & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 225-232). US: Springer.

Herbert, S., & Pierce, R. (2008). An "Emergent Model" for rate of change. *International Journal of Computers for Mathematical Learning*, *13*, 231–249. https://doi.org/10.1007/s10758-008-9140-8

Herbert, S., & Pierce, R. (2011). What is rate? Does context or representation matter? *Mathematics Education Research Journal*, *23*, 455-477. https://doi.org/10.1007/s13394-011-0026-z

Herbert, S., & Pierce, R. (2012). Revealing educationally critical aspects of rate. *Educational Studies in Mathematics*, *81*, 85-101. https://doi.org/10.1007/s10649-011-9368-4

Johnson, H. L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *The Journal of Mathematical Behavior*, *31*(3), 313-330. https://doi.org/10.1016/j.jmathb.2012.01.001

Jones, S. R. (2017). An exploratory study on student understandings of derivatives in real-world, non-kinematics contexts. *Journal of Mathematical Behavior, 45*, 95-110. https://doi.org/10.1016/j.jmathb.2016.11.002

Jones, S. R., & Watson, K. L. (2018). Recommendations for a "target understanding" of the derivative concept for first-semester calculus teaching and learning. *International Journal of Research in Undergraduate Mathematics Education*, *4*(2), 199-227. https://doi.org/10.1007/s40753-017-0057-2

Kertil, M. (2014). *Pre-service elementary mathematics teachers' understanding of derivative through a model development unit* (Unpublished doctoral dissertation). Middle East Technical University, Ankara, Turkey.

Kertil, M., Erbaş, A. K., & ve Çetinkaya, B. (2017). İlköğretim matematik öğretmen adaylarının değişim oranı ile ilgili düşünme biçimlerinin bir modelleme etkinliği bağlamında incelenmesi [Pre-service elementary mathematics teachers' ways of thinking about rate of change in the context of a modeling activity]. *Turkish Journal of Computer and Mathematics Education,* *8*(1), 188-217. https://doi.org/10.16949/turkbilmat.304212

Kertil, M., Erbas, A. K., & Cetinkaya, B. (2019). Developing prospective teachers' covariational reasoning through a model development sequence. *Mathematical Thinking and Learning, 21*(3), 207-233. https://doi.org/10.1080/10986065.2019.1576001

Kertil, M., & Küpcü, A. R. (2021). Prospective mathematics teachers' ways of reasoning on differentiability and corner points in a real-life context. *International Journal of Mathematical Education in Science and Technology*, *52*(9), 1361-1384. https://doi.org/10.1080/0020739X.2020.1766138

Maaß, K. (2006). What are modelling competencies? *ZDM - The International Journal on Mathematics Education*, *38*(2), 113–142. https://doi.org/10.1007/BF02655885

Mkhatshwa, T. P. (2018). Business calculus students' interpretations of marginal change in economic contexts. In Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.), *Proceedings of the 40th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 564-571). Greenville, SC: University of South Carolina & Clemson University.

Mkhatshwa, T. P. (2020). Calculus students' quantitative reasoning in the context of solving related rates of change problems. *Mathematical Thinking and Learning*, *22*(2), 139-161. https://doi.org/10.1080/10986065.2019.1658055

Mkhatshwa, T., & Doerr, H. M. (2016). Opportunity to learn solving context-based tasks provided by business calculus textbooks: An exploratory study. In T. Fukawa-Connelly, N. E. Infante, M. Wawro, & S. Brown (eds.), *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1124–1132). Pittsburgh, PA: SIGMAA.

Mkhatshwa, T., & Doerr, H. (2018). Undergraduate students' quantitative reasoning in economic contexts. *Mathematical Thinking and Learning*, *20*(2), 142-161. https://doi.org/10.1080/10986065.2018.1442642

Niss, M., & Jensen, T. H. (Eds.) (2011). Competencies and Mathematical Learning – Ideas and inspiration for the development of mathematics teaching and learning in Denmark. Denmark: Roskilde University.

Organisation for Economic Co-operation and Development [OECD] (2018). *Preparing our youth for an inclusive and sustainable world*. *The OECD PISA global competence framework*. Paris: OECD

Rodríguez-Nieto, C. A., Rodríguez-Vásquez, F. M., & García-García, J. (2021). Pre-service mathematics teachers' mathematical connections in the context of problem-solving about the derivative. *Turkish Journal of Computer and Mathematics Education, 12* (1), 202-220. https://doi.org/10.16949/turkbilmat.797182.

Rowland, D. R., & Jovanoski, Z. (2004). Student interpretation of the terms in first-order ordinary differential equations in modeling contexts. *International Journal of Mathematical Education in Science and Technology*, *35*(4), 505-516. https://doi.org/10.1080/00207390410001686607

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks, CA: Sage.

Teuscher, D. & Reys, R. E. (2012). Rate of change: AP calculus students' understandings and misconceptions after completing different curricular paths. *School Science and Mathematics*, *112*(6), 359–376. https://doi.org/10.1111/j.1949-8594.2012.00150.x

Thompson, P.W. (1994a). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, *26*, 229- 274. https://doi.org/10.1007/BF01273664

Thompson, P.W. (1994b). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181–236). New York: State University of New York Press.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. A. Chamberlin, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOMe [WISDOMe monograph volume 1]* (pp. 33–57). Laramie: University of Wyoming.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

Saldanha L. & Thompson P. (1998). Re-thinking covariation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berenson & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education – North America* (Vol. 1, pp. 298-304). Raleigh, NC: North Carolina University

Wilhelm, J. A., & Confrey, J. (2003). Projecting rate of change in the context of motion onto the context of money. *International Journal of Mathematical Education in Science and Technology*, *34*(6), 887-904. https://doi.org/10.1080/00207390310001606660

Yin, R. K. (2011). *Qualitative research from start to finish*. New York: The Guildford Press.

Yoon, H., Byerley, C. & Thompson, P.W. (2015). Teachers' meanings for average rate of change in U.S.A. and Korea. In T. Fukawa-Connelly, N. E. Infante, K. Keene, & M. Zandieh (eds.), *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1124–1132). Pittsburgh, PA: SIGMAA.

Zandieh, M. (2000). A theoretical framework for analyzing students' understanding of the concept of derivative. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.*), Research in collegiate mathematics education* (vol. IV, pp. 103–127). Providence, RI: American Mathematical Society.

Zandieh, M. J., & Knapp, J. (2006). Exploring the role of metonymy in mathematical understanding and reasoning: The concept of derivative as an example. *The Journal of Mathematical Behavior*, *25*(1), 1-17. https://doi.org/10.1016/j.jmathb.2005.11.002