A Study of Slope Explanations across a Standards-Based Textbook Series

Courtney Nagle
Penn State Behrend
crt12@psu.edu

Deborah Moore-Russo
The University of Oklahoma
dmr@ou.edu

Thomas Fisher
tomfisher2718@gmail.com

This study analyses a standards-based textbook series to investigate how slope is presented across seven sequential textbooks in a secondary mathematics curriculum. Analysis of the expository content across the textbooks shows that slope is covered in all seven books, is frequently described as a *Constant Parameter* or *Ratio*, is frequently used in real-world contexts, and is often described alongside covariational language of how inputs and outputs change in relation to one another. Results indicate that *Determining Property* and *Steepness* components of slope are less prevalent and disproportionately unlikely to be accompanied by covariational language and applications compared with the other slope components. Although many consistencies were found across the textbook series, differences in emphases from one textbook to another are also discussed and connected to the recommendation that future work around slope investigate its development across the secondary curriculum. Results are discussed relative to students' preparation for specific calculus topics that build on prior knowledge of slope. Implications for both teachers and researchers are provided.

Mathematics textbooks serve as narratives presenting mathematical ideas in an influential and purposeful order (Dietiker, 2013). Research studies have consistently documented that, textbooks are typically the primary resource for mathematics instruction and learning (Beaton et al.,1996; Beagle, 1973; Biefhoff, 1996; Elsaleh, 2010; Fan & Kaeley, 2000; Grouws et al., 2004; Mullis & Martin, 2008; Pepin et al., 2001; Stein et al., 2007). Textbooks serve as the primary link bridging the intended and attained curriculum (Schmidt et al., 2001), helping "teachers identify content to be taught [and] instructional strategies appropriate for a particular age level" (Thompson et al., 2012, p. 254). This study seeks to describe how a particular topic, slope, is developed not in a single textbook but across an entire textbook series using research findings on slope and covariational reasoning as lenses to frame the study. The following research questions were used for the study.

1. Which *components of slope* and which *types of approaches* (visual only, nonvisual only, or both visual and nonvisual) are emphasized in the expository material within each textbook and across the series?
2. Which *levels of covariational reasoning* are emphasized in the expository material related to slope within each textbook and across the series; how is covariational reasoning developed in relation to the slope components and the type of approach?
3. What is the role of *contextual applications* in developing slope; how frequently do contextual applications occur with the various slope components, covariational reasoning levels and types of approaches?

Literature Review

This section provides an overview of relevant research related to textbooks and slope to help position our study within past work in these areas.

Textbooks

Textbooks play an influential role in mathematics education (Fan et al.,, 2013; Pepin et al., 2013). Textbooks reflect “significant views of what mathematics is…and the ways that mathematics can be taught and learnt” (Pepin et al., 2001, p. 166). They play a role in how teachers shape and sequence their instruction (Davis, 2009). Fan and Kaeley (2000) suggest that textbooks send “pedagogical messages” to teachers, since teachers using different textbooks display differences in their teaching strategies. While teachers may have access to a variety of resources, the textbook is typically the only common resource for students (Lepik et al., 2015). Textbooks influence how students learn and how they consider and solve problems (Massey & Riley, 2013). Schmidt et al. (1997) argue that to be commercially viable, textbooks are likely to reflect the intentions of the national curriculum. They suggest that textbooks often shape key instructional decisions such as the time allocated to topics, the topics that are included and omitted, and the goals for instruction (Schmidt et al., 1997).

Slope

The concept of slope is foundational in many areas. In algebra, it is used to distinguish the covariational relationships of linear and nonlinear functions (Carlson et al., 2002; Teuscher & Reys, 2010). In calculus, students must use slope to understand the limit definition of a derivative (Sofronas et al., 2011; Zandieh & Knapp, 2006) as well as related applications of derivatives, such as the Mean Value Theorem and linearization (Bateman et al., 2021). In statistics, slope is a key parameter for regression lines (Nagle et al., 2017). In science, students need to understand functional applications of slope since physical quantities are often defined as gradients (e.g., velocity, acceleration) and are represented with line graphs (Planinic et al., 2012). An understanding of slope as a physical application is also needed in science and engineering (e.g., to consider cone slopes in volcanology as well as safety considerations related to landslides and riverbank slopes).

Stump (1999) distinguished between contextual applications that promoted thinking of slope as a dynamic, functional relationship versus a static, physical or geometric quantity. Students are often only able to apply slope in particular contexts (Byerley & Thompson, 2017). For instance, a student could focus on slope as the angle of inclination of a line or as the uniform difference quotient in ordered pairs in a table of values for a linear function without seeing that the two are connected. Students frequently develop meanings for slope that are isolated (Dolores-Flores et al., 2019) and struggle to work with different representations of slope (Adu-Gyamfi & Bossé, 2014; Tanışlı & Bike Kalkan, 2018). Although there are multiple ways to consider slope, a prevalence in thinking of slope as a ratio has been reported (Nagle et al., 2013; Stump, 1999).

Studies of slope in the standards and curricular documents have been conducted in Mexico, South Africa, and the U.S. (Moore-Russo et al., 2011, Stanton & Moore-Russo, 2012, Nagle & Moore-Russo, 2014, Dolores Flores et al., 2020). Recently, a comprehensive study conducted in the U.S. concluded that calculus students and high school teachers held similar meanings of slope, which the researchers described as “unproductive for understanding calculus” (Frank & Thompson, 2021, p. 561). Frank and Thompson’s (2021) analysis included analysis of ten precalculus textbooks. The researchers concluded that the texts they analyzed assumed students connected slope to the notion of constant rate of change and that the texts did not support students’ variational reasoning in the context of slope (Frank & Thompson, 2021). Moreover, Frank and Thompson (2021) point out, “While an expert can conceptualize slope as a ratio that measures the constant rate of change of one quantity with respect to another, many students conceptualize slope as an index of steepness of a line and thus experience difficulty interpreting slope as a constant rate of change" (p. 554). Frank and Thompson's (2021) work points to the importance of studying the interplay between students' instruction on slope and their opportunities to develop covariational reasoning. To understand how students do reason about slope as a constant rate of change, it is important to consider the extent to which textbooks include covariational reasoning with notions of slope. We investigate the interplay of slope coverage and covariational reasoning across a mathematics textbook series in this study.

Methods

An overview of the data, the processes for collecting, coding, and analyzing the data, and a detailed description of the data coding categories follows.

Purpose of Study

We expand upon previous work to better understand how slope is presented in textbooks, concentrating on the development of slope across a secondary textbook series. Past research indicates that slope is a topic that spans the secondary mathematics curriculum (Nagle & Moore-Russo, 2014, Stanton & Moore-Russo, 2012) and which plays a key role in understanding average and instantaneous rates of change (Zandieh & Knapp, 2006). Moreover, past research suggests that it is not only teachers' personal content knowledge that impacts their slope instruction, but classroom textbooks are also likely to play a role in how this topic is taught (Nagle & Moore-Russo, 2015).

This supports the need for research to describe how slope is developed not just in a particular lesson or course, but rather there is a need for a more expansive view of how slope develops across courses and textbooks. Frank and Thompson's (2021) findings that multiple precalculus textbooks did not support students' covariational reasoning in the context of slope warrants further investigation of the treatment of slope in textbooks found earlier in the curriculum, and across an entire textbook series, to fully understand the extent to which students are given opportunities to conceptualize slope as a rate of change. As we seek to better prepare students for calculus by investigating their experiences with slope and rate of change, we must consider those ideas developed leading up to and including their experiences in precalculus. Collectively, these opportunities to conceptualize slope and rate of change inform the understanding students bring to calculus.

Data Used for Study

We chose to investigate slope across all seven textbooks in the University of Chicago School Mathematics Project (UCSMP) secondary mathematics curriculum. The UCSMP was selected because it is a research-based curriculum project that is standards-driven and that has undergone extensive field testing and revisions. The UCSMP curriculum has been studied extensively in research (e.g., Dietiker & Richman, 2021), including studies related to algebra achievement (Thompson & Senk, 2001). It is part of a curriculum developed by UCSMP written to address the U.S. Common Core State Standards; this curriculum claims to emphasize applications, digital resources, and mastery learning (UCSMP, n.d.). The UCSMP is widely accepted as a high-quality, reform-oriented textbook series. Still, the current in-depth investigation of slope development within this series can be informative to these and other curriculum developers, offering insight into characteristics of this concept they may not have considered. The seven textbooks analyzed included: *Pre-Transition Mathematics* (PTM); *Transition Mathematics* (TM); *Algebra* (A); *Geometry* (G); *Advanced Algebra* (AA); *Functions, Statistics, and Trigonometry* (FST); and *Precalculus and Discrete Mathematics* (PC). Note that the textbooks' abbreviations are used in the tables and figures below. Since this study specifically focused on linear slope, the coding of the textbooks excluded examples of variable or instantaneous slope that did not have explicit connections to linear slope.

Data for this study included all the expository material (i.e., the components of the textbook that conveyed information intended to facilitate student learning) within the textbook series. The different types of expository material analyzed included: chapter overviews, explanatory dialogue examples, and activities. Each chapter began with a two-page overview intended to motivate the topics that followed. Within chapters, each section typically followed a similar format with explanatory dialogue punctuated with examples. The explanatory dialogue was text that introduced new terminology and definitions, reviewed foundational ideas, and provided general explanations. The examples were used to illustrate, clarify, and extend the ideas and relationships provided in the explanatory dialogue. They were either fully complete or mostly complete with a few missing details to prompt student thinking. Some sections included activities, often utilizing digital resources, which guided students through a series of steps with embedded explanations and guided questions.

Data Coding and Analysis

We chose to analyze the data systemically in a deductive process using three existing concept-driven coding frameworks; each had been used in prior research studies involving analysis of mathematical texts. This required first identifying the unit of analysis. The unit of analysis was easily defined for examples and activities (each example or activity was a single unit of analysis). For the chapter overview and the explanatory dialogue, a unit of analysis was distinguished as all the content included within a single heading or separated by examples or activities. While most units of analysis included one to two paragraphs of mathematical expository content, some were as short as two sentences and others extended to three or more paragraphs. For the first pass through the data, the first and third authors identified all units of analysis that involved the topic of slope in the textbook series.

Next the research team extensively reviewed each of the three coding categories, looking at multiple representative examples of each code. The third author was the primary coder. The first and third authors practiced coding together for weeks, using other resources besides the textbook series, and then the coding process took place for three months. The first author met weekly with the third author to confirm coding selections, which were taken only from the expository sections of the textbooks. Each textbook took one to three weeks to code in full, with the duration depending on the specific textbook. For example, the A textbook had many more references to slope than the G textbook and, therefore, took much longer to code.

Once coding was complete, the data, which were in spreadsheets, were sorted and prepared for analysis. The first and second authors worked on the data analysis and write up of the results. The sorted data were used to study each of the seven textbooks. Next, the coded data for the entire textbook series was studied to describe the consistency and progression of the data in light of the three coding categories to look for longitudinal trends across the series.

Coding Categories

Data was coded for slope components, covariational reasoning, and use of applications.

Table 1
Slope Component Coding (adapted from Nagle & Moore-Russo, 2013b)

Slope Code	Approach	Description
Constant Parameter	*Visual* (CP-V)	Defining parameter of linear graph (with a *y*-intercept) that indicates a uniform "straightness" of the line's entire graph; no matter which segment of the line is considered the "straightness" is constant due to similar triangles
	Nonvisual (CP-N)	Defining parameter of linear relationship (with a *y*-intercept) indicating constant rate of change between two covarying quantities; slope calculations remain constant between any two points or on any increment of change in independent variable
Ratio	*Visual* (R-V)	Ratio calculated by rise/run or vertical change divided by the horizontal change between any two graphed points
	Nonvisual (R-N)	Ratio calculated for any two ordered pair points (x_1, y_1) and (x_2, y_2) using the difference quotient $(y_2 - y_1)/(x_2 - x_1)$
Behavior Indicator of line or linear relationship	*Visual* (BI-V)	Indicator of (increasing, decreasing, horizontal, or vertical) behavior of linear graph; correlates sign of slope to directions of rise and run to determine graphical behavior
	Nonvisual (BI-N)	Indicator of increasing, decreasing, or constant behavior of linear relationship; correlates sign of slope to relationships between *change in y* and *change in x*
Steepness of line's angle of inclination with horizontal	*Visual* (S-V)	Measure of steepness of linear graph (how inclined, tilted, slanted, or pitched a line is *seen* as being); relates slope to angle of elevation of linear graph
	Nonvisual (S-N)	Measure of how extreme a linear rate of change is *calculated* as being (e.g., relates magnitude of |slope| with corresponding magnitude of $\|y_2 - y_1\|$); relates slope to calculation of $\tan\theta$
Determining Property between lines	*Visual* (DP-V)	Property that determines if linear graphs will intersect and how (e.g., if slopes are negative reciprocals, the lines intersect at right angles)
	Nonvisual (DP-N)	Property that determines whether two linear relationships that form a system of equations will have solutions and how many solutions will result

Slope conceptualization-approach pairs. Stump's (1999, 2001a, 2001b) initial research determined that slope is a multifaceted notion, which is conceptualized in many ways. Moore-Russo and colleagues (2011) introduced conceptualizations of slope as the ways that people think about and make meaning of the topic. The categories have been revisited and revised in research that bridges secondary to postsecondary mathematics (Nagle et al., 2013, 2019) resulting in a more nuanced conceptual framework using five connected components, each with visual and nonvisual approaches (Nagle & Moore-Russo, 2013b). In Table 1, we adopt a revised framework omitting the *Calculus* component since our study focuses on the development of slope in a context prior to calculus. Furthermore, we include both the *Ratio* and *Constant Parameter* components of slope to more completely delineate the nuances of slope development around these two closely connected components. The descriptions for the five conceptualization codes and the two approaches in this coding scheme were adapted from Authors' (2013b) coding. The component coding used for this study is found below in Table 1, with as many codes being assigned as were evidenced in each unit analyzed.

Table 2
Levels of Covariational Reasoning (Carlson et al., 2002)

Level	Description	Sample Slope Language at Corresponding Level
L1: Coordination	Coordinate *change* in one variable with change in second variable	Slope describes how the temperature changes over time.
L2: Direction	Coordinate *direction of change* in one variable with change in second variable	A positive slope indicates that as time increases, the temperature also increases.
L3: Quantitative Coordination	Coordinate *amount of change* in one variable with change in second variable	The slope of -4 indicates that the temperature decreases by 4 degrees every hour.
L4: Average Rate	Coordinate *average rate of change* of function with uniform changes in input variable	Between 2:00 and 5:00, the temperature decreased by an average of 4 degrees every hour.
L5: Instantaneous Rate	Coordinate *instantaneous rate of change* of function with continuous changes in independent variable	Evaluating the derivative function at $t=3$ indicates that temperature was changing at a rate of -4 degrees per hour at 3:00.

Covariational reasoning. Covariational reasoning relates to the "mental coordination of two varying quantities while attending to the ways in which they change in relation to one another" (Carlson et al., 2002, p. 354). Slope is a topic that describes the covariational relationship between the dependent and independent variables in a linear relationship. To understand the development of slope reasoning across the curriculum, it is vital to consider how these components are built from an underlying conception of covariational reasoning.

Carlson and colleagues (2002) describe five hierarchical levels of covariational reasoning, outlined in Table 2. Our study focuses on the development of slope prior to calculus, including slope of linear functions (i.e., contexts where the average and instantaneous rate of change are the same) and average rate of change (i.e., slopes of secant lines) of nonlinear functions. Although the Precalculus textbook did introduce the concept of a derivative to define an instantaneous rate of change of a nonlinear function (i.e., slope of a tangent line), these references to slope were not coded. Hence, we collapsed the L4 and L5 categories of covariational reasoning in our coding.

Contextual applications. Since this text series is described by its publisher as one that values contextual applications, we used binary coding for *applications* or *no applications* to determine the prevalence of connections when the concept of slope appears in expository materials in the textbook series. Following Zhu and Fan's (2006) coding of textbook problems, we considered applications to be any situations that involved some context to the real world or everyday life, even if the situations were more contrived than authentic. Problems that were purely abstract or theoretical, with no mention of context (e.g., inclusion of only equations or graphs) were coded as *no applications*. Occurrences coded as *applications* were further distinguished using Stump's (2001b) categories of *functional* versus *physical*. Table 3 below shows representative examples of the coding used.

Table 3
Types of Applications

Application Type	Description	Sample Occurrences in Series
Functional	Consideration of slope as a property of dynamic, functional situations (e.g., distance versus time) involving constant rates of change	*A tortoise is one of the slowest reptiles. In one test, a tortoise walked about 0.11 meters per second. This means that in t seconds, at this rate a tortoise walks about d meters, where d=0.11t.* (TM, p. 524)
Physical	Consideration of slope as a property of a static, physical object (e.g., the pitch of a roof or the steepness of a ramp) with uniform inclination	*By federal law, every new public building must have an access ramp with slope 1/12 or less. What is the maximum angle of elevation for a ramp?* (FST, p. 322)

To illustrate how the various coding constructs come together within a single slope occurrence, consider the following expository excerpt from one of the guided examples in the A textbook (p. 584) that employs question prompts in its worked-out solution.

Find all solutions to the system $y = \begin{cases} -x+4 \\ -x-2 \end{cases}$.

Solution: Graph both equations using a graphing calculator.

1. *How many times do the lines appear to intersect?*
2. *What is the slope of each line?*
3. *If two lines in a plane do not intersect, they are called ________ lines.*

Now look at the table of values on the graphing calculator.

4. *Is there an ordered pair common to both lines?*
This is an example of a system of equations for which there is no solution. The solution set is ∅.

The codes assigned to this guided example are now explained. The example first brings students' attention to the parallel relationship between the graphs of the respective lines. By expecting students to recognize slope as the constant parameter *m* in the given linear equations and calling attention to the two slopes, which are the same value, the example brings together two components each with different approaches, namely C-N and DP-V. By drawing attention to the system of equations as having no solutions for the same slope value, the example further brings in DP-N. This occurrence has no *covariational reasoning* since students were not asked to consider how *x* and *y* covaried in the two linear relationships. The occurrence does not include any real-world context; so, it has *no applications*.

Results and Discussion

We now turn to the results of the study. The findings and related discussion are presented in three sections that correspond with the three research questions.

Slope Components

Across the entire series, 201 expository units were identified and coded as addressing slope (see Table 4). All seven textbooks in the series addressed slope, even if not explicitly using the term when first introduced. As anticipated, the number of slope occurrences was highest in the A and AA textbooks, with these two texts accounting for 55% of all occurrences across the series.

Table 4
Relative Frequency of Slope Occurrences across the Textbook Series (n = 201)

Percentage of All Slope Occurrences Across Entire Series	Textbook						
	PT	TM	A	G	AA	FST	PC
	5%	14%	35%	10%	20%	7%	8%

Textbooks. Information about the frequency of each slope component within each textbook and across the series is provided in Table 5 by *visual* approach only, *nonvisual* approach only, and *both* visual and nonvisual approaches. Over the series, about 90% of all slope occurrences included either the *Constant Parameter* component, *Ratio* component or both components. Moreover, one of these two was the most prevalent component identified for each textbook, and this was particularly true in the textbooks with higher numbers of slope occurrences. Table 5 indicates that both the *Constant Parameter* and *Ratio* components were assigned to 50% or more of the slope occurrences in four of the seven textbooks (i.e., TM, A, G, and AA). There was a marked prevalence of *nonvisual* only approaches across all textbooks in the series for the most frequently occurring slope components (i.e., *Constant Parameter* and *Ratio*). The instances that involved *visual* only approaches typically occurred in the G textbook with the *Behavior Indicator*, *Steepness*, and *Determining Property* components.

Component-Approach Pairs. The total number of *visual* only and *nonvisual* only approaches for all slope components tended to vary greatly, with a marked prevalence of *nonvisual* only approaches across all textbooks in the series for the most frequently occurring slope components (i.e., *Constant Parameter* and *Ratio*). Use of the *Behavior Indicator*, *Steepness*, and *Determining Property* components frequently involved *visual* only approaches. Occurrences either involving *visual* only approaches or *both* nonvisual and visual approaches of the *Behavior Indicator* component were more prevalent than for the other slope components. Connections between BI-N and BI-V in analyzed units were often facilitated by explanations that incorporated two or more representations of linear functions (e.g., the equation $y = -2x+7$ and the corresponding linear graph) when analyzing what the slope indicates both about the rate of change of y with respect to x (e.g., as x increases by 1, y decreases by 2) and about the graphical representation of that relationship (e.g., as a decreasing line that moves to the right 1 unit as it moves down 2 units). Note that in situations such as this, BI-N was coded since the corresponding directions of change of the two covarying quantities were linked (L2 covariational reasoning) and connected to the increasing or decreasing behavior of the linear graph (BI-V). However, these occurrences often stopped short of relating the reasoning about direction of change and the increasing or decreasing nature of the function itself (e.g., if $x_1 < x_2$, then $f(x_1) < f(x_2)$). Other than *Behavior Indicator*. There were strikingly few occurrences incorporating both *visual* and *nonvisual* aspects for any of the slope components.

Table 5
Frequency of Slope Components by Approach within Occurrences by Textbook

Textbook (number of slope occurrences)	Slope Components (by *Visual* only, *Nonvisual* only, or *Both* Approaches)														
	CP			R			BI			S			DP		
	V	N	both	V	N	both	V	N	both	V	N	both	V	N	both
PTM (n=11)	0	0	2	0	9	0	1	4	0	0	0	1	0	0	0
TM (n=28)	0	14	3	0	19	0	2	6	5	3	0	0	1	0	0
A (n=70)	5	45	8	4	40	5	7	14	10	4	1	1	2	1	4
G (n=21)	0	11	0	2	10	2	3	0	0	3	0	0	10	0	0
AA (n=40)	3	27	0	0	21	4	1	8	4	1	1	1	8	1	0
FST (n=14)	0	9	1	2	3	0	0	2	2	0	2	0	0	0	0
PC (n=17)	0	6	0	0	9	2	1	2	5	0	0	1	1	0	0
Series (n=201)	8	112	14	8	111	13	15	36	26	11	4	4	22	2	4

The number of slope occurrences, by textbook, are provided in Figure 1. In each cluster, the bars represent the frequency of the five different slope components. The frequencies indicate that textbook A included the most overall slope occurrences followed by AA. This was not a surprise, given the emphasis of slope in high school Algebra classes.

The *Constant Parameter*, *Ratio*, *Behavior Indicator*, and *Steepness* components of slope each occurred more often in A than any of the other textbooks. The prevalence of *Constant Parameter*, *Ratio*, and *Behavior Indicator* references of slope in the textbook series are consistent with the emphases noted in studies of the curricular standards documents used in the U.S. (Stanon & Moore-Russo, 2012; Nagle & Moore-Russo, 2014). *Determining Property* is the only slope component that did not occur most often in A, with more occurrences in both the G and AA textbooks. Figure 1 also illustrates that the emphasis of slope components within each textbook remains relatively similar. The *Constant Parameter* and *Ratio* components of slope are the two most frequently occurring components in five of the seven textbooks. In the first and last textbooks in the series (i.e., PTM and PC), *Ratio* is the most dominant component, but *Constant Parameter* drops below *Behavior Indicator* in frequency in both textbooks. *Behavior Indicator* is the second or third most prevalent slope component in every textbook except for G. In G, *Determining Property* receives more relative emphasis compared to the other components than it does in the other six textbooks.

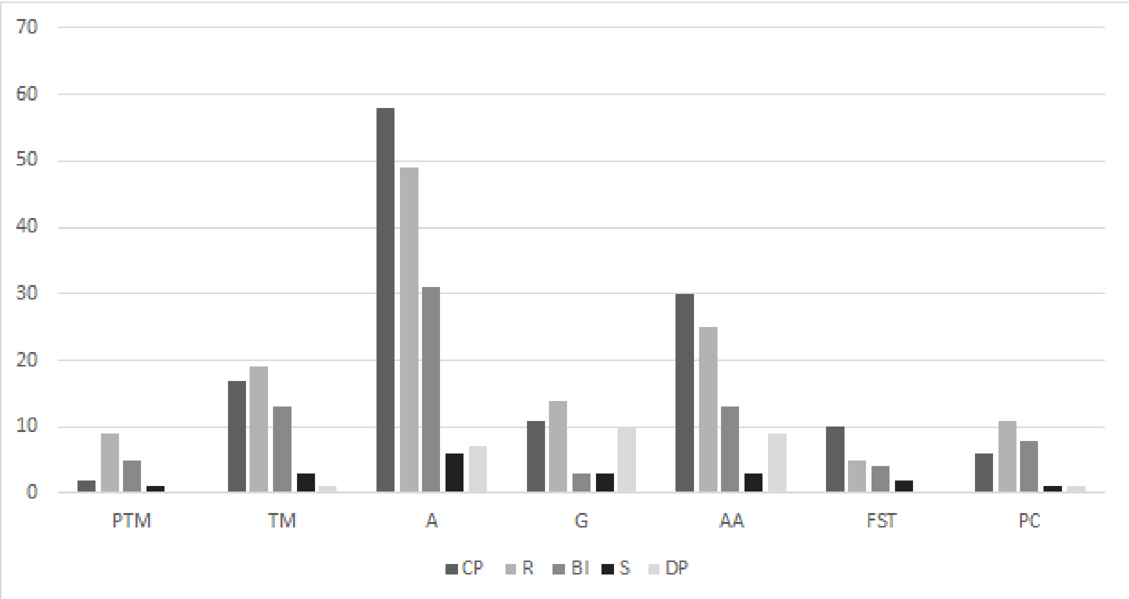

Figure 1. Frequency of slope component occurrences by textbook

Steepness occurs at low frequencies in all seven textbooks. The infrequency of *Steepness* through the textbook series is in line with previous research that found it to be the least frequent way that slope was put forward in states' standards documents (Stanton & Moore-Russo, 2012).The lack of slope as relating to *Steepness* is interesting and in contrast to past work that has pointed to building notions of slope as related to inclination and angles of elevation as important mechanisms for connecting students' intuitive experiences with slope (Nagle & Moore-Russo, 2013a; Stump, 2001b).

Previous research (Nagle et al., 2013) on college calculus students' notions of slope found that although *Constant Parameter* and *Ratio* were prevalent, both components seemed to be grounded in notation and procedures rather than deep understanding. In the previous study, students' responses to the question *How is slope used?* were most often consistent with the *Constant Parameter* component but were often trivial (e.g., slope is m in $y=mx+b$) lacking any evidence of real understanding. Students' responses to the question *What is slope?* typically involved *Behavior Indicator*, *Ratio*, and *Steepness Indicator* components. They often described slope as a tool to determine the increasing or decreasing nature of a linear graph or a quantity that could be calculated. They rarely responded with any mention of a relationship between covarying quantities. When asked on an additional prompt to provide examples of slope, the students generally used the *Ratio* component with some version of the mnemonic "rise over run" or a *Ratio* calculation involving two ordered pairs (e.g., 5-4/6-3=1/3). The examples rarely connected slope as a *Ratio* with any covariational language about the corresponding change in quantities (Nagle et al., 2013).

Considering the results that slope is most frequently developed as *Constant Parameter* and *Ratio* in this textbook series and students' trivial use of these components of slope in past research, additional analysis on how the textbook series develops these two components with relation to the other slope components and in tandem with covariational reasoning is warranted.

To better understand how the various components of slope are developed and connected, we also considered which slope components were developed concurrently within a single occurrence. A total of 21 unique combinations of slope components (e.g., *Constant Parameter* with *Ratio*. *Constant Parameter* with both *Ratio* and *Behavior Indicator*) were identified across the series. Each of these combinations and the corresponding number of occurrences coded with each unique combination of slope components is provided in Table 6.

Using both Figure 1 and Table 6, we now consider when the slope components are introduced. Coverage of slope appears to be focused on fewer slope component combinations in the early curriculum and again in the later curriculum, with a greater diversity of component combinations in the middle curriculum, particularly in the two algebra (A and AA) textbooks, where slope is emphasized. In the PTM textbook, slope is first introduced, using the *Ratio* component primarily. This extends to the formal definition of slope as a constant value which defines a linear relationship in the TM textbook; hence, there are more occurrences of the *Constant Parameter* component, often combined with the *Ratio* component. Starting in the TM textbook and continuing into the A textbook, slope as a *Behavior Indicator* receives more emphasis, and the *Behavior Indicator* component is often built upon the *Constant Parameter* and *Ratio* components. The TM and A textbooks emphasize that slope is used to indicate the increasing or decreasing nature of a linear relationship by building the notion of slope as a *Constant Parameter* that applies to the entire linear relationship that is calculated using a *Ratio*. In the A and AA textbooks, we see the number of occurrences of all components increase. In both the A and AA textbooks, the notion of slope is extended and used for a variety of purposes (including *Behavior Indicator* and *Determining Property*). In the G textbook, the number of occurrences drops; however, it is here and in the AA textbook that we see slope being used as a *Determining Property*. In the later curriculum (FST and PC), when slope is used as an established concept, coverage is again focused on fewer component combinations and more occurrences of single slope components. 64% of the occurrences involving slope in FST involve a single slope component. We recognize that in these instances it might be assumed that students have already established the other connections and thus they may not be reiterated when these concepts reappear later. The *Steepness* component is not concentrated in a single textbook but rather occurs in low numbers across all textbooks.

Table 6
Frequency Heat Map of Slope Component Combinations across Textbooks and Series

	Total Number of Occurrences including each Slope Component Combination							
Component Combinations	PTM (n=11)	TM (n=28)	A (n=70)	G (n=21)	AA (n=40)	FST (n=14)	PC (n=17)	Across Series
CP-R	2	8	21	5	11	2	4	53
CP-R-BI		2	15	1	6	1		25
CP		2	8	1	3	6	1	21
R-BI	4	6	6				3	19
R	4	2	3	2	2		3	16
CP-DP		1	5	2	4		1	13
BI		2		1	2	3	4	12
CP-BI		2	4		3			9
R-DP			1	4	2			7
CP-R-BI-S			1	1	1			3
CP-BI-S		1	2					3
DP				1	2			3
CP-R-DP				1	1			2
CP-S			1		1			2
R-BI-S			1				1	2
S-DP				2				2
S	1	1						2
R-S					2	1		3
CP-R-S		1				1		2
CP-R-BI-DP			1					1
BI-S			1					1
Total per Textbook	11	28	70	21	40	14	17	201

Our findings highlight the prevalence of *Constant Parameter* and *Ratio* components across the textbook series. The slope component combinations displayed in Table 6 provide additional insight about how frequently these components were developed alone or in relation to other components. Of the 201 occurrences, *Constant Parameter* appeared by itself in 21 occurrences, and *Ratio* appeared by itself in 16 occurrences. They appeared together, with or without other components, in 86 occurrences. *Constant Parameter* appeared with at least one other component (that was not with *Ratio)* in 27 occurrences. *Ratio* appeared with at least one other component (that was not with *Constant Parameter)* in 31 occurrences. *Behavior Indicator* appeared by itself in 12 occurrences and appeared with other components in 65 occurrences. *Behavior Indicator* was combined with *Constant Parameter* in 12 occurrences, with *Ratio* in 21 occurrences, with both *Constant Parameter* and *Ratio* in 29 occurrences. Since *Determining Property* and *Steepness* only appeared 28 and 19 times respectively across the textbook series, they obviously had fewer combinations with other slope components. *Determining Property* appeared in 3 occurrences by itself, and *Steepness* appeared in 2 occurrences by itself.

Covariational Reasoning

Across the textbook series, 53% of all expository content related to slope exhibited some level of covariational reasoning. As might be expected from expository content related to slope, L3 reasoning that described slope as a ratio of amount of change in two covarying quantities was the most included (38% of all occurrences across the series). This was followed by L2 reasoning that related the sign of slope to the directions of change of covarying quantities (7% of all occurrences) and L1 reasoning that simply linked slope to two covarying quantities without mention of either the direction or amount of that change (5% of all occurrences). Only 3% of the occurrences incorporated L4/L5 reasoning, typically to describe the slope of the secant line through the endpoints of a real-world functional relationship in terms of the average rate of change of the covarying quantities (e.g., from April 1st to May 1st, the length of day has been changing at an average of 2.13 minutes/day).

Covariational reasoning levels incorporated within the expository slope content varied from one textbook to another, as demonstrated in Figure 3. One observation is that the later curriculum (starting at the G text and beyond) incorporates a higher percentage of slope occurrences with no explicit covariational reasoning. The G textbook is particularly dominated by slope references that do not incorporate covariational reasoning. In the later curriculum, slope has already been formally defined and is often referenced without describing it as the constant rate of change of covarying quantities. When covariational reasoning is present, Figure 3 supports a subtle shift to more occurrences at higher levels (L4/L5) in the later textbooks in the series. Again, this is not surprising but does support the important link between students' development of covariational reasoning required for success in later mathematics (Carlson et al., 2003) and their experiences working with slope. The small uptick in L2 reasoning visible in the PC textbook can be attributed to the focus on increasing and decreasing functions presented in that textbook.

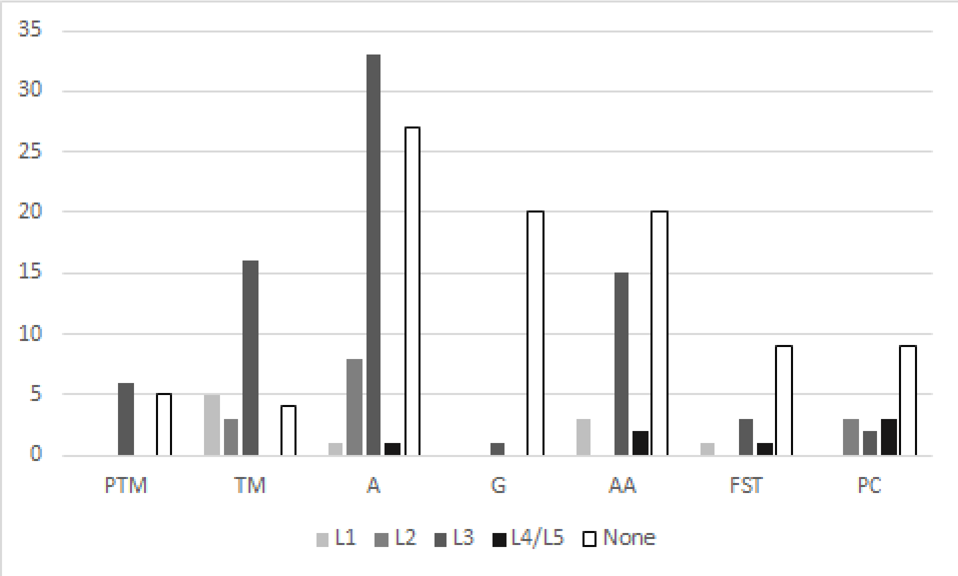

Figure 3. Frequency of occurrences within each textbook by covariation level

We now consider which levels of covariational reasoning accompany the development of each slope component. Table 7 provides the percentage of all slope occurrences for each component broken into the various levels of covariational reasoning. For instance, row 1 highlights that 44% of all occurrences involving *Constant Parameter* included no covariational reasoning. The remaining 56% did include covariational reasoning, with the majority involving L3 reasoning (45% total). A relatively small percentage of the *Constant Parameter* occurrences included L1 (6%), L2 (4%), and L4/L5 (1%) reasoning.

Table 7
Relative Frequency of Covariational Levels by Slope Component

Component	None	L1	L2	L3	L4/L5
CP (n=134)	44%	6%	4%	45%	1%
R (n=132)	38%	3%	3%	52%	5%
BI (n=77)	29%	4%	17%	44%	6%
DP (n=28)	89%	4%	0%	7%	0%
S (n=19)	37%	5%	16%	37%	5%
All Occurrences	47%	5%	7%	38%	3%

A few trends are noticeable when considering the relative frequency of the covariational reasoning levels in conjunction with each slope component. Notably, *Determining Property* stands out for seldom being developed in conjunction with covariational reasoning. This might be due to its occurrence later in the curriculum, when fewer slope references involve covariational reasoning. While *Behavior Indicator* was the slope component most often involving covariational reasoning, *Ratio* was most likely to incorporate higher levels of covariational reasoning (L3 or L4/L5). Still, it is somewhat surprising that 44% of all occurrences involving the *Behavior Indicator* component incorporated L3 reasoning since *Behavior Indicator* aligns well with L2 reasoning focused on coordinating direction of changes. However, our prior analysis indicated that *Behavior Indicator* was often linked with *Ratio*, which would facilitate extensions beyond simply coordinating directions of change between the covarying quantities to quantifying amounts of change (i.e., a *Ratio* of amounts of change).

Table 8 provides the frequencies of each level of covariational reasoning for slope occurrences with, *visual* only, *nonvisual* only, and *both* visual and nonvisual approaches. This data indicates that *visual* only approaches, although uncommon, tended to occur when there was no covariational reasoning. Slope occurrences incorporating L1 and L4/L5 covariational reasoning were approximately equally split between *nonvisual* only and *both* visual and nonvisual approaches, but both categories had few occurrences (10 and 7, respectively). The 14 slope occurrences incorporating L2 reasoning tended to use *both* visual and nonvisual approaches, often connecting functional behavior or relationships (increasing or decreasing relationship) to a graphical interpretation ("going up"). Most slope occurrences that incorporated covariational reasoning were at the L3 level (76 of 107). For them, reasoning was more frequently connected to *nonvisual* only slope approaches despite the opportunity to visually connect slope as a measure of steepness related to the amount of change in the dependent variable for a given change in the independent variable. In other words, *visual* approaches to slope might help students see that a steeper line has a greater slope since larger quantitative changes in outputs occur than would occur for a line that was not as steep, given equivalent changes in the corresponding inputs for the two lines.

Table 8

Frequency of Covariational Reasoning Levels by Type of Approach

	Visual only (n=11)	*Nonvisual* only (n=106)	*Both* visual & nonvisual (n=84)
L1 (n=10)	1	5	4
L2 (n=14)	2	3	9
L3 (n=76)	1	52	23
L4/L5 (n=7)	0	4	3
None (n=94)	7	42	45

Contextual Application

Contextual applications were present in 53% (106 of 201 occurrences) of all the expository slope occurrences across the series. This suggests that the textbooks drew heavily on real-world situations to build meaning for slope. However, like covariational reasoning, the prevalence of *applications* varied across the series (see Table 9, columns 1 and 2). In particular, the two early textbooks relied more heavily on *applications* than the others. The textbooks with lower percentages of covariational reasoning for their slope occurrences seem to coincide with those that involved fewer *applications*. This makes sense given real-world contexts often facilitate the use of covariational language. Slope was rarely connected to contextual applications in the FST textbook with only one such connection in the G textbook. While this might be expected in geometry, it is rather surprising for a textbook that covers functions, statistics, and trigonometry.

Table 9

Relative Frequency of Contextual Applications by Textbook, Type of Approach, and Covariational Reasoning Level

Occurrences by Textbook	Contextual Applications	Type of Approach	Contextual Applications	Covariational Reasoning	Contextual Applications
PTM (n=11)	100%	V (n=11)	18%	L1 (n=10)	60%
TM (n=28)	86%	N (n=106)	65%	L2 (n=14)	43%
A (n=70)	57%	Both (n=84)	42%	L3 (n=76)	89%
G (n=21)	5%			L4/L5 (n=7)	86%
AA (n=40)	53%			None (n=94)	21%
FST (n=14)	14%				
PC (n=17)	41%				

The results provided in Table 9 in columns 3 and 4 show the relative frequency of contextual applications for slope occurrences that included *visual* only, *nonvisual* only, and *both* visual and nonvisual approaches. Results highlight that nearly two-thirds of slope occurrences without *visual* approaches had contextual applications. Looking at this from a different vantage point, only about one third of all contextual application occurrences involved any visualization of slope (either *visual* only or *both* visual and nonvisual). The small percentage of contextual applications with visual approaches prompted the researchers to further investigate the applications using Stump's (1999) distinction between dynamic, *functional* contexts (e.g., money versus time) and static, *physical* contexts (e.g., pitch of a roof). Only four contextual application slope occurrences included *physical* contexts, with the PTM, A, G, and FST texts each including one physical application involving a pyramid, roof, water slide, and ramp, respectively. The remaining 102 contextual applications of slope all involved *functional* situations. This contrasts with Stump's (1999) findings that teachers in the study who mentioned real-world examples, when describing classroom instruction on slope, more frequently referenced *physical* (versus *functional*) situations.

Information in columns 5 and 6 of Table 9 suggests a link between the development of covariational reasoning and the use of contextual applications. Relatively few of the slope occurrences with no covariational reasoning incorporated real-world contexts (21%) while the percentage of occurrences incorporating L1, L3, and L4/L5 reasoning were 60%, 89%, and 86%, respectively. Contextual applications were only used in 43% of the occurrences incorporating L2 reasoning, which is lower than the other three levels of covariational reasoning. However, L2 reasoning was often built visually in conjunction with describing the direction of change of two covarying quantities (e.g., as the x-values get larger, the increasing graph suggests the *y*-values also get bigger), and contextual applications were helpful, but not necessary, to facilitate this type of discussion. Overall, the data supports that the expository content in the textbook series frequently used

Recall that a slope occurrence might be coded with more than one slope component. To look at the contextual applications by slope component, it is important to recognize this overlap when interpreting Table 10. The results indicate *Determining Property* rarely occurred with *applications*, compared to the other slope components. Discussions and sample problems in the expository text using slope to determine relationships between lines typically occurred without any applied context, with no consideration of what a parallel (or perpendicular) relationship would mean in an applied real-world situation.

Table 10
Relative Frequency of Contextual Applications by Slope Component

Slope Component	Contextual Applications
CP (n=134)	56%
R (n=132)	66%
BI (n=77)	68%
DP (n=28)	4%
S (n=19)	63%

Summary

The UCSMP curriculum develops slope across all seven of its secondary mathematics textbooks and utilizes real-world applications frequently. Our analysis highlighted the importance of considering more than just students' experiences with slope in precalculus (e.g., looking not only at precalculus textbooks but also prior books in the curriculum series) when determining their preparation for calculus. A natural progression of covariational reasoning and slope occurrences was seen as we looked across the curriculum. A glance at the precalculus curriculum alone might have suggested a lack of covariational reasoning and few combinations of slope components to prepare students for calculus. However, when viewed considering what occurs across the entire curriculum of the series used in this study, it seems likely that the term *slope* is often used in precalculus textbooks as a taken-as-shared term (with a robust, but previously developed, meaning) without a thorough rehashing all the notions of slope developed in the courses leading up to precalculus.

Our analysis suggests that slope is primarily and consistently developed with *Constant Parameter*, *Ratio*, and *Behavior Indicator* interpretations in the UCSMP textbook series. Moreover, these three components of slope are often connected to each other and frequently incorporate covariational reasoning. Together, these ideas suggest that the UCSMP positions students to be prepared to understand the definition of a derivative as a function which provides the instantaneous slope (or rate of change) of the original function at a given point. The series' grounding across the seven textbooks of slope as a *Behavior Indicator* with connections to other slope components also suggests students should be well positioned to understand the first derivative test's approach of using the sign of the derivative for determining open intervals over which a function is increasing and decreasing.

Slope was less frequently described as a *Determining Property*. Occurrences of slope involving *Determining Property* were rarely connected to covariational reasoning and real-world contexts. The lack of emphasis on determining property could impact students' understanding of both antiderivatives and the Mean Value Theorem. As students in calculus develop an understanding of the relationship between families of antiderivatives of a function, they need a notion of *Determining Property* connected with covariational reasoning. They need to understand that two linear functions with the same slope will have the same change in the output variable for a set change in the input variable and should be able to connect this to corresponding graphs that are parallel because they produce the same rise for a set run. This extends to antiderivatives in calculus where a student can interpret two antiderivatives as having the same derivative function, and, hence, the same instantaneous rate of change, and parallel tangent lines, at points with the same input values, as illustrated in Figure 4. Extending this thinking across all x-values in the domain of the two functions would lead to the notion of a family of antiderivatives as consisting of all vertical translations of a given function. Calculus also builds on students' notions of *Determining Property*, often using both *visual* and *nonvisual* approaches, when the Mean Value Theorem is introduced, since it requires considering equivalent average and instantaneous rates of change visualized as slopes of secant and tangent lines, respectively.

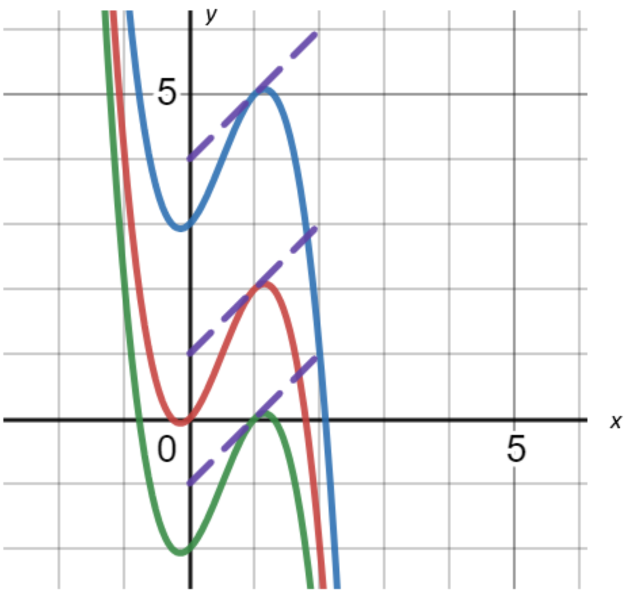

Figure 4. A family of antiderivatives with parallel tangent lines for any common input

Slope was infrequently connected to *Steepness* in this textbook series. While often oversimplified as an application of slope in a static, physical setting (e.g., building a ramp), it is important that students develop notions of slope as *Steepness* that are connected to their other, more dynamic notions of slope (Nagle & Moore-Russo, 2013a). From a calculus perspective, a derivative is often conceptualized as a measure of a function's sensitivity to change. This requires students to have a strong foundation connecting slope as a *Ratio* and *Steepness*, where steepness is not just a physical attribute but also a description of a linear relationship's sensitivity to change in outputs for a change in inputs. Note that this is achieved by also building students L3 covariational reasoning. In calculus, this idea is extended to studying concavity of nonlinear functions where we want to compare how this steepness changes to relate physical characteristics (concavity) to functional characteristics (e.g., rate of change increasing/decreasing). A strong foundation connecting slope as a *Ratio* and *Steepness* that allows for an awareness of the sensitivity to change in outputs for a change in inputs, is also important in understanding linearization in calculus. Finally, *Steepness* is important to be able to interpret regression lines in statistics and has been shown to be an area where students struggle to apply their prior knowledge of slope (Nagle et al., 2017).

Conclusions

This study only considers a single textbook series, and there are findings that are specific to the series. However, it also provides a window as to what might occur in other series. Future research should look at the development of slope across the most popular textbook series, including those that have not received as many accolades as the UCSMP curriculum. Future research should also consider how the curricular material is enacted by the instructor; it would be interesting to know if teachers who use UCSMP textbooks supplement them to augment the students' exposure to certain slope components or more real-world problems.

The UCSMP curriculum has been lauded as a high-quality, reform-oriented series that emphasizes contextual applications. Recently, Rezat et al. (2021) described mathematical content as one of three objects of change through curriculum materials (with pedagogical approaches and mathematics as a subject). In-depth investigations of how a particular mathematical concept is developed across the series can provide new insight for the curriculum developers to consider when evaluating whether their materials are achieving the desired changes. In terms of this series' coverage of slope in its expository material, our findings support the textbooks' claims regarding emphasis on contextual applications. However, there are slope components, such as *Determining Property* and *Steepness*, that may need additional attention and intentional coverage by secondary teachers using this curriculum. While the series did incorporate several contextual applications (with a large portion that were *functional*, rather than *physical* contexts), there are textbooks in the series that have fewer *applications*. While that might be expected of a *Geometry* textbook, it is rather surprising for the *Functions, Statistics, and Trigonometry* textbook. This information can support curriculum developers in considering not just contextual *applications* of the concept of slope, but also in considering whether those applications accompany the various components at different points in time across the secondary curriculum.

There are findings in this study that reach beyond the UCSMP curriculum that can be of value both for the research community as well as for practicing instructors. First, students' exposure to slope in several courses across the secondary curriculum, not just in precalculus courses, inform the ideas of slope students bring to a calculus course and will influence how prepared students are to build upon their ideas of slope in a meaningful way. Secondly, this study reiterates how important it is to have a robust notion of slope, which is likely why the topic occurs across the curriculum. Secondary instructors should consider how students are developing their understanding of slope across grade levels. Without an intentional consideration of where in the curriculum different notions of slope are developed, it is more likely that students will arrive in calculus classes "capable of using the slope formula to find the rate of change; however, ...unable to interpret the meaning of rate of change in a contextual situation, with a given graph, or more importantly with nonlinear situations" (Teuscher & Reys, 2012, p. 373). The robustness of students' notions of slope also depends on the extent to which the various components have been developed in tandem, whether *visual* or *nonvisual* approaches have been emphasized, how *covariational reasoning* has been incorporated, and the extent to which students have had opportunities to reason about slope in real-world *applications*.

Third, and finally, although slope is sometimes oversimplified as a physical property of a linear graph, our analysis of slope across this textbook series has illustrated that it is a multi-faceted concept, with different components and approaches emphasized at various stages of the precalculus curriculum, and with important implications for students' preparation for calculus. Thus, improving students' preparation for calculus requires taking a holistic view of developing a flexible, robust understanding of slope across the curriculum; this will help us know possible ways students might build understanding of derivatives, derivative applications, and even antiderivatives from their existing notions of slope.

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