The Role of Representation in Supporting Prospective Teachers' Understandings of Fractions as Measures

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We discuss the evolution of an instructional unit for supporting prospective elementary and middle grades' teachers' conceptions of fractions as measures. We first studied prospective teachers' conceptions of fractions using a task-based written assessment with discrete, linear, and circle representations. We designed an instructional unit that involved systematically introducing and connecting non-standard circles representations to other representations of fractions. We describe the assessments, learning outcomes of the instructional unit, limitations, and future directions.

Consider the following comment expressed by a prospective teacher about fractions representations: "[D]ots make no logical sense to me, and the shapes [bars] are easy until a fraction is more than a whole. What is that supposed to look like?". This statement illustrates a challenge for mathematics teacher educators. Although a conception of a fraction as a *measure* (Lamon, 2007) is emphasized early in the United States' Common Core State Standards (CCSSM, 2010), many elementary and middle grades students and prospective teachers (PTs) remain focused on *part-whole* meanings of fractions (DeWolf & Vosniadou, 2015; Newton, 2008; Norton & Wilkins, 2010; Olanoff et al., 2014). Being constrained to only a part-whole conception can be problematic when teaching children about fractions larger than one. Our focus is on understanding how to support PTs in constructing measurement conceptions of fractions when their part-whole conceptions may be well-established. How are conceptions of fractions as measures associated with particular fraction representations, and how can understanding this relationship be useful for improving instruction?

We begin by describing our theoretical framework and constructs for understanding fractions as measures and for supporting teacher learning. We next describe how we built off of extant research and written instruments to develop an assessment of how PTs' ways of experiencing written fractions tasks vary across three representations: dots, bars, and circles. We discuss the results of administering the assessment, in particular how it demonstrated persistent and statistically significant challenges with tasks involving a circular representation. Rather than avoiding circular models in instruction, we created a teaching innovation purposefully incorporating them as part of an instructional unit. We detail how systematic engagement with the three representations within the instructional unit provided opportunities for PTs to understand fractions as measures and discuss implications for teacher educator practice.

Theoretical Framework

We adopt a constructivist epistemology in thinking about PTs' fractions meanings as the product of their organizing mental structures (*schemes*) to fit their experiences (Glasersfeld, 1995). The construct of scheme refers generally to the way researchers model how individuals operate mentally in service of a goal. A scheme consists of three parts – recognition of a situation, operations (mental actions), and an expected outcome. Individuals' schemes become established as they become refined and generalized through their use, via processes of assimilation and accommodation (Piaget, 1970). When a scheme is interiorized, the situation, operations, and anticipated result of operating are experienced altogether as a unified and connected structure (a concept) that can itself be operated upon (Hackenberg, 2010b; Piaget, 1970).

Fractions Schemes

We focus on four specific schemes pertaining to fractions identified by Steffe and Olive (2010) – the parts-out-of-wholes fraction scheme (PWS), the partitive unit fraction scheme (PUFS), the general partitive fraction scheme (PFS), and the iterative fraction scheme (IFS). The PWS involves *partitioning* a whole into discrete pieces that can be disembedded (removed from the whole without modifying the whole) and double-counted to form a *numerosity* of part(s) within a numerosity of a whole. The remaining three schemes describe increasingly powerful ways of reasoning about fractions as measures. The PUFS builds upon the PWS as the individual conceives of the *size* of a disembedded part and its relation to the size of the whole (i.e., that *iterating* the amount of $1/n$ n times results in the size of 1). The PFS extends this notion to the size of a composite (but proper) fraction. An individual with an iterative fraction scheme (IFS) understands the size of fraction (m/n) (including when m is larger than n) as the result of coordinating mental operations to include partitioning the size of '1', disembedding a unit fractional size ($1/n$), and iterating the disembedded fractional unit m times.

In order for an individual's fraction scheme to become interiorized as a fraction concept, their fraction scheme must be *reversible*. For instance, an individual with a reversible IFS could reverse his or her ways of operating to determine the size of '1' from a given fractional size, whether or not it was larger than one. Reversing the PFS involves forming the size of '1' from a given (composite) proper fraction size, and reversing the PUFS involves forming the size of '1' from a given unit fraction size. Reversing the PWS involves forming the numerosity of the whole from a given proper fraction (e.g., reasoning that if three parts represents the fraction 3/7, then the whole must be 7 parts). By being able to reverse these schemes, individuals are more capable of sophisticated fraction reasoning. In the next section we describe specific examples of tasks used to assess these four fractions schemes and theoretical explanations for potential varying responses.

Task Structures

Figure 1 displays four task structures, two involving proper fractions (Task PFS1 and Task PFS2) and two involving fractions larger than one (Task IFS1 and Task IFS2). Consider that if a task involves a discrete representation for a unit fraction (e.g., a dot or a chip) then forming a size (via a counting measure) is indistinguishable from forming a numerosity. Task PFS1 theoretically requires a PWS in the discrete representation and a PFS in the bar and circular representations. In each of the three representations, the correct response to Task PFS1 is 2/5, but an individual with a PFS might instead respond with a slightly different fraction (such as 4/7 or 3/8) in the bar and circle models without rulers or protractors available to make precise measurements. Task PFS2 theoretically requires a reversible PFS in the bar and circular representations. To form the size of a unit fraction from the proper fraction involves intermediately forming the size of the whole. In the dots representation, the task requires a reversible PWS, as the task requires forming the numerosity of the whole given the numerosity of the parts.

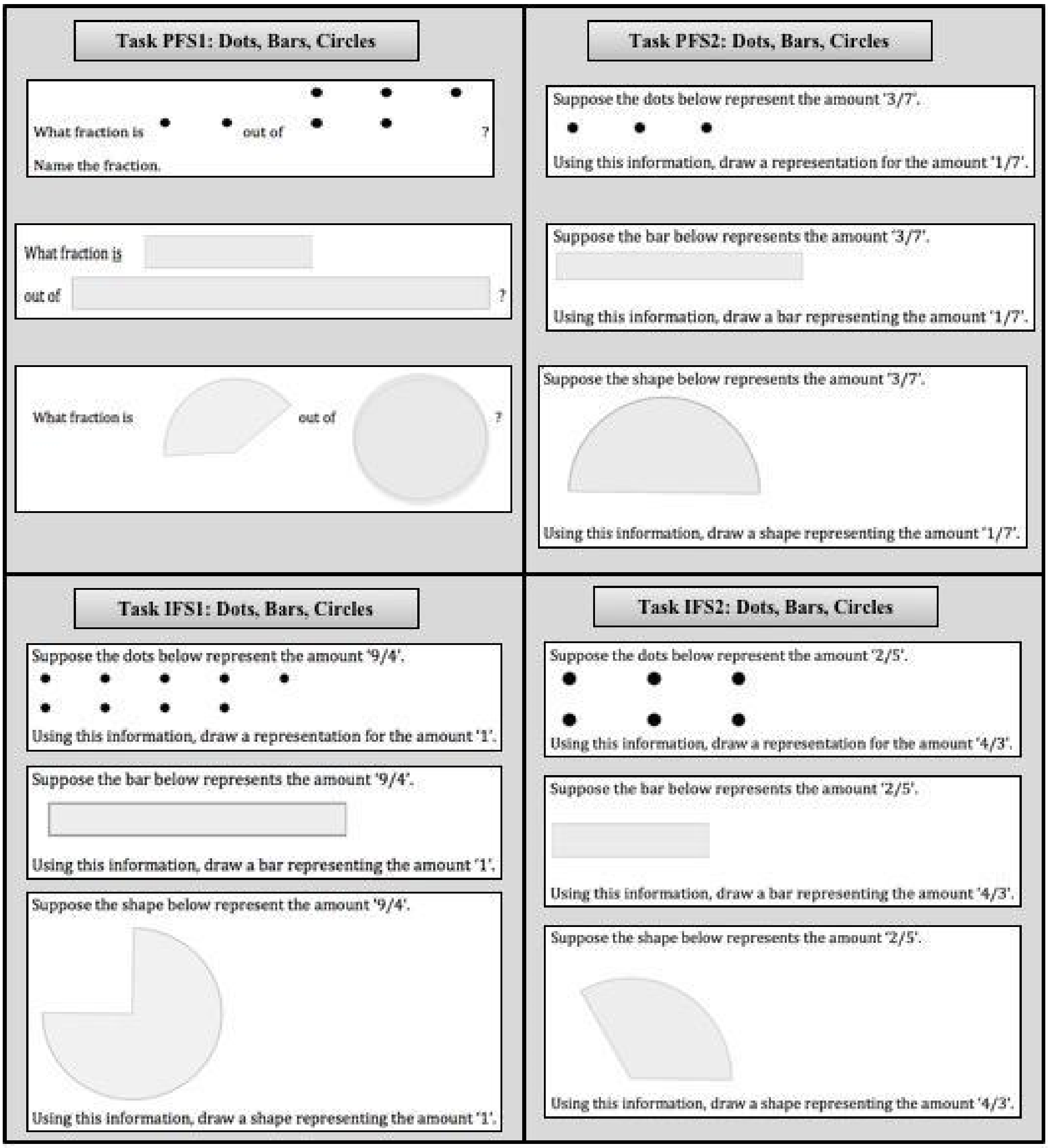


Figure 1. Four fractions task structures with dots, bars, and circles

Theoretically, Task IFS1 requires a reversible IFS, as it asks for the size of '1' from a given size larger than one. To form this size, one could partition the given amount into nine equivalent one-fourths and then iterate that one-fourth amount four times. However, a PT could potentially solve the task in the discrete representation using a ratio understanding instead. A PT might coordinate partitioning and iterating to solve the task in the bar representation, but not the circular representation, because this reasoning could be impeded by an established understanding of a circle as necessarily the size of '1'.

Task IFS2 can be solved by repeated use of an IFS. One could first partition and iterate to form the size of '1' (the first use of an IFS) and then partition and iterate to form a fractional size larger than one (next use of an IFS). PTs might instead approach the task by finding a common denominator by which to determine equivalent fractions without forming the size of '1'. Lovin et al. (2016) noted that PTs may rely on such procedures when encountering fractions larger than one because they have yet to coordinate three levels of units: the unit fraction, the composite unit fraction, and the whole. When attempting to iterate a unit fraction beyond the whole, they lose track of the size of the whole (Steffe & Olive, 2010).

Assessment Methods

Using Written Items to Assess Fractions Schemes

We administered written assessment to 76 PTs in an Elementary Math Methods course at a U.S. university, prior to class discussion of fractions (Boyce & Moss, 2017). Our assessments build off the learning trajectory for PTs' understandings of fractions as measures described by Lovin, Stevens, Siegfried, Norton, and Wilkins (2016). Norton and Wilkins (2010) designed written assessments of middle grades students' fractions schemes using rectangular and circular representations of fractions. The rectangular item assessment was validated via clinical interviews with sixth-grade U.S. students (Wilkins, Norton, & Boyce, 2013). Lovin et al. (2016) used the written items to assess the fractions schemes of PTs before and after instruction in mathematics courses for elementary PTs. In examining PTs' written responses, Lovin et al. (2016) noted that PTs often set up proportions and used division to correctly solve fractions tasks in ways that would not be available to elementary or middle grades students. They mentioned that PTs' use of such procedures to find equivalent ratios may have been confounding researchers' assessments of some of the PTs' fractions understandings (potentially producing both false positives and false negatives). This could lead to difficulty in assessing and supporting PTs' learning, particularly for fostering PTs' self-monitoring of their understanding of the mathematical goals and constraints that their prospective students may face.

Difficulties in making inferences from written items exist for elementary and middle grades students as well. For instance, Steffe and Olive (2010) describe the operation of *segmenting*, in which a whole is separated into a numerosity of pieces without necessarily disembedding a piece or attending to the relationship between the size of the pieces and the size of the whole. Researchers of elementary and middle grades students' fractional knowledge have noted that segmenting activity is sometimes very difficult to distinguish from iterating and partitioning activity, based on observations of students working with linear representations (Norton, 2008; Steffe & Olive, 2010).

We adapted Norton and Wilkins' (2010) items and methods to explore relationships between the form of fractions' task representations (discrete, rectangular, or circular) and PTs' fractions conceptions prior to instruction in their college course. We explored how PTs' ways of operating with fractions in different representations are connected, so that we, as mathematics teacher educators (MTEs), could plan to introduce and moderate perturbation (Glasersfeld, 1995) that could help them to construct more powerful fractions schemes. With this aim, we modified the assessment approach of Norton and Wilkins (2010) and Lovin et al. (2016) to isolate differences in PTs' responses to fractions tasks across representations.

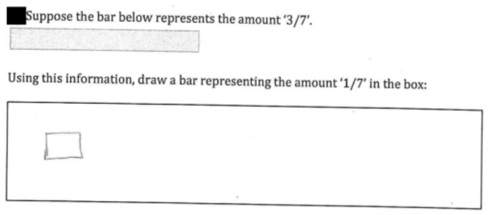
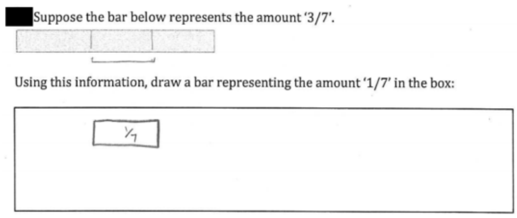
To reduce the length of the assessment and to isolate differences in task representation and structure, each of the PTs completed one of three forms. On the first form, there were four tasks with dots followed by four structurally identical tasks with bars. On the second form, the four dots tasks were followed by four circles tasks, and on the third form, the four bars tasks were followed by the four circles tasks (see Figure 1). These tasks use the same wording and form of items developed by Norton and Wilkins (2010), and the process for scoring items also followed their approach. If PTs had constructed schemes in the same developmental sequence as sixth-grade students, their mean scores would have the following relationship: PFS1 > PFS2 > IFS1 > IFS2 within each task representation (Norton & Wilkins, 2010).

The four tasks in each representation were consistently given in the sequence: IFS1, PFS2, PFS1, IFS2. Each of the three assessment forms included an identical ninth question, intended to assess PTs experiences of the cognitive demands of their previous eight tasks. The PTs were asked to rank the previous eight tasks they had completed in order from least difficult (1) to most difficult (8) and to then explain their ranking decisions. Figure 2 displays sample responses each scored as '0' or '1' for two items in the bars and circles representations. For instance, for Task PFS2, to assign a score of '1', we were looking for evidence that the individual had partitioned the given size into three equally-sized pieces and drawn one of those pieces. Indication that the individual had instead partitioned the given amount into seven pieces would suggest assigning a score of '0' for Task PFS2. Note that we coded Task IFS2 with '1' even if the PT determined the (approximately) correct fractional size by first using a procedure to determine equivalent fractions. We did not consider this as counter-indication of their ability to reason about the task by utilizing an iterative fraction scheme.

We calibrated our scoring by discussing our inferences and interpretations of ten randomly selected responses to each of the 12 items. As we independently coded the remaining responses, we also assigned '0.6' and '0.4' to indicate "leaning" toward indication or counter-indication, respectively. The (linear) kappa scores for the two raters across the 12 items ranged between '.44' and '1', with a mean kappa score of '.75', suggesting substantial inter-rater agreement (Landis & Koch, 1977). Agreement was strongest for the dots tasks, for which the PTs' responses were less ambiguous (kappa > .9 for each of the four tasks). The bars and circles tasks had the lower kappa values, as we had to make inferences about the PTs' markings about their intent to create a correct fractional size because they were not provided with a ruler or protractor with which to make exact measurements.

After computing a satisfactory kappa, we reconciled our scores. We assigned a reconciled score of '1' if we had each marked either a '1' or a '0.6', and we assigned a reconciled score of '0' if we had each marked either a '0' or a '0.4'. If one rater had marked '0.4' and the other '0.6', then we assigned a reconciled score of '0.5'. For the remaining responses, we returned to the data to decide on a score of either '0' or '1' for each item. Whereas Lovin et al. (2016) further used the sum of four reconciled scores on similar items to assign an overall '1' or '0' regarding an individual's construction of a fractions scheme, our item scoring remained focused at the item level.

We used the Wilcoxon rank sum (Wilcoxon, 1945) to test whether there were significant pairwise differences in mean scores for each item for this population of PTs. We tested for differences in mean scores across task representations (dots, bars, and circles), controlling for the task types (PFS1, PFS2, IFS1, and IFS2), and we also tested for pairwise differences in mean scores across items within each of the three representations. The null hypotheses were that there would not be significant pairwise differences in mean scores across task types or representations.

<table>
<tr><th></th><th>Responses scored as ‘0’</th><th>Responses scored as ‘1’</th></tr>
<tr><td rowspan="2">Task PFS 2</td><td>
</td><td>
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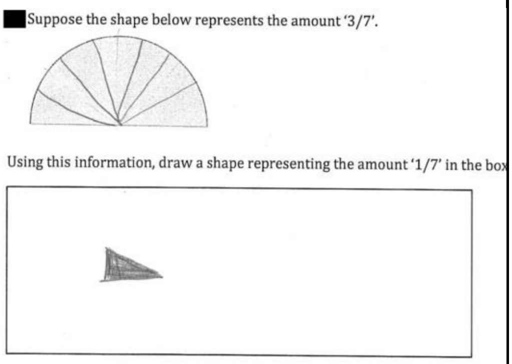
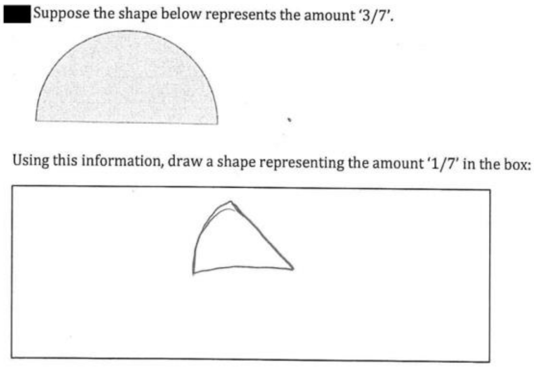
<tr><td>
</td><td>
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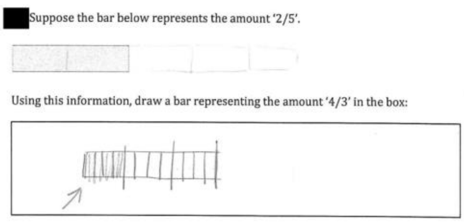
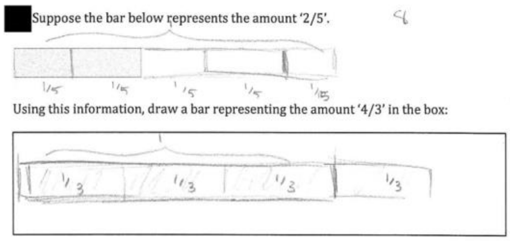
<tr><td rowspan="2">Task IFS 2</td><td>
</td><td>
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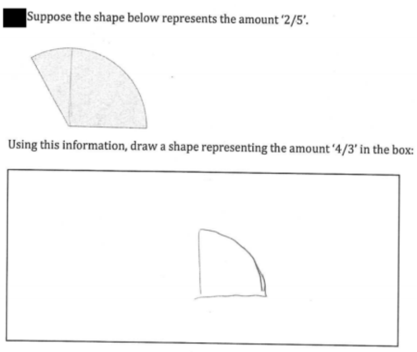
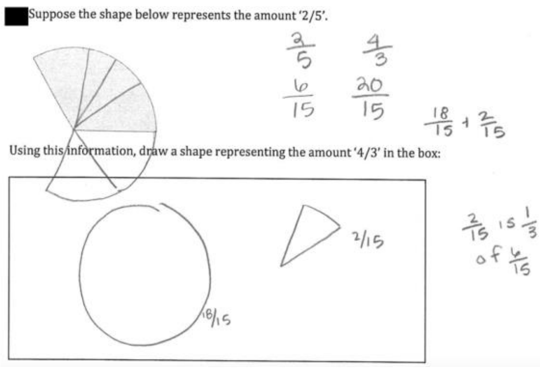
<tr><td>
</td><td>
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Figure 2. Sample responses coded with ‘0’ or ‘1’ in the bars and circles representations

Assessment Results

Table 1 displays the mean scores and difficulty rankings for each of the 12 tasks. Within each of the three representation types, the Wilcoxon signed rank test indicated significant differences (at alpha = .05 level) between both the mean performance and difficulty rankings on the two IFS items. Task IFS2 was more difficult than Task IFS1 (and was also more difficult than the PFS items), and this was in concordance with the PTs' perceptions of the two items' difficulty. The difference was not significant for the two PFS items across all three representations (p=.500 for dots, p=.355 for bars, and p=.232 for circles). For dots and bars, the perception of the difficulty of Task PFS2 was significantly greater than Task PFS1; for circles the difference in the perception of difficulty was marginally nonsignificant (p = .012, p=.009, and p=.073, respectively).

Table 1
Results Across Task Representation

	PFS1	PFS2	IFS1	IFS2
Dots Mean Score	.906[a]	.937[ab]	.760[b]	.344[b]
(Mean Dots Difficulty Rank)	(2.062)[ab]	(2.917)[b]	(3.771)[b]	(5.833)[b]
Bars Means Score	.522[ac]	.622[a]	.656	.300[c]
(Mean Bars Difficulty Rank)	(4.467)[a]	(3.333)[c]	(4.467)[c]	(6.333)
Circles Mean Score	.766[c]	.681[b]	.457[b]	.128[bc]
(Mean Circles Difficulty Rank)	(4.553)[b]	(4.021)[bc]	(5.553)[bc]	(6.745)[b]

Note. The score was from 0 (counter-indication of scheme) to 1 (indication of scheme). Difficulty ranking was from 1 (easiest) to 8 (hardest). Wilcoxon signed rank test was used, with alpha = .05.

[a] denotes significant difference between dots and bars.

[b] denotes significant difference between dots and circles.

[c] denotes significant difference between bars and circles.

Across each of the four items, scores on dots tasks were significantly higher than circles tasks. Though Task IFS2 was ranked as the most difficult across all three representations, it was significantly less likely to be answered correctly in the circles representation than in the bars or dots representations. These results affirmed that task representation is an important consideration in assessing PTs' fractions knowledge, which was corroborated by the PTs' explanations for their difficulty rankings.

A common explanation for the PTs' assignment of difficulty rankings was that PTs found Task IFS2 "confusing"—particularly in the circles representation. We inferred that this was often because the PTs were unfamiliar with a whole circle representing a size other than '1' (see Figure 3 for a sample response). Supporting this inference, PTs were more likely to correctly answer Task PFS1 (in which the whole circle represented the amount '1') as representing 2/5 in the circles representation than in the bars representation.

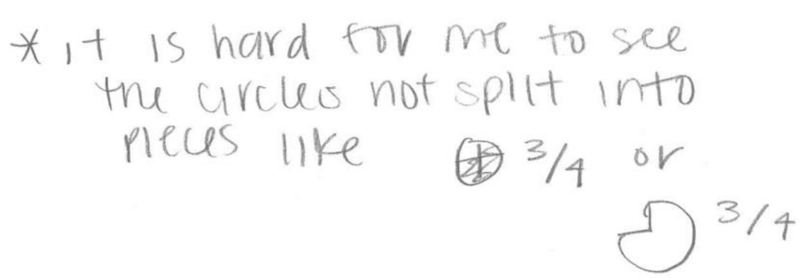

Figure 3. One PT's explanation of her difficulty rankings.

The results generally concur with the learning trajectory described by Norton and Wilkins (2010) and Lovin et al. (2016), in that the PTs in our study were more likely to make sense of PFS tasks than IFS tasks. However, some PTs' familiarity with part-whole interpretations of fractions and proportions resulted in their being able to correctly solve tasks with dots but not structurally identical tasks with bars or circles. Thus, PTs' ranking of task difficulty aligned best with the empirical trajectory of middle grades students' fractions schemes for tasks involving the circle representations. Overall, the results of the assessment suggest that PTs enter their elementary mathematics course well-suited to appreciate the challenges elementary students face in constructing fractions schemes, and that instructors might support PTs in understanding fractions as measures by incorporating non-standard circular representations of fractions.

Development and Implementation of the Instructional Unit

Perspectives on PTs' Fractions Learning

It is important to recognize that PTs' prior mathematics learning experiences, including interactions with teachers, other students, and mathematics content, can influence their perceptions of mathematics and, in some cases, bring on feelings of mathematics anxiety (NCTM, 2014; NRC, 2012). Mathematics anxiety in PTs can affect motivation towards learning and teaching mathematics (e.g., Brady & Bowd, 2005). Thus, PTs might become frustrated and feel anxiety if they realize that their approach for making sense of fractions does not always work. PTs' willingness to work through periods of frustration or cognitive roadblocks can be supported by the MTE communicating high expectations, being supportive, and being responsive to PTs' emotions, which are termed *mathematical caring relations* (Hackenberg, 2010a).

PTs may also question the motivation to learn something new about an elementary or middle school topic they were not taught during their school experience (Phillip, 2007, p. 439). We do not believe that PTs are mathematically incapable if they enter our course without having constructed fractions as measures, and we wanted to avoid such misconstruation. Our goal as mathematics educators is to encourage PTs to have growth mindsets, as opposed to fixed mindsets when it comes to learning mathematics. Students with fixed mindsets believe that they cannot do mathematics and give up easily, whereas those with growth mindsets are persistent and try very hard to solve mathematics problems (Boaler, 2016; Dweck, 2007). Blackwell et al. (2007) showed that students' mindsets can change from fixed to growth, and that those with a growth mindset have a much more positive and successful approach to learning.

Figure 4 depicts the instructional unit we developed, with these perspectives in mind, for systematically introducing representations to support understanding fractions as measures.

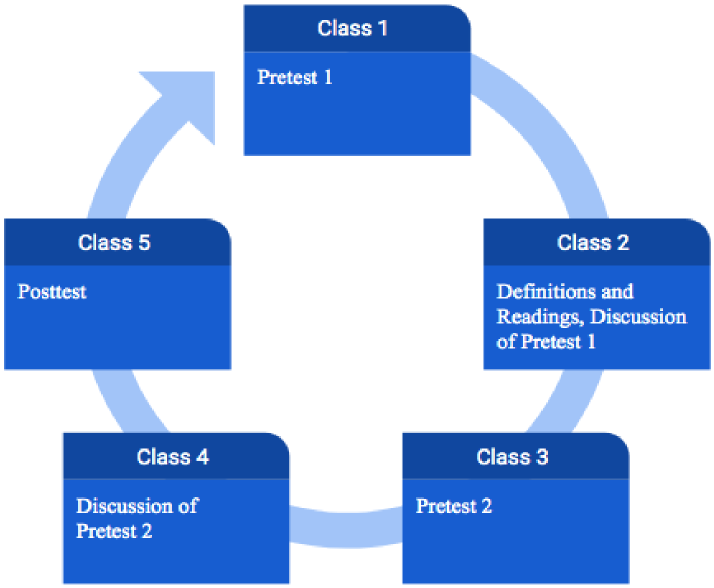

Figure 4. The instructional unit to investigate how to support our PTs in constructing fractions as measures.

By beginning with tasks that involved discrete and linear representations that were more accessible to students, we planned to maintain the PTs' familiarity and competence, and avoid increasing their anxiety. We introduced circular representations later in the instructional unit, after PTs had experienced and discussed their reasoning about fractions as measures in linear representations, to help them distinguish iterating and partitioning activity from segmenting or counting activity. By sequencing the representations and interspersing discussion about iterating and partitioning activity we hoped to bring successively greater focus to the distinguishing features of conceptions of fractions as measures.

Context

The PTs, seniors in college, were enrolled in a mathematics methods course offered in a College of Education that was taught by the second author. This semester-long course is required for prospective middle grades teachers who will earn a specialization in mathematics teaching (grades 5-8). The aim of the course is to help PTs think about teaching and learning mathematics in the middle grades in ways that will enable them to make good instructional decisions. Goals for the course span across several interrelated areas that include understanding mathematics to teach, understanding oneself as a learner of mathematics, understanding children as learners, and developing knowledge of curriculum and planning. PTs had not previously completed college coursework focusing on fractions, as the prerequisites included courses such as calculus, statistics, and linear algebra. Because the class size was only 6, the instructor was able to attend closely to the differences and changes in their reasoning during the instructional unit.

Class 1: Pre-test 1. In response to the difficulties our PTs in prior elementary mathematics methods courses had expressed with circles representations, instead of beginning with written assessments of PTs’ fraction schemes using all three representations, we modified the assessment to first include only the discrete and linear representations. The instructor (second author) explained to the class that this “pre-test” would inform her knowledge about their understanding of fractions so that she could introduce them to ideas about fractions that they might not have previously learned. The instructor also informed the PTs that these pre-tests would not be graded and that the class would discuss and work through the tasks together during subsequent class meetings. All six PTs completed the pre-test in the allotted 20 minutes.

Pre-test 1 included the PFS1, PFS2, IFS1, and IFS2 tasks given in sequence for dots and then for bars (Appendix A). We independently coded their responses, using the previously described methods (see Figure 2). Overall, the results were similar to those described in Table 1, in that the PTs were more likely to correctly solve tasks in the discrete representation than with bars or circles, and they were more likely to correctly solve tasks PFS1 and PFS2 than tasks IFS1 and IFS2. We use pseudonyms when referring to individual PTs. Figure 5 displays examples of PT work from Pre-test 1 that was coded with ‘0’, suggesting that this PT, Margaret, had not yet constructed an IFS. In the work in Figure 5, Margaret describes that she did not know where to begin on the discrete task; she drew a bar much longer than 9/4 on that task.

Whereas in the previous administration of the assessment only about one-third of elementary PTs correctly solved task IFS2 in the linear representation, three of the six prospective middle grades teachers were able to do so on the pre-test (Donna, Helen, and Kelly). These three PTs each appealed to their knowledge of ratios to form equivalent fractions rather than reversing and composing partitioning and iterating operations (for example, see Donna’s work in Figure 6). These three PTs also created correctly sized representations for task IFS2 in the dot representation.

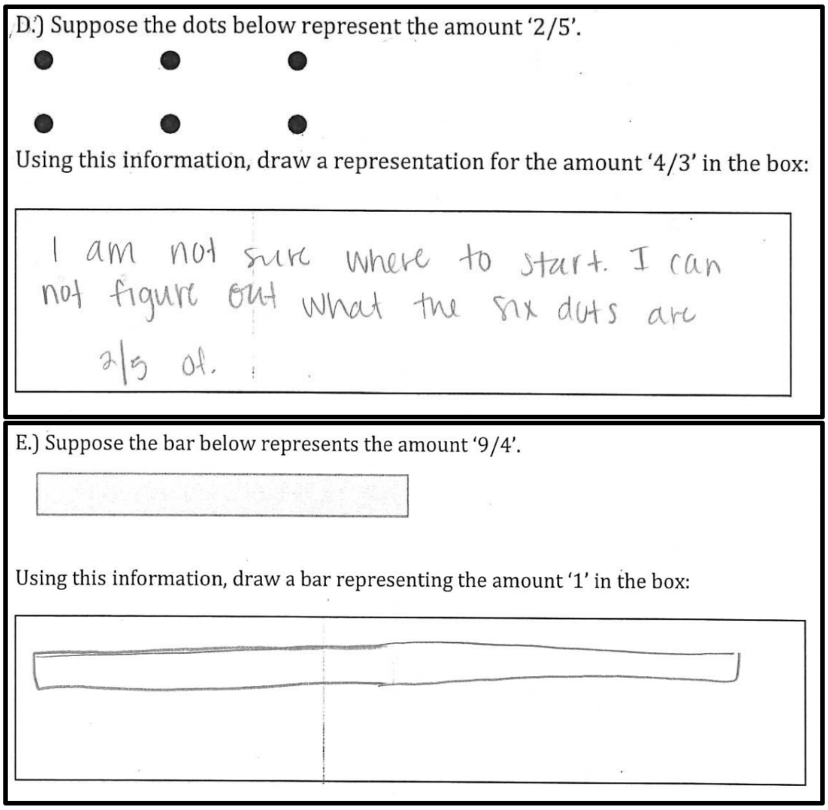

Figure 5. Margaret's work on discrete and linear tasks on Pre-test 1.

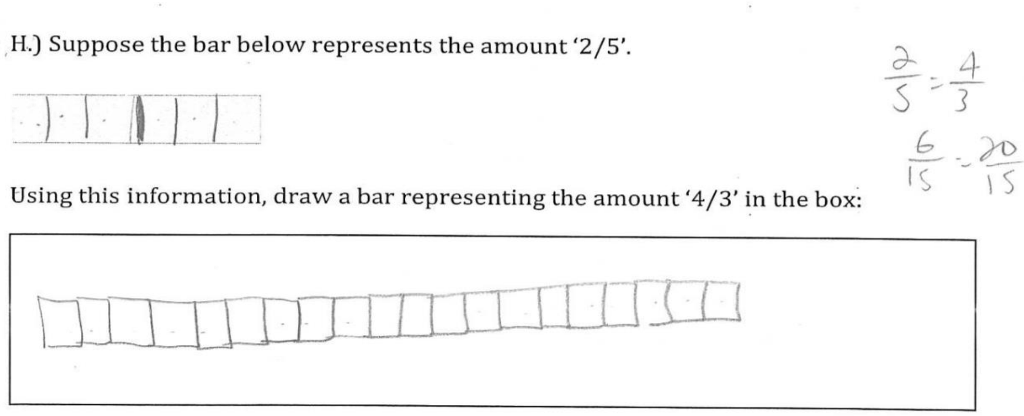

Figure 6. Donna's work showing use of ratios and equivalent fractions on IFS2 on Pre-test 1.

Two of the remaining three PTs did not create correctly sized representations for IFS2 in both the dot and bar representations. Margaret responded that she was unsure where to start on both. Sarah incorrectly attempted to use fraction division to solve the task in the dot representation (see Figure 7). In the bar representation, she created a bar that was 4/3 the size of the given bar (treating the given bar as 1). The sixth PT, Cody, correctly solved IFS2 in the dots representation by appealing to a ratio understanding, but he created a bar that was too long when responding to IFS2 in that representation.

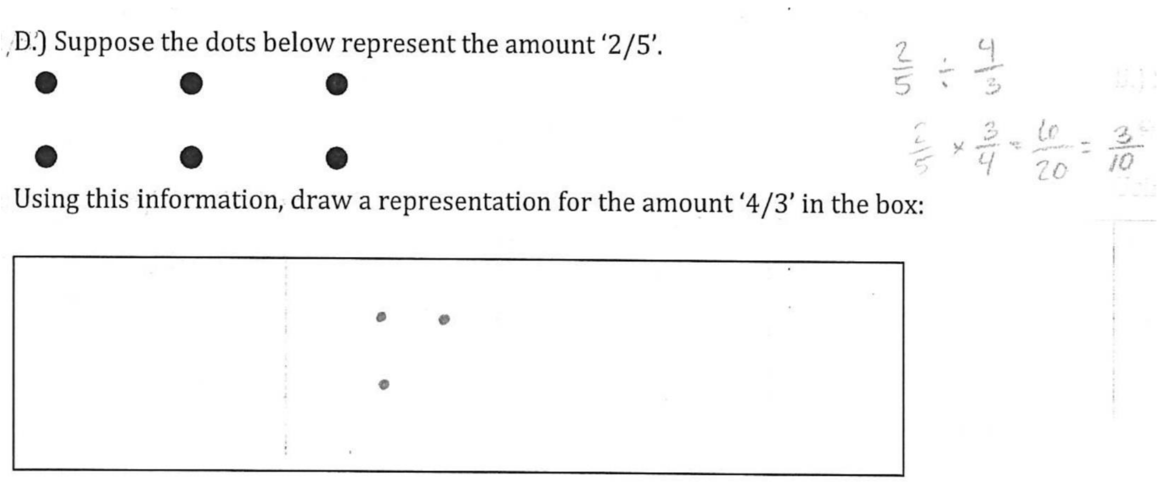

*Figure 7***.** Sarah's use of arithmetic operations on Task IFS2 on Pre-test 1.

Class 2: Definitions and Reading, Discussion of Pre-test 1. Anticipating that PTs lacked the experience and familiarity with thinking of fractions as measures, the instructor assigned them to read the article, *Creating, Naming, and Justifying Fractions* (Siebert & Gaskin, 2006), and write their own definitions of partitioning and iterating operations to help them conceptualize fractions as measures. The example in Figure 8 shows how Donna wrote what it means to partition and iterate with fractions and drew a picture to explain her thinking.

Partitioning	**Iterating**
When you have a larger amount and it needs to be broken down into smaller amounts, that are equal, to be viewed in a new light. For example if you have a bar that equals 4/9 and want to find the size of one 9th you can partition the bar into 4 equal parts to show 1/9. 4/9 4 pieces where each piece = 1/9	When you can now take these new smaller pieces, like the 1/9 and you can put multiple 1/9 pieces together to create an even larger amount than what you started with. Below you can see six 1/9 pieces that come to create 6/9.

Figure 8. Donna's explanation of partitioning and iterating operations.

Once PTs had a personal understanding of partitioning and iterating operations with fractions, they had opportunities to share their definitions and also revise and discuss their responses to the tasks from Pre-test 1. Key terms in PTs' definitions for partitioning included "breaking down", "cutting", and "equal parts", and key terms for iterating included "multiple pieces", "same size", "copying", and "repeating". For instance, Kelly wrote:

> Partitioning is "cutting" smaller, equal-sized amounts from a larger whole amount. For instance, to get ¼, we would "cut" or partition the larger whole amount of four into four smaller, equal amounts.

> Then one of these equal parts would be shaded to represent ¼. Iterating is "copying" or the repetition of a smaller amount that is then combined to create a larger amount. For instance, if we were to find a whole bar (one bar) when given 2/5 of a bar, we would partition the bar into two equal pieces. We would then iterate a 1/5 piece five times to find the length of 1 whole.

As PTs discussed the discrete and linear tasks, the instructor encouraged them to manipulate counters and Cuisenaire Rods to help them carry out iterating and partitioning operations in activity. Figure 9 shows Sarah and Donna working with the manipulatives to represent 4/3 to solve IFS2.

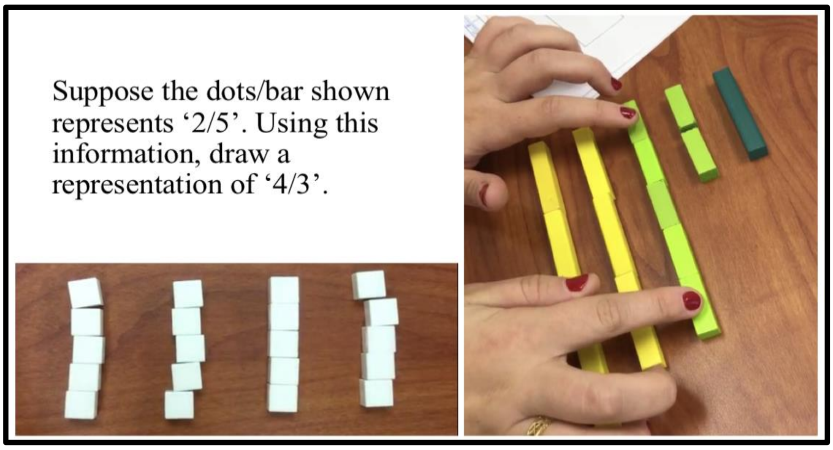

Figure 9. Sarah's working with counters (contrast with Figure 7) and Donna's working with Cuisenaire rods (contrast with Figure 6) on task IFS2.

Sarah started by partitioning the given representation of 2/5 as a 2×3 array (see Figure 6) into two 1×3 arrays, each representing the size of '1/5'. She next formed the size of '1' by iterating the size '1/5', vertically, forming the 5×3 array of blocks on the left. Next, she partitioned the '1' into three one-thirds, each represented by a column of blocks. Last, she iterated one-third four times from left to right, forming four copies of the first column of blocks to represent the size '4/3'. Donna explained how she solved the problem using Cuisenaire rods:

> We are going to start here. This is our bar representing two-fifths (points to dark green bar). First, we are going to partition it into two pieces (points to two light green bars below dark green bar), which will each be one-fifth. Now we want to iterate our one-fifth five times (points to the five light green bars) to give us a whole, which will be five-fifths. So now we are also trying to get into four-thirds, so we are going to take out our one, which is five-fifths and partition it into three pieces (points to the three yellow bars). That will give us these three pieces, which are each one-third, but we want four-thirds. So, we will take our one-third and iterate it four times, so we have one, two, three, four times (points to the four yellow bars) to give us four-thirds.

As PTs used the manipulatives to solve these problems, they created representations of the fractions which helped them visualize the operations and units and begin to construct PFS and IFS schemes. During this discussion, the class conveyed that they had never used the discrete representation of fractions and had used linear representations, but usually in the form of a number line. Thus, conceptualizing fractions as measures using dots and bars was new to them.

Class 3: Pre-test 2. Pre-test 2 included the PFS1, PFS2, IFS1, and IFS2 tasks containing only circle representations (Appendix B). The instructions and expectations for PTs were similar to Pre-test 1. This time, PTs seemed to demonstrate partitioning and iterating. However, the responses to tasks with non-standard circle representations indicated that although they were attending to the sizes of fractional parts, their meanings for those sizes were affected by their expectations that a whole would consist of a full circle.

On Pre-test 2, Helen, Kelly, Margaret, and Sarah were able to correctly solve Task IFS1; however, everyone had difficulty thinking through Task IFS2 (see Figure 10). Both tasks involve non-standard circle representations, meaning that a whole circle does not represent '1'. In IFS1, '9/4' is represented with an amount that looks like '3/4' in a standard circle representation. In IFS2, '2/5' is represented with an amount that looks like '2/6' with a standard circle representation.

Kelly's work on IFS1 displayed in Figure 10 correctly depicts the amount '1'. She partitioned the '9/4' into nine equal pieces. A piece that represents '1' would have to be the size of 4 of these pieces (as drawn in Figure 10). On Task IFS2, Kelly's work shown in Figure 10 indicates that she partitioned the given '2/5' into two '1/5' pieces. Her written explanation correctly suggests iterating a total of five times to form five one-fifth pieces as the size of the whole, and the accompanying picture shows that *six* one-fifth pieces would correspond with a circle. However, she represented the size of '1' as the circle in her subsequent partitioning for the size of '1/3'.

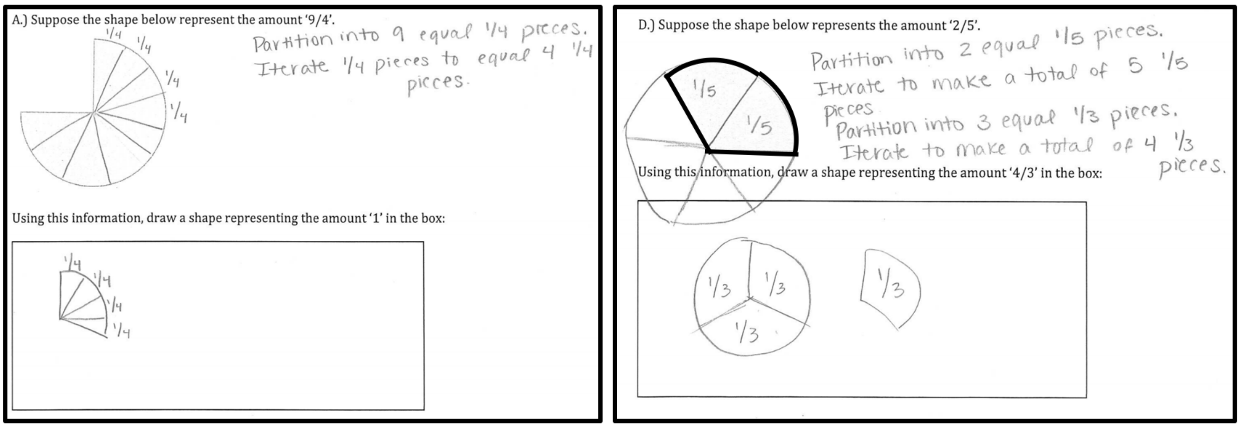

Figure 10. Kelly's work on the circle tasks IFS1 (left) and IFS2 (right) on Pre-test 2.

Class 4: Discussion of Pre-test 2. Once again, we wanted to give the PTs the opportunity to share their thinking and work through these problems using fraction circles (manipulatives). The fraction circles worked well for representing most of the tasks on Pre-test 2. Figure 11 (top) shows Kelly using fraction circle manipulatives to work through Tasks IFS1. Kelly represented the given part, '9/4' with the three-fourths of the fraction circles manipulatives (yellow). Then, she laid nine of the twelfths fraction circles pieces (pink) over the yellow to show that it could be partitioned into nine equal parts. Lastly, she showed that '1' is four of the one twelfth parts. The reasoning on IFS1 using the manipulatives (Figure 11) is the same as the reasoning using drawings (Figure 10).

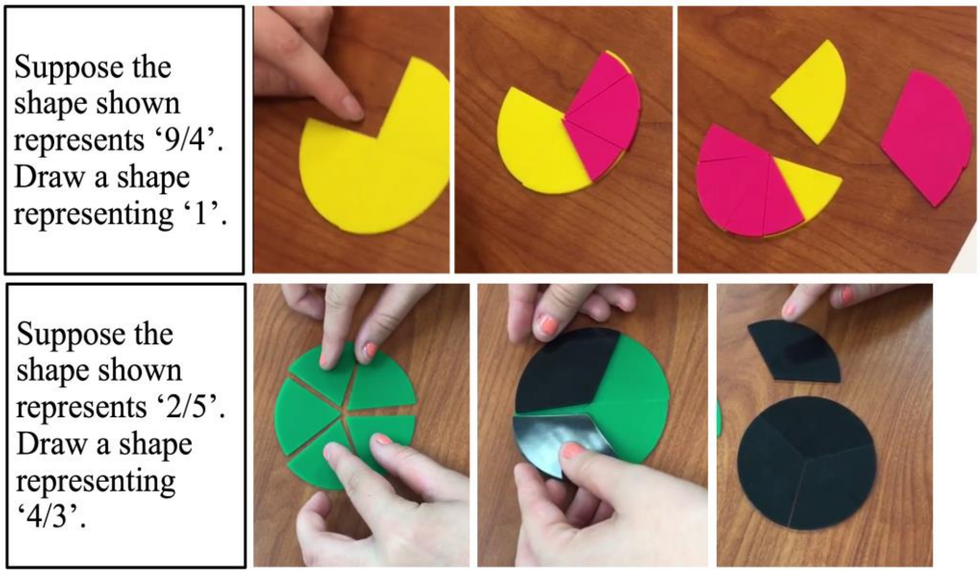

*Figure 11***.** PTs' use of fraction circles on Tasks IFS1 (top, Kelly) and IFS2 (bottom, Donna).

There remained issues with representing Task IFS2 from Pre-test 2. Like Kelly, Donna formed a whole that was the size of a full circle instead of partitioning and iterating 2/5 to form the correct size of the whole (Figure 11, bottom). Donna represented fifths with the green fraction circle pieces, but the given amount (Figure 10, highlighted in black) is a different size than two-fifths of fraction circle pieces. So, in this case the issue was representing '2/5' using pieces that are two-fifths of a standard circle representation, rather than representing the amount in Figure 10. Using the manipulatives, Donna partitioned the full circle created by the fifths into thirds and then included an additional '1/3' piece (black) to show '4/3'. Donna's explanation of Task IFS2, included below, otherwise indicates correct reasoning:

> We are going to start with our shape, '2/5' (points to two green 1/5 pieces). We are going to partition this shape into two '1/5'. We are going to want to end up with '4/3'. We are going to take our '1/5' and iterate it five times, so we have '1/5', '1/5', '1/5', '1/5', '1/5' (creates a full circle with the green pieces). So now we have five '1/5' pieces which equals our whole. So now we are going to partition this (full circle) into three pieces because we want '4/3'. So, we are going to take our whole and turn it into '3/3' (lays the black '1/3' pieces on top of the green circle). This is '1/3', this is '1/3', and this is a '1/3'. So, now we have another whole which is '3/3' or '1'. Now we are going to partition this whole back into '1/3' pieces and then we want '4/3' so we can iterate this '1/3' piece, four times. We have one, two, three, and four '1/3' pieces. So, we end up with '1' and '1/3' (points to a full circle made up of the black pieces and a '1/3' black shape), which is also '4/3'.

Class 5: Post-test. The post-test included tasks with discrete, linear, and circle representations that were similar to the pre-test items (see Appendix C). The PTs' responses on the linear items indicated they were now able to coordinate iterating and partitioning to form correct fractional sizes. For instance, the work displayed in Figure 12 by Margaret is indicative of an iterative fraction scheme, unlike her previous work (contrast with Figure 5).

The bar drawn below is 7/3 as long as a whole candy bar. Draw the whole candy bar.

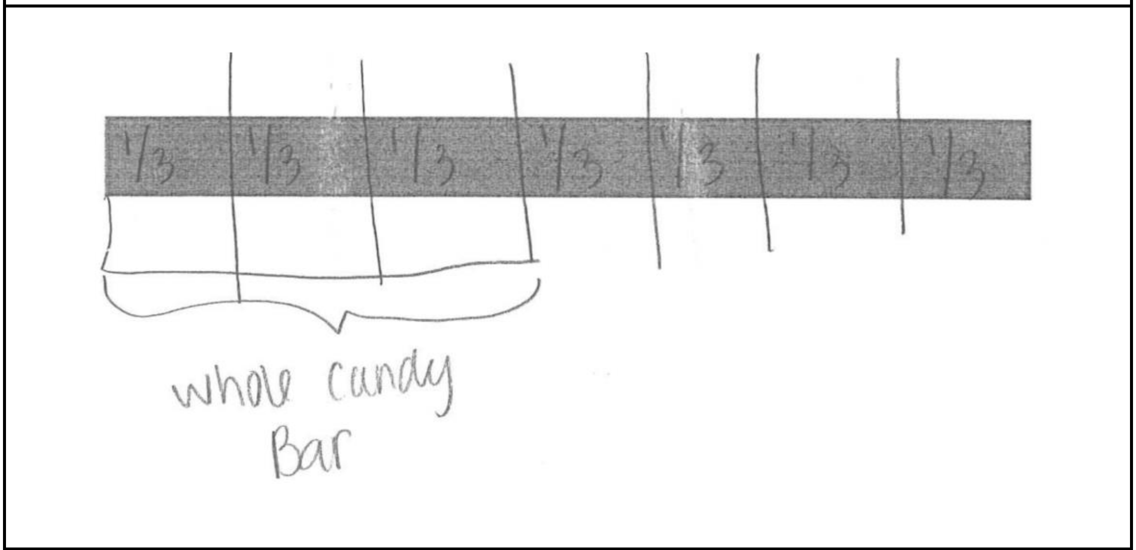

Figure 12. Post-test response from Margaret indicative of a (reversible) iterative fraction scheme.

We included one item with a non-standard circle representation, associated with a partitive fraction scheme (see Figure 13), but we chose a non-standard representation for which we expected less difficulties due to PTs being accustomed to a complete circle representing a whole.

Your piece of pie is 2/7 the amount shown below. Draw your piece of pie.

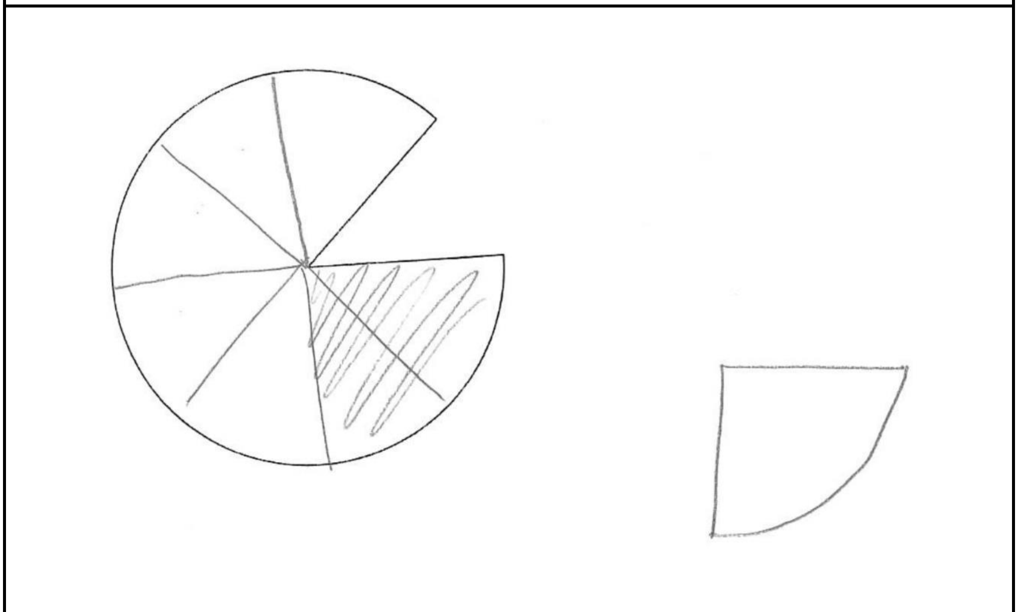

Figure 13. Post-test response from Kelly indicative of a partitive fraction scheme

Because the represented whole is 7/8ths of a circle, an approach for segmenting a circle into eighths results in correctly segmenting the non-standard whole into sevenths. Indeed, all PTs were assessed as forming the correct fractional size of 2/7.

On the post-assessment we also included new discrete items. For instance, the task depicted in Figure 14 is similar to IFS2 (compare with Figure 5). In contrast to the results of analyzing results of PTs' responses across representations depicted in Table 1 on the initial assessment, this task was more difficult for the PTs than similar tasks with bars and circles. All six of the PTs produced the correct size in response to the tasks depicted in Figure 12 and Figure 13, but only three (Donna, Sarah, and Kelly) responded correctly to the task depicted in Figure 14. They each identified one-third as 4 chips and then explained that the whole contained 3 times 4 = 12 chips. Margaret left this item blank, while Cody and Helen answered, "16".

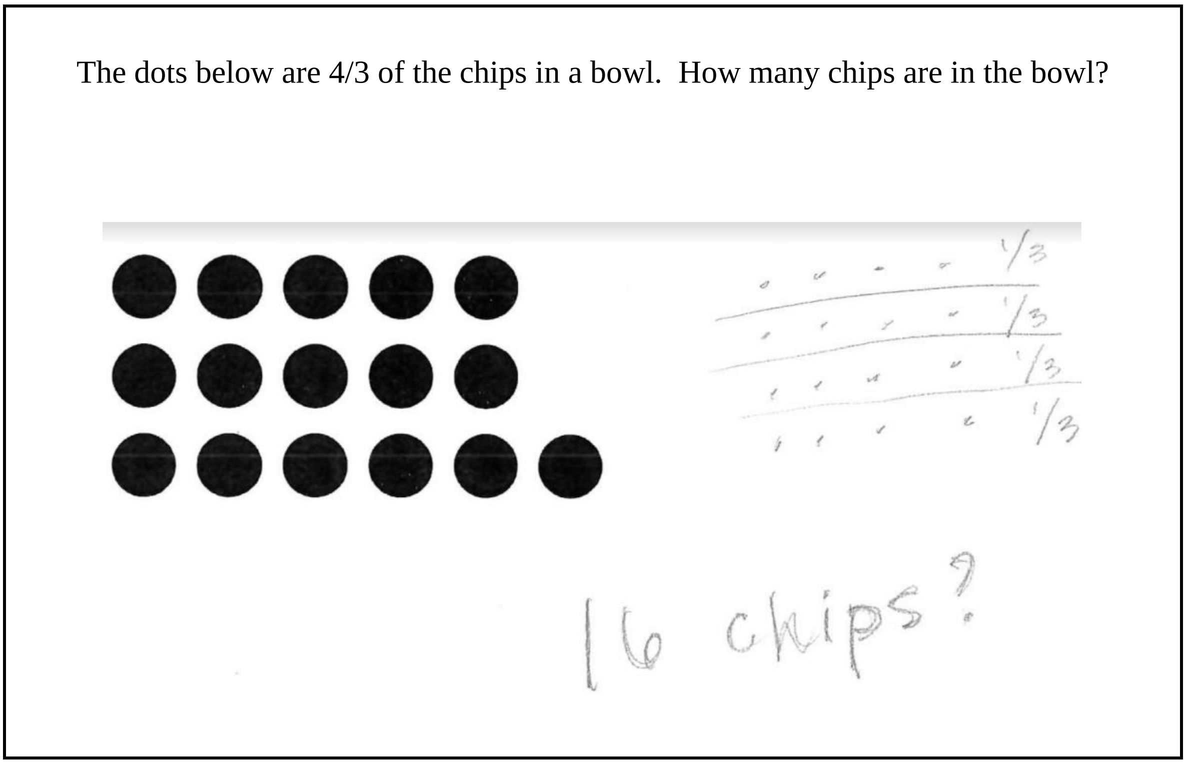

Figure 14. Helen's post-test response counter-indicative of a (reversible) iterative fraction scheme

On the Pre-tests, in each of the dots, bars, and circles representations, the tasks we presented did not explicitly provide or ask for values of measures: counts, lengths, or areas, respectively. PTs instead had to act upon the representation to form a quantity, by forming sizes to iterate (and/or count). The task prompts each included the instruction to "draw a representation for [an] amount". In contrast, the discrete task in Figure 14 asks the PT to determine "how many chips are in the bowl", rather than to "represent the chips in the bowl." This is perhaps a subtle difference, but one that may be useful for MTEs to investigate further. It may be that the wording of the task fostered reasoning about numerosities, rather than reasoning about quantities.

Conclusions

In this paper, we narrated a process of using assessments to explore differences in PTs' reasoning about fractions as measures across task structures and representations. We explained how in the discrete representation, the concept of a proper fraction as a size is a ratio conception. In a linear (bar) representation, PTs may be able to apply part-whole reasoning to produce correctly sized amounts, as via segmenting the bar representation can be discretized. It is in the circular representation that the part-whole conception was least fruitful, for the images of benchmark fractions, such as '¼' or '⅛', are associated with the size of '1' being a full circle, and the tasks on the assessment included non-standard representations for the size of '1'.

Introducing non-standard circular representations is particularly useful for inducing perturbation, for bringing attention to mathematical norms and encouraging PTs' reflection on unitizing, partitioning, and iterating operations. But what is the right amount of perturbation? For students yet to construct fractions as measures in the linear (bar) context, the complexity of simultaneously working with non-standard circular representations may be too demanding and contribute to math anxiety.

We designed and implemented an instructional unit for prospective teachers in a middle-grades mathematics methods course. The sequence of perturbations began with assessments of the PTs' reasoning about discrete and linear (bar) fractional representations. We introduced fractions as measures and the roles of unitizing, iterating and partitioning in forming sizes *consistent* with standard representations (dots and bars). We then isolated and fostered reflection on iterating and partitioning by tasking students to revisit their earlier responses. We next asked them to form sizes *inconsistent* with standard representations (i.e., non-standard circular representations). This sojourn was temporary. Its purpose was deepening the PTs' understandings of iterating and partitioning in the linear bars representation, which is what they will be teaching to students. Each of the six PTs' responses on linear (bar) task on the post-assessment was consistent with their having constructed an IFS, highlighting the promise of the instructional unit.

Implications for Teacher Educator Practice

The main implication of this paper is to suggest that systematically attending to both task structure and representation may be helpful for supporting PTs in their construction of fractions as measures. In future work, we would also like to include post-assessments involving number line representations. We conjecture that having isolated the mental operations of iterating and partitioning with the linear (bar) representation, PTs will be able to assimilate tasks involving the number line representation without perturbation. Because understanding fractions as points on a number line is one of the important learning goals for elementary and middle grades students, this is an important connection.

A limitation of this study is the small class size (only 6) for the testing of the teaching innovation. However, the instructional methods we described lend themselves to scaling up to a larger group of students, as the main activities for the PTs involved their reading, reflecting, video-recording their explanations of their thinking, and discussing their viewing of others' explanations. It is an open question how and whether additional support may be needed for prospective elementary teachers (as the instructional unit involved prospective middle grades teachers). In our future research, we are planning to scale up the instructional unit to a larger class of prospective K-8 teachers to better understand particular needs or refinements.

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Appendix A: Pre-test 1 Items

A.) Suppose the dots below represent the amount ‘9/4’.

● ● ● ● ●

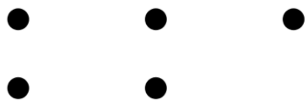
● ● ● ●

Using this information, draw a representation for the amount ‘1’ in the box:

B.) Suppose the dots below represent the amount ‘3/7’.

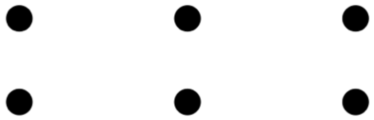
● ● ●

Using this information, draw a representation for the amount ‘1/7’ in the box:

C.) What fraction is  out of  ?

Name the fraction in the box:

D.) Suppose the dots below represent the amount '2/5'.



Using this information, draw a representation for the amount '4/3' in the box:

E.) Suppose the bar below represents the amount ‘9/4’.

Using this information, draw a bar representing the amount ‘1’ in the box:

F.) Suppose the bar below represents the amount ‘3/7’.

Using this information, draw a bar representing the amount ‘1/7’ in the box:

G.) What fraction is 

out of  ?

Name the fraction in the box:

H.) Suppose the bar below represents the amount ‘2/5’.

Using this information, draw a bar representing the amount ‘4/3’ in the box:

I.) Please rank each of the tasks A – H from easiest (1) to hardest (8). Use each of the numbers 1-8 exactly one time. Explain your reasoning for the rankings.

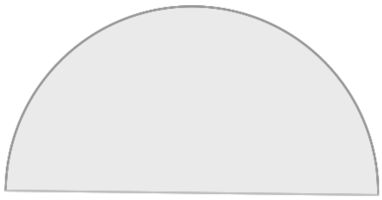
Task	A	B	C	D	E	F	G	H
Difficulty Rank								

Appendix B: Pre-test 2 Items

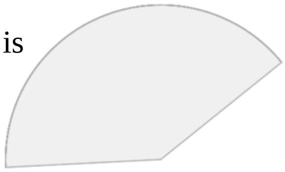
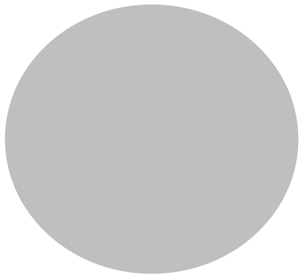
A.) Suppose the shape below represent the amount ‘9/4’.

Using this information, draw a shape representing the amount ‘1’ in the box:

B.) Suppose the shape below represents the amount ‘3/7’.

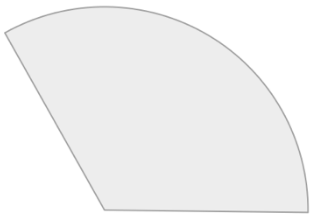


Using this information, draw a shape representing the amount ‘1/7’ in the box:

C.) What fraction is  out of  ?

Name the fraction in the box:

D.) Suppose the shape below represents the amount ‘2/5’.



Using this information, draw a shape representing the amount ‘4/3’ in the box:

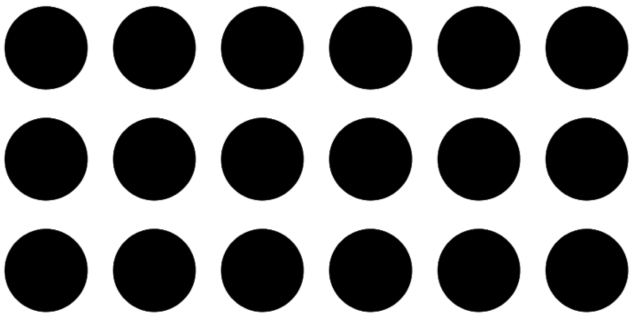
E.) Please rank each of the tasks A – D from easiest (1) to hardest (4). Use each of the numbers 1-4 exactly one time.

Task	A	B	C	D
Difficulty Rank				

Appendix C: Post-test Items

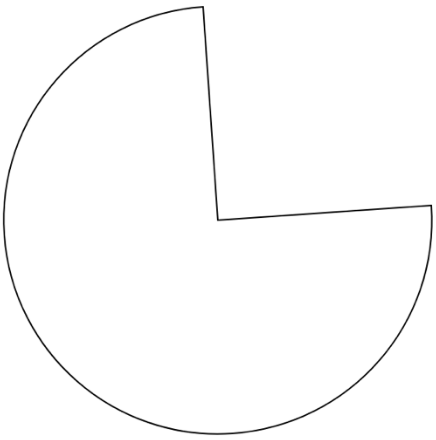
Question 1:

The dots below are 3/5 of the pennies in a piggybank. How many pennies are in the piggybank?

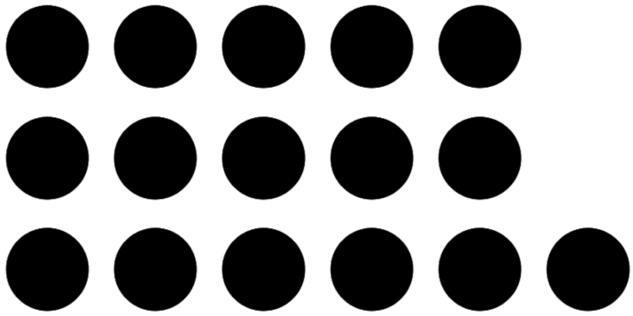


Question 2:
The stick shown below is 2/3 of a whole stick. How many 1/9 sticks can you make from the 2/3 stick?

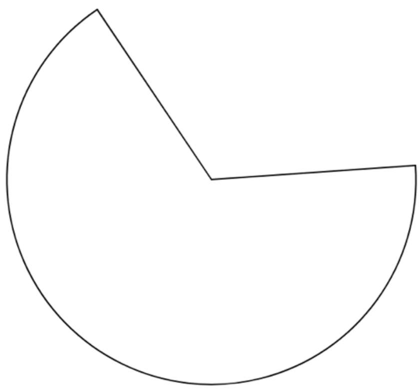
Question 3: The pizza shown below is 3/4 of a whole pie. If each person wants 1/8 of a whole pie, how many people can share the amount shown below?



Question 4: The dots below are 4/3 of the chips in a bowl. How many chips are in the bowl?

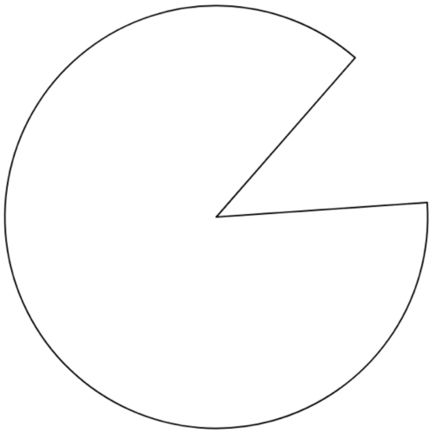


Question 5: The pizza shown below is 2/3 of a whole pizza.   If each person wants 1/9 of a whole pizza, how many people can share the amount shown below?



Question 6: The stick shown below is 3/5 of a whole stick. How many 1/20 sticks can you make from the 3/5 stick?

Question 7: Your piece of pie is 2/7 the amount shown below.   Draw your piece of pie .



Question 8: The bar drawn below is 7/3 as long as a whole candy bar. Draw the whole candy bar.

Question 9: The dots below are 3/5 of the pieces of candy in a box. If each child gets 1/20 of a box of candy, how many children can share the amount below?