Middle School Mathematics Teachers' Concept Definitions of Geometric Objects*

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This study examines middle school mathematics teachers' concept definitions of five geometric objects (prism, pyramid, cone, cylinder, and sphere). The study included 99 middle-grade mathematics teachers in three provinces and districts. The data were gathered through six open-ended questions determined to reveal the participants' Concept Definition (CD) of Geometric Objects (GOs). Via content analysis, codes from the data and categories from the codes were obtained. Results showed that participants could not make the CDs of GOs correctly, generally drew square or rectangular objects, and only included the horizontal and perpendicular states of the GOs (prototype). Participants generally could not establish a relationship between "cylinder and prism" and "cone and pyramid." Some participants saw themselves as inadequate in defining the GOs. Indeed, teachers' explanations and drawings reveal that their professional experiences have been ineffective in defining those concepts.

Keywords: geometric objects, middle school mathematics teachers, concept definition, prototype

Introduction

Geometry is one of the most important fields of mathematics for making sense of the world (National Council of Teachers of Mathematics (NCTM), 2000). Developing geometric thinking is essential for high-level mathematics (Clements & Battista, 1992). Geometry increases students' spatial awareness, develops reasoning skills, and activates, challenges, and informs them (French, 2004).

More abstract than other mathematical subjects, the subject of Geometric Objects (GOs) requires students to use their spatial abilities more. Since students often struggle with geometry, researchers have conducted many studies on different topics related to geometry teaching and learning. Students must have sufficient geometric knowledge and skills to prepare for higher education and future careers (Russell, 2018). However, TIMSS (2003) reports that students exhibit the lowest achievement in geometry among mathematics subjects (Martin et al., 2008). This may be because students struggle to grasp the static views of GOs and perceive three dimensions (Accascina & Rogora, 2006).

Theoretical Framework for Geometry Teaching

Through teacher training programs and professional development, teachers need help to improve their geometry teaching competence (Yi et al., 2020). Only teachers with sufficient pedagogical knowledge can design desired learning environments (Putnam et al., 1992). Geometry's interdisciplinary nature requires teachers to understand various mathematical topics such as properties of shapes, structures, vectors, symmetry, cartesian equations and diagrams, similarity, and congruency (Sunzuma & Maharaj, 2019). Teachers

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also need presentation knowledge to enable students to understand the subjects and knowledge of student learning regarding difficulties and mistakes (Gökkurt, 2014).

In the teaching process, many teaching activities, such as choosing appropriate learning activities, asking productive questions, and evaluating student learning, depend on the teacher having sufficient pedagogical content knowledge (Ball & McDiarmid, 1990)—how a teacher transforms content knowledge into teaching in the context of students' learning difficulties, previous understanding, and teaching-related concepts (Shulman, 1986). Sunzuma and Maharaj (2019) found that in-service teachers did not sufficiently learn some geometry subjects and, therefore, had to skip them when teaching. They even found that 47.5% of these teachers were not ready to teach due to an inadequacy in teaching geometry. Similarly, studies have revealed that pre-service teachers struggle to define some geometry concepts (Çakmak et al., 2014; Gutierrez & Jaime, 1999; Koç & Bozkurt, 2011; Ulusoy, 2020). Teachers' fears of being inadequately equipped in geometry negatively affect their teaching of even basic geometry subjects (Niyukuri et al., 2020). This fear and inadequacy may cause teachers and students to have similar misconceptions about geometry (Swafford et al., 1997).

Concept Definition (CD)

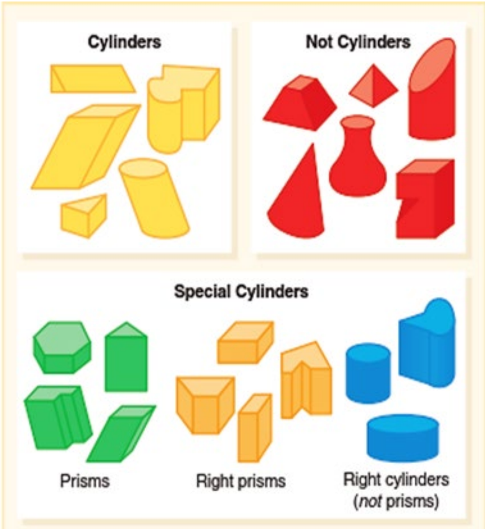
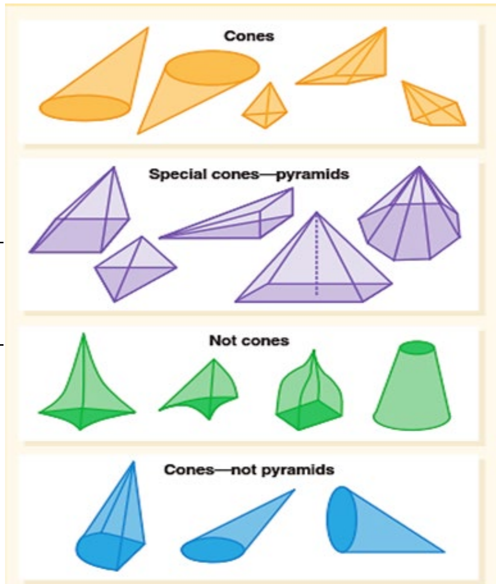
While mathematics aims to develop different thinking and solution strategies in problem situations, obtaining meaningful results can only be possible by making these structures and processes understandable (Schoenfeld, 1992). Individuals need to gain conceptual knowledge of how concepts interrelate. Vinner (1983) explains the concept as a set of words and symbols clearly defining that expression. Having detailed knowledge about a subject requires understanding the subject's concepts well. Since CD forms the basis of the subject, it also provides the opportunity to switch between subjects. Each subject has its own CDs and conceptual framework. Understanding the definitions of the concepts in the subjects helps to avoid simple memorization and visual learning. Thus, the definition process, essential for transparent mathematics education, must be encouraged (Erdem & Man, 2018; Martín-Molina et al., 2020).

Understanding geometric concepts is essential to successfully learning geometry (Marchis, 2012). Developing geometric thinking skills and high-level geometric thinking involves a sufficient understanding of CDs (Linchevsky et al., 1992). While geometry specifically defines cylinder, prism, pyramid, cone, and sphere, students do not have a solid conceptual background in geometry, these shapes, and their definitions (Clements & Battista, 1992; Clements et al., 1998; Fujita & Jones, 2007; Gündüz et al., 2017; Gutierrez & Jaime, 1999; Linchevski et al., 1992; Meng, 2009; Mistretta, 2000; Ulusoy, 2020). Overcoming these difficulties will only be possible with the support of teachers with deep knowledge of geometry concepts and definitions.

The Concepts of Geometric Objects (GOs)

Van de Walle et al. (2008, pp. 412–413) presented the GOs in two main categories, "Cylinders" and "Cones," and explained them as definitions and classifications in Table 1.

Table 1. Concept definitions and explanatory drawings of “Cylinders” and “Cones”

Main Concept	Sub-Concept	Definition	Drawing
Cylinders	*Cylinder*	Two congruent, parallel faces called bases. Lines joining corresponding points on the two bases are always parallel	
	Right cylinder	A cylinder with elements perpendicular to the bases. A cylinder that is not a right cylinder is an oblique cylinder.	
	Prism	A cylinder with polygons for bases. All prisms are special cases of cylinders.	
	Rectangular prism	A cylinder with rectangles for bases.	
	Cube	A square prism with square sides.	
Cones	*Cone*	A solid with exactly one face and a vertex that is not on the face. Straight lines (elements) can be drawn from any point on the edge of the base to the vertex. The base may be any shape at all. The vertex need not be directly over the base.	
	Circular cone	Cone with a circular base	
	Pyramid	Cone with a polygon for a base. All faces joining the vertex are triangles. Pyramids are named by the shape of the base: triangular pyramid, square pyramid, octagonal pyramid, and so on. All pyramids are special cases of cones.	

The Purpose and Importance of the Present Study

While some posit that GOs are among the most important and challenging to learn and teach issues of geometry (Clements et al., 1998; Matos, 1994; Linchevski et al., 1992; Gündüz et al., 2017), there are very few studies examining the current state of middle school mathematics teachers' CDs for each GO. Teachers' determinations of the CDs of GOs directly reflect improving their students' understanding. Because teachers and students have similar misconceptions about geometry (Swafford et al., 1997), teachers' competence or inadequacy in CDs of GOs causes students to perform similarly. When mathematics subjects are taught and CDs are explained in a topical introduction, the incorrect or incomplete presentation of definitions may cause problems regarding following applications or connected topics.

The results of a recent study (Ulusoy, 2019) with prospective teachers give clues about why the definitions of GOs need further investigation. After participants were asked to define the cylinder and prism in written form and present examples from daily life, the class deliberated on the correctness of the examples. Results suggested that, while prospective teachers demonstrated prototypical circular cylinders, some drew the cone and pyramid as prisms. In this context, this present study investigates middle school mathematics teachers' definitions of the concepts of five GOs through the questions:

- Are middle school mathematics teachers able to define the concepts of GOs?
- Do middle school mathematics teachers make sense of the conceptual relationships between GOs?

Method

Research Design and Participants

This study employs a survey-based case study methodology wherein a phenomenon is analyzed in a versatile, systematic, and in-depth way within its real-life framework (Cohen et al., 2000; Yin, 2011).

Since teaching GOs at the conceptual level starts in middle school, this study involves 99 Turkish middle-school math teachers in three different provinces and districts in Turkey. The respective teachers participated voluntarily. To protect the participants' identities, codes such as M1, M2, and M3 were used instead of names. Data on the teachers' professional experiences are given in Table 2. Participant diversity was achieved by observing participants' experiences and work in schools in different socio-economic environments. In later reports of the findings, teachers' opinions are juxtaposed with their years of professional experience in parentheses.

Table 2. Professional experience of the participants

Professional experience	N
0-5 years	39
6-10 years	24
11-15 years	27
16-20 years	6
21-25 years	3

Data Collection

Since research questions should serve the purpose of the study, the respective questions were derived from middle school textbooks and resource books from the Ministry of National Education (MoNE). After the questions were determined, the opinions of two mathematics education research experts and two Turkish teachers were solicited to determine whether the study served its purpose. A pilot study was conducted to test the survey instrument's form and to give the researchers experience. The pilot application was conducted with four middle school mathematics teachers who did not participate in the full study. Processes ensured that factors such as these four participants' working environment, conditions, and years of professional experience differed. The pilot study was also based on participants' volunteerism. To ensure validity in qualitative research, some additional measures (triangulation, participant confirmation, colleague confirmation, etc.) were used to confirm the data and results (Yin, 2011). Participants were given 30-60 minutes to answer the research questions:

- *What is a Prism? Define and show it by drawing.*
- *What is a Pyramid? Define and show it by drawing.*
- *What is a Cone? Define and show it by drawing.*
- *What is a Cylinder? Define and show it by drawing.*
- *What is a Sphere? Define and show it by drawing.*
- *What is the knowledge of in-service teachers about the GOs? Explain separately.*
    - *If it is sufficient, why?*
    - *If it is insufficient, why?*

Participants were asked to answer the written interview questions. The researchers assumed the role of guiding the research environment and preventing any negativity that would overshadow transparency. The researchers explained that the names of the participants in the study would be kept confidential.

Data Analysis

Qualitative content analysis was used to reach common concepts or expressions explaining the obtained raw data and creating meaningful sections (Yin, 2011). The content analysis process provided an important advantage and convenience for organizing qualitative data (Bogdan & Biklen, 2007). Two mathematics education researchers thoroughly analyzed and coded the study's data. Categories were created from the codes. Miles and Huberman's (1994) reliability formula was applied (i.e., reliability = consensus / (agreement + disagreement). The reliability of this independent coding was calculated as 0.84. Inconsistencies were eliminated by establishing links between the comments. Since it was anticipated that the participants who made incomplete and/or wrong definitions would exceed the others, two incomplete definitions and one correct definition were given for each category. In addition, participants' opinions regarding each code were shown as frequency and percentage. The researchers made different interpretations by presenting their assumptions, criticisms, and suggestions regarding the results.

Findings

The frequency and percentage results of codes/categories and some participants' opinions regarding each category are included in this section.

Table 3. Frequency and percentages of participant opinions by codes/categories

Category	Code	Frequency	Percentage
Conceptual knowledge of prism	*Not getting the definition right*	51	51.5
	Draw a square or rectangular prism	63	63.6
	Inability to relate with cylinder	99	100
	Just draw a right prism	96	97
Conceptual knowledge of pyramid	*Not getting the definition right*	39	39.4
	Draw a square or rectangular pyramid	75	75.8
	Inability to relate with cone	99	100
	Just draw a right pyramid	98	99
Conceptual knowledge of cone	*Not getting the definition right*	48	48.5
	Inability to relate with the pyramid	84	84.8
	Just draw a right cone	99	100
Conceptual knowledge of sphere	*Not getting the definition right*	72	72.7
	Relating with the circle	2	2
	Inability to draw the shape correctly	18	18.2
Conceptual knowledge of cylinder	*Not getting the definition right*	39	39.4
	Associating with a rectangle	39	39.4
	Inability to relate with prism	98	99
	Just draw a right circular cylinder	98	99
Feeling inadequate to make a definition	*Having difficulty in making a definition*	39	39.4

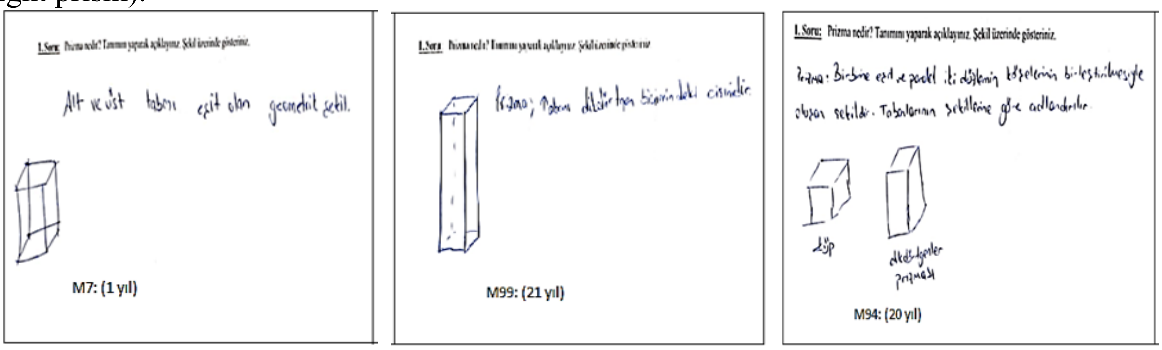
As can be seen in Table 3, more than half of the participants (51.5%) could not correctly define the concept of the prism. Most of the participants (63.6%) drew a square or rectangular prism. In addition, none of the participants could relate the prism with the cylinder, and almost all the participants (97%) just drew the right prism. While those who could not make the CD of the pyramid correctly (39.4%), most of the participants (75.8%) drew the pyramid as a square or rectangular pyramid, and none of the participants could relate the pyramid with the cone. Almost all the participants (99%) drew only the right pyramid. The rate of those who could not get the CD of the cone correctly was 48.5%, and most of the participants (84.8%) could not relate the cone with the pyramid, and all of the participants (100%) drew the right cone. The rate of those who failed to define the concept of the sphere correctly was 72.7%; only 2% could relate the sphere to the circle, and 18.2% could not correctly draw the sphere's shape. The concept of the cylinder was correctly defined by 39.4% of the participants could not correctly define the concept of the cylinder. Of the participants, 39.4% could not relate the cylinder with the rectangle, and 99% could not relate the cylinder with the prism. Almost all the participants (99%) drew the cylinder as a right circular cylinder. It has also been concluded that 39.4% of the teachers considered themselves inadequate regarding CDs of GOs.

Conceptual knowledge of prism

A cylinder with polygons for bases. All prisms are special cases of cylinders (Van de Walle et al., 2008). The number of participants who could not define the prism correctly was 51 (51.5%). The opinions of the sample participants, who made two incomplete definitions and one correct definition, respectively, are as follows:

M7: It is a geometric shape whose lower and upper bases are equal… (1 year)
M99: It is a rectangular object with a base… (21 years)
M94: It is a shape formed by joining the vertices of two equal and parallel planes. It is named according to its bases… (20 years)

Participants drew square or rectangular prisms, special forms of prisms. Since the prism definitions were incorrect, so too were their respective drawings (usually a square or right prism).



Incomplete definitions	Correct definition

Figure 1. Prism definitions and drawings by M7, M99, M94

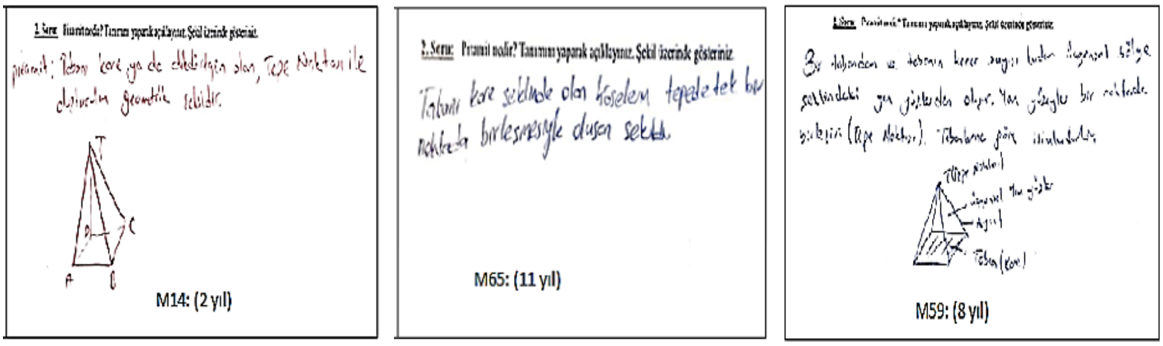
Despite the relation that all prisms are special forms of the cylinder, none of the participants could explain the prism by relating it to the cylinder. The congruent and parallel nature of the bases was insufficiently considered. While the prisms do not need to be right, almost all the participants (97%) drew a right prism. The remaining participants did not draw any shapes. As a result, none of the participants could draw an oblique prism and did not mention it.

Conceptual knowledge of pyramid

Cone with a polygon for a base. All pyramids are special cases of cones (Van de Walle et al., 2008). Thus, this notion of the cone led to the pyramid concept, which 39.4% of the participants could not correctly define.

M14: The base is a rectangular or square geometric shape with an apex … (2 years)
M65: It is a shape with a square base and formed by joining its vertices at an apex … (11 years)

M59: Consists of a base and side faces in the form of triangular regions as many as the number of sides on the base. Connect the side surfaces at one point (apex). They are named according to their bases… (8 years)



Incomplete definitions	Correct definition

Figure 2. Pyramid definitions and drawings by M14, M65, M59

Most of the participants (75.8%) used only the square or rectangular pyramid when drawing. M35 (4 years) included a hexagonal pyramid in the drawing, while M38 (4 years) included a pentagonal pyramid. None of the participants related the pyramid and the cone. When the participants were asked to draw, almost all (99%) of the participants drew a right pyramid. M20 (2 years) drew a right rectangular pyramid, while M35 (4 years) drew a right hexagonal pyramid.

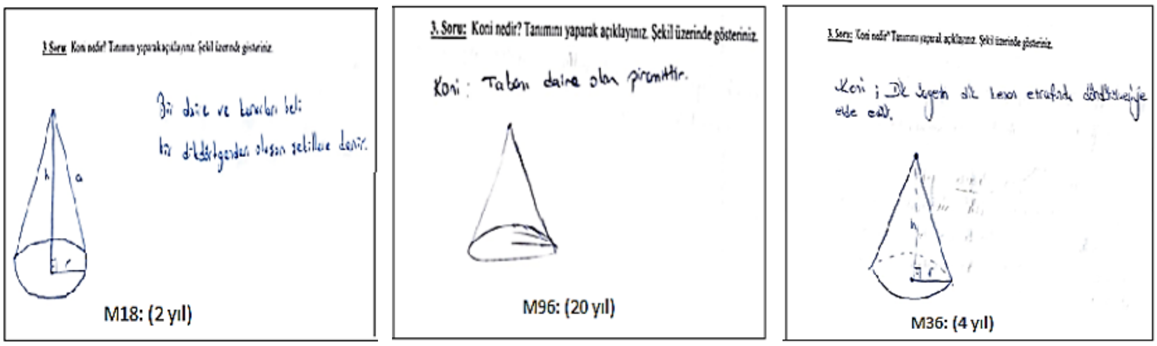
Conceptual knowledge of cone

A solid with exactly one face and a vertex that is not on the face (Van de Walle et al., 2008). Of the participants, 48.5% could not correctly define the cone.

M18: It is called a shape consisting of a circle and a rectangle with its sides ... (2 years)

M96: It is a pyramid with a circle base ... (20 years)

M36: It is obtained by rotating the right triangle around its vertical side ... (4 years)



Incomplete definitions	Correct definition

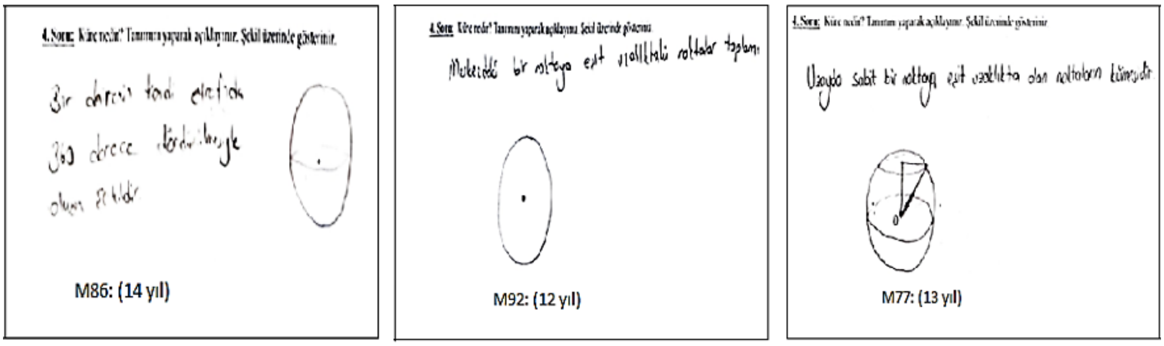
Figure 3. Cone definitions and drawings by M18, M96, M36

Most of the participants (84.8%) could not establish a connection between the cone and the pyramid. The rest (15.2%) established the relationship reversely. While it can be argued that the pyramid is a special cone, some of the participants have stated that the cone is a special pyramid. All the participants (100%) have included right cones in their drawings.

Conceptual knowledge of sphere

The sphere is a finite part of space bounded by a surface called a spherical surface, and all points on it are equidistant from the center of the sphere (Beman & Smith, 1900). Most of the participants (72.7%) could not correctly define the concept of the sphere.

M86: It is a shape formed by rotating a circle around itself 360 degrees… (14 years)
M92: All points equidistant to a point in the center… (12 years)
M77: It is the set of points in space that are equidistant from a fixed point ... (13 years)



Incomplete definitions	Correct definition

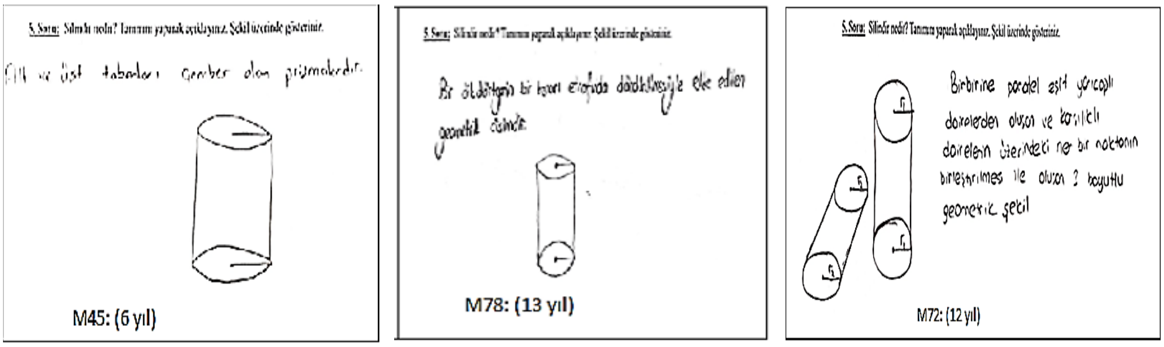
Figure 4. Sphere definitions and drawings by M86, M92, M77

While most participants could not correctly define the sphere concept, 18.2% could not draw the sphere correctly. Only 2% of the participants demonstrated a relationship between the sphere and the circle.

Conceptual knowledge of cylinder

Two congruent, parallel faces, called bases. Lines joining corresponding points on the two bases are always parallel (Van de Walle et al., 2008). While prisms are special forms of cylinders, there remains a persistent misperception that a cylinder is a special form of prism.

M45: They are prisms whose lower and upper bases are circles ... (6 years)
M78: It is a GO obtained by rotating a rectangle around one side ... (13 years)
M72: 3-dimensional geometric shapes consisting of circles of equal radii parallel to each other and formed by joining each point on mutual circles ... (12 years)



Incomplete definitions	Correct definition

Figure 5. Cylinder definitions and drawings by M45, M78, M72

While 39.4% of the participants could not define the cylinder correctly, they related it to the rectangle, and 99% tried to describe the cylinder without relating it to the prism. Additionally, almost 99% of the participants drew a right circular cylinder.

Feeling inadequate to make a definition

Of the teacher participants, 39.4% stated that they did not find their knowledge sufficient to make the CDs of GOs.

M43: Insufficient; because we find it difficult to define in general. We do not participate in any training after the university is over. The missing parts continue to be incomplete… (6 years)

M63: I think there is no problem-solving questions. However, there is a deficiency when asked directly for the definition. We also do not internalize the definition very much… (10 years)

M68: We are inadequate because I think that usually, the emphasis is on operations due to mathematics. I think that the verbal part of our field is not used much and that teaching with definitions is incomplete. For this reason, I think that definition is lacking not only in geometric bodies but also in other mathematical terms ... (12 years)

M86: If I need to evaluate myself first, although it remains in my mind as a subject, I have missing points in the definitions. The reason is that the subject requires the ability to think in three dimensions and I did not learn permanently while learning in my past period. While teaching mathematics subjects, I explain and understand most subjects with the knowledge I learned as a child… (14 years)

M83: We teachers do not have enough knowledge. After being assigned, we can forget about topics. We should study these topics better than university books… (14 years)

M89: I think it's insufficient. We are generally not familiar with the recognition ... (15 years)

Discussion

The current study was conducted to determine the CDs of middle school mathematics teachers regarding five GOs (prism, pyramid, cone, cylinder, and sphere). In this section, similar results related to all GOs are presented and discussed under certain headings. In this way, the reader can see the common difficulties related to GOs together.

Deficiency in concept definition

CDs are essential for students to understand a given concept (Edwards & Ward, 2008). However, the study revealed that middle school mathematics teachers generally could not correctly define GOs. This result parallels studies revealing that mathematics teachers and pre-service teachers' knowledge of GOs is incomplete (Gökkurt, 2014; Gökkurt & Soylu, 2016; Koçak et al., 2017; Marchis, 2012; Linchevski et al., 1992; Gutierrez & Jaime, 1999; Ulusoy, 2020). Teacher participants did not mention the details, the special states of the objects, or directly put the visual states of the objects into the definition. Instead of making general definitions of objects, teachers made explanations by considering the current appearance of each object. In other words, teachers could not make a generalized definition for all prisms. Notably, providing special definitions instead of general definitions accompanied by visuals is effective in this result (Çakmak et al., 2014; Yemen-Karpuzcu & Işıksal-Bostan, 2013).

The quality of the education teachers received during their university years may have played a role in the emergence of these results. Many studies emphasize the effect of university education on teachers' effective educational activities (Erdem & Soylu, 2013; Gürbüz et al., 2013; Hill et al., 2005; Smith, 2000). Teachers have often forgotten some mathematics-specific knowledge over time due to lack of use. The fact that many teachers do not follow current research may also be effective in the emergence of this result. Teaching procedural knowledge rather than conceptual knowledge in middle school may cause teachers to forget CDs. Teachers may also see their knowledge as sufficient for middle-school mathematics and choose not to improve themselves. The central examination-oriented education during the middle school years may also not give the necessary importance to the CDs. These exams, which generally require a quick solution to the given problems rather than conceptual content, can provide such an education in schools. In the study of Gökkurt et al. (2015), prospective teachers' difficulties with conceptual questions rather than operational questions support this claim.

Teachers' explanations based on the prism while defining the cylinder, in general, caused them to be mistaken. The most important reason for this misconception is that most of the participants know that "the cylinder should only be circular." It can be said that the knowledge that the bases of the cylinder, which can be any shape, are congruent and parallel to each other is poorly known by most teachers. A similar situation has arisen for the definition of the cone. Teachers generally stated that cones can only be circular. On the other hand, when the sphere was defined, it was observed that many participants made the error of dimension. Those participants defined the circle (points in a plane) instead of the definition of the sphere (points in space).

Drawing only right forms of GOs

Almost all the teachers drew the right forms of GOs. Most of the participants used square or rectangular shapes while drawing prisms and pyramids (Çakmak et al., 2014; Gökbulut & Ubuz, 2013; Gökkurt, 2014, Ulusoy, 2019). Teachers' getting used to drawing basic prototype drawings may cause students to make similar drawings. For example, a triangular prism or triangular pyramid is rarely drawn. In this study, no teacher drew a hexagonal prism or pentagonal pyramid. This is mirrored in textbooks, where examples of prisms and pyramids are generally in prototypical form.

Superficial knowledge about GOs and routine leads to many of these study results. To draw different visuals, it is necessary to think differently and to think relationally between objects. Karpuz et al. (2014) explained that the difficulties in solving the questions that do not contain shapes are caused by the students' inability to draw or incorrectly draw the shape representing the conceptual knowledge, the drawn geometric shape not meeting the generalizability condition, and being under the cognitive influence of prototype shapes. The consistency of presented visuals in different sources (textbooks, supplementary course resources, internet, etc.) (prototype) prevents thinking and drawing differently.

Ulusoy (2019) found that prospective teachers typically drew a prototypical circular cylinder. Some in this study drew the cone and pyramid as an example of the prism. In this sense, while drawing visuals in the classroom, different representations should also be included to not contradict the definition. While showing different examples, care should be taken to be correct and not to cause misconceptions. Weber et al. (2008) underlined that forming an appropriate concept image in students depends on giving correct examples. Erdem (2023) and Gecu-Parmaksiz & Delialioglu (2019) emphasized that providing different forms of geometric shapes to children is very important for them to recognize and identify the shapes better. As Monaghan (2000) suggested, prototypes limit the concept by presenting limited visuals. For this reason, it can be said that giving as many accurate and different examples of a concept as possible will provide more effective learning. However, the same type of drawings that teachers have made for the prism and pyramid are also found in circular cylinders and cones.

In this study, no participant drew an oblique cone. Only one participant (M72, 12 years) drew the shape of the oblique cylinder. This participant has defined a cylinder as "a three-dimensional shape consisting of circles of equal radii parallel to each other and formed by joining each point on the opposing circles." In addition, it was observed that while drawing the sphere, teachers generally tried to draw the shape correctly but could not give three-dimensional depth. Teachers who drew two-dimensional visuals drew a circle instead of a sphere. It can be argued whether the lack of knowledge or the lack of drawing skills is effective in teachers' drawings in this way.

Inability to relate with other GOs

Interestingly, no participant could correctly relate the prism to the cylinder. When this result is detailed, it can be understood that the teachers do not know that the prism is a special cylinder. In other words, it can be said that the participants did not know that "prism is a cylinder with a polygon at the base." A similar result has been found in the pyramid and cone relationship. No participant accurately mentioned the relationship between a pyramid and a cone. It was understood that the participants did not know "a pyramid is a special cone" or "pyramid is a cone with a polygonal base." It can be said that the most important reason why teachers cannot establish a relationship between these objects is that their conceptual knowledge is quite insufficient. Based on the definitions made, it can be said that these participants saw these objects as geometric concepts independent of each other. The visual in Figure 6 will be useful in terms of establishing the correct relations between these objects.

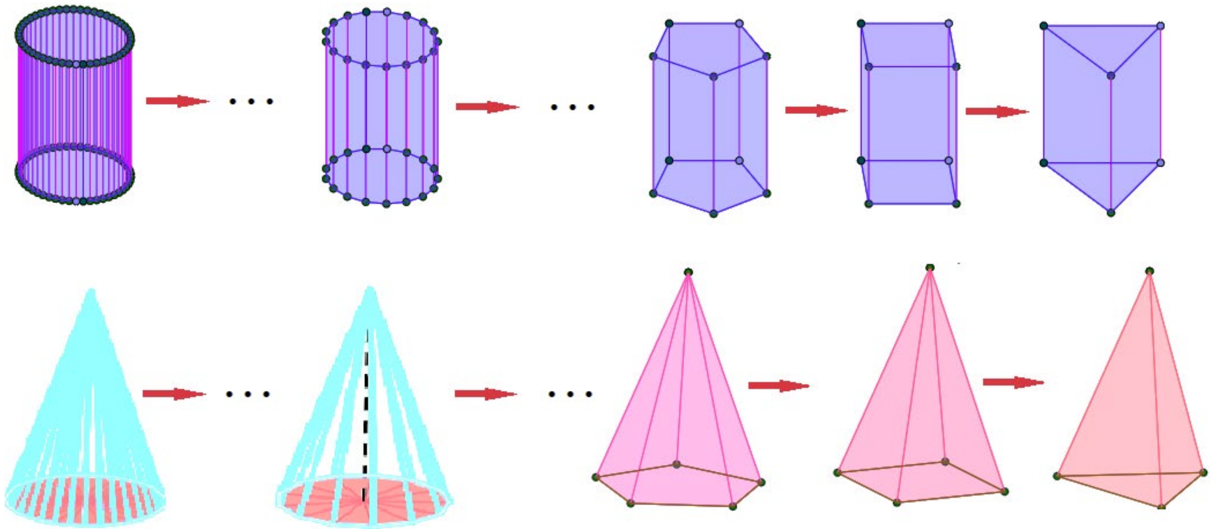


Figure 6. The transition from “Circular cylinder to the prism”, “Circular cone to the pyramid” respectively

Few of the participants who linked these objects found an inverse relation. These participants referred to the opposite relationship between the cylinder being a special prism and the cone being a special pyramid. Except for a few participants, the relation/difference between the sphere and the circle was not mentioned. Teachers’ transferring this incorrect or incomplete knowledge to students without correcting it will naturally pave the way for students to learn incompletely or incorrectly. Teachers may not have considered the prerequisite principles of mathematics while learning these concepts and teaching afterward. Prerequisite relation can be explained as having enough knowledge of some concepts for teaching mathematical concepts. In Figure 7, basic models of the prerequisite principle are given visually

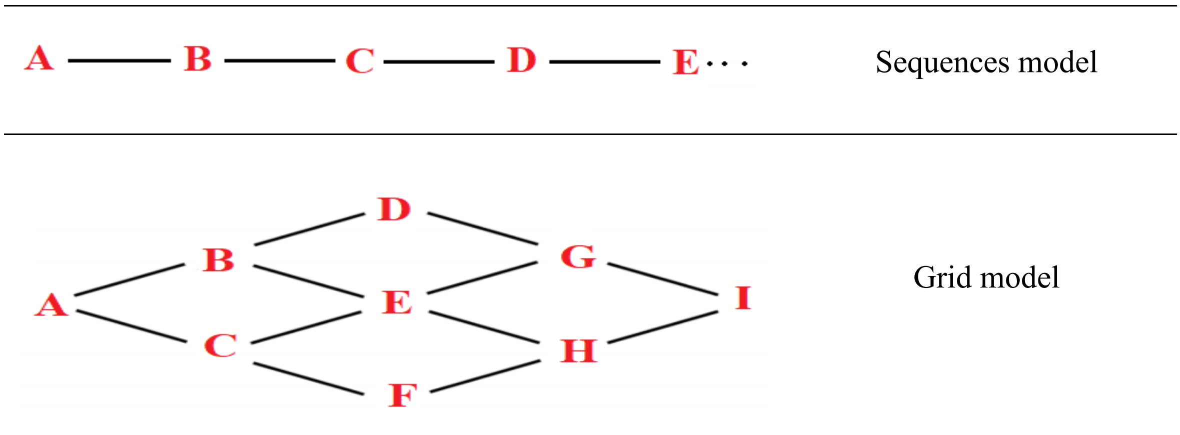


Figure 7. Prerequisite models for teaching mathematical concepts

Feeling an inability to define concepts

A substantial number (39.4%) of the participants found themselves inadequate to define the concepts of GOs. A similar result was found in the study by Çakmak et al. (2014) with prospective teachers. Some of the pre-service teachers pointed out that GOs were abstract and could not be internalized sufficiently. Others stated that they did not improve themselves after graduating from the university and did not follow the current situation. The sad but courageous and sincere statement of M86, "*While teaching mathematics subjects, I explain and understand most subjects with the knowledge I learned as a child…*" summarizes this issue. It coincides with the finding that teachers' performance in defining concepts is quite low, with the result that they see themselves as inadequate. The sincere expressions of the teachers can be interpreted as their desire to have more detailed and accurate knowledge about these objects.

Since mathematics teaching often emphasizes operations, teachers may feel inadequate. Studies (Erdem, 2023; Johann et al., 2005; Gökkurt, 2014; Hiebert & Waerne, 1996; Baumert et al., 2010) also report that mathematics teaching is based on operational knowledge. The low weight of the concept lessons may have distracted teachers from CDs for many years. Some teachers also stated that they do not have difficulties in the parts of GOs that require processing, but they have difficulty in abstract contents related to the definition or concept.

Is the professional experience effective?

Statistical analysis of the effect of professional experience on CD was not one of the main problems of the study. Since the contents of the questions are different from each other, the knowledge of each participant varies from question to question. For example, a participant with knowledge regarding the prism may have insufficient knowledge regarding the sphere or vice versa. A participant with a poor drawing performance gave better answers while defining a concept.

Altogether there seem to be too many confounding variables to form cogent opinions regarding the overall success of a participant regarding all GOs, based on the correct, incorrect, or incomplete answer to a question. It was thought that determining the relationship between the performances of the participants regarding each GO and their professional experiences separately would distract from the main purpose of the study. Many participants teaching for many years gave more incomplete and erroneous answers than teachers with a few years of experience, making it necessary to provide report sub-titles such as "Is the professional experience effective?" Many participants could not provide correct definitions of GOs, despite their professional experience. For example, M99 (21 years) defined an "object with a circle base" for the cylinder, while M25 (3 years) wrote, "a closed GO whose lower and upper base planes are equal and parallel shapes." Similar results were obtained in the studies of Gökkurt (2014) and Yıldızlı & Sarı (2017).

Limitations

After the results were revealed and interpreted in the research, it was revealed that there were some limitations. For example, the relationship between the CD performances of the participants and their professional experience could be statistically examined. In this way, it would be possible to contribute to the literature about whether professional experience has an effect on teaching CD. One of the most important results of the research is that the participants do not know the relationship between GOs. For example, the participants did not mention their knowledge that the prism is a special cylinder. Perhaps if they had been asked such a question, they could have correctly written the expected relationships. It can be said that this is the most important limitation of the current research. In this context, the question of whether there are relationships between GOs could be added to the written interview form. Again, instead of asking whether the middle school mathematics teachers' knowledge of CD about GOs is sufficient, it could be asked what the participants' state of this knowledge was. However, this question was not asked in this way, as there was a concern about whether the participants would give sincere answers in this case. Based on the answers given, it can be said that the participants (some of them sincerely stated that they are not enough) do not consider themselves sufficient in the CD of GOs.

Conclusions and Suggestions

For the reader to understand the study more clearly and easily, the results are summarized in one sentence below, and the suggestions are presented at the end.

The conclusions are:

- It has been revealed that the definitions of GOs are incomplete and/or wrong.
- It was observed that only right and prototype forms of GOs were drawn.
- It has been determined that no relationship can be made between GOs.
- It was found that the teachers felt inadequate in making the definitions of GOs.
- It can be said that professional experience does not have much effect on defining GOs.

The suggestions are:

- Teaching GOs should be carried out more conceptually during university years. Thus, surface learning leaves its place in detailed learning.
- The parts of textbooks related to GOs should be reviewed and definitions should be revised.
- Some participants stated that not participating in the training in which new teaching methods were explained after their university years was also effective in their lack of CD. In this context, Conceptual relations between GOs should be taught to the teachers through the seminars to be held.
- Teaching GOs should be done in the form of cylinders, prisms, cones, pyramids, spheres, respectively. In this way, the misconceptions of "cylinder is a special prism" and "cone is a special pyramid" can be avoided.
- In addition to the right drawings of GOs, it is important to draw their oblique, upside down and sideways versions.
- A statistical relationship between mathematics teachers' professional experiences and their CD performance can be investigated by studying a large sample.

References

Accascina, G. & Rogora, E. (2006). Using Cabri 3D diagrams for teaching geometry. *International Journal for Technology in Mathematics Education, 13*(1), 1-12.

Ball, D. L., & McDiarmird, G. (1990). The subject matter preparation of teachers. In W. R. Houston (Ed.), *Handbook of research on teacher education* (pp. 437–449). New York, NY: Macmillan

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., Klusmann, U., Krauss, S., Neubrand, M., & Tsai, Y. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal, 47*(1), 133–180.

Beman, W.W. & Smith, D.E. (1900). *New plane and solid geometry.* Boston: Ginn & Company.

Bogdan, R.C. & Biklen, S. K. (2007). *Qualitative research for education: An introduction to theory and methods* (5th Ed.). Boston: Pearson Education.

Clements, D. H. & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420-464). *New York: Macmillan.*

Clements, D. H., Sarama, J., & Battista, M. T. (1998). Development of concepts of geometric figures in a specially designed Logo computer environment. *Focus on Learning Problems in Mathematics*, *20*, 47-64.

Cohen, L., Manion, L., & Morrison, K. (2000). *Research methods in education*. London and New York: Routledge Falmer.

Çakmak, Z., Konyalıoğlu, A. C., & Işık, A. (2014). The investigation of pre-service elementary mathematics teachers' content knowledge on three dimensional objects. *Middle Eastern & African Journal of Educational Research*, *8*, 28-44.

Edwards, B., & Ward, M. (2008). The role of mathematical definitions in mathematics and in undergraduate mathematics courses. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 221–230). Washington, DC: Mathematical Association of America.

Erdem, E., & Soylu, Y. (2013). Prospective teachers' opinions about CSSE and field examination. *Çankırı Karatekin University Journal of Social Sciences, 4*(1), 223-236.

Erdem, E., & Man, S. (2018). Conceptual knowledge of middle school mathematics teachers on radian and on π in special. Ege Journal of Education, 19(2), 488-504.

Erdem, E. (2023). *Teaching mathematics with concrete suggestions [Somut önerilerle matematik öğretimi].* Ankara: Pegem Akademi Press.

Ferraro, D., & Van de Kerckhove, W. (2006). *Trends in international mathematics and science study (TIMSS) 2003 nonresponse bias analysis, technical report.* Washington, DC, U.S. Dept. of Education, Institute of Education Sciences.

French, D. (2004). *Teaching and learning geometry: Issues and methods in mathematical education*. New York: Continuum

Fujita, T. & Jones, K. (2007). Learners' understanding of the definitions and hierarchical classification of quadrilaterals: towards a theoretical framing. *Research in Mathematics Education, 9*, 3-20.

Gecu-Parmaksiz, Z., & Delialioglu, O. (2019). Augmented reality-based virtual manipulatives versus physical manipulatives for teaching geometric shapes to preschool children. *British Journal of Educational Technology*, *50*(6), 3376–3390.

Gökbulut, Y. & Ubuz, B. (2013). Prospective primary teachers' knowledge on prism: generating definitions and examples. *Elementary Education Online*, *12*(2), 401-412.

Gökkurt, B. (2014). *An examination of secondary school mathematics teachers' pedagogical content knowledge on geometric shapes.* Doctoral dissertation, Atatürk University, Erzurum, Turkey

Gökkurt, B., Şahin, Ö., Soylu, Y., & Doğan, Y. (2015). Pre-service teachers' pedagogical content knowledge regarding student mistakes on the subject of geometric shapes. *Elementary Education Online*, 14(1), 55-71.

Gökkurt, B. & Soylu, Y. (2016). Examination of middle school mathematics teachers' pedagogical content knowledge: the sample of cone. *Elementary Education Online*, *15*(3), 946-973.

Gutierrez, A. & Jaime, A. (1999). Pre-service primary teachers' understanding of the concept of altitude of a triangle. *Journal of Mathematics Teacher of Education*, 2(3), 253-275.

Gündüz, N., Bulut, A. S., & Dündar, S. (2017). Öğretmen adaylarının geometrik cisimler konusundaki bilgi düzeyleri, problem çözme düzeyleri ve tutumlarının incelenmesi. *Inonu University Journal of the Faculty of Education*, *18*(2), 1-15.

Gürbüz, R., Erdem, E., & Gülburnu, M. (2013). An investigation on factors affecting classroom teachers' mathematics competence. *Kırşehir Faculty of Education Journal, 14*(2), 255-272.

Hiebert, J. & Waerne, D. (1996). Instruction, understanding and skill in multidigit addition and instruction. *Cognition and Instruction, 14,* 251-283.
Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.
Johann, E., Ansie, H., & Marietjie, P. (2005). Undergraduate students' performance and confidence in procedural and conceptual mathematics. *International Journal of Matheinalical Edication in Science and Technology,* 36(7), 701-712.
Karpuz, Y., Koparan, T., & Güven, B. (2014). Using figure and concept knowledge in Geometry. *Turkish Journal of Computer and Mathematics Education*, *5*(2), 108-118.
Koç, Y., & Bozkurt, A. (2011). Evaluating pre-service mathematics teachers' comprehension level of geometric concepts. In B. Ubuz, (Ed.), *The Proceedings of the 35th annual meeting of the international group for the psychology of mathematics education* Vol. 1. (p. 335). Ankara, Turkey.
Koçak, M., Gökkurt-Özdemir, B., & Soylu, Y. (2017). An investigation the pedagogical content knowledge of preservice elementary mathematics teachers' about the concept of cylinder. *Cukurova University Faculty of Education Journal, 46*(2), 711-765.
Linchevsky, L., Vinner, S., & Karsenty, R. (1992). To be or not to be minimal? Student teachers views about definitions in geometry. *In W. Geeslin & K. Graham (Eds.), Proceedings of the 16th international conference for the psychology of mathematics education,* Durham USA, pp. 48–55.
Marchis, I. (2012). Preservice primary school teachers' elementary geometry knowledge. *Acta Didactica Napocensia, 5*(2), 33-40.
Martin, M.O., Mullis, I.V.S., & Foy, P. (2008). *TIMSS 2007 International Science Report: Findings from IEA's Trends in International Mathematics and Science Study at the Fourth and Eighth Grades*. Chestnut Hill, MA: Boston College.
Martín-Molina, V., González-Regaña, A. J., Toscano, R., & Gavilán-Izquierdo, J. M. (2020). Differences between how Undergraduate Students Define Geometric Solids and what their Lecturers Expect from them through the Lens of the Theory of Commognition. *Eurasia Journal of Mathematics, Science and Technology Education*, *16*(12), em1917.
Matos, J. M. (1994). Cognitive models of the concept of angle. *In Proceedings of the 18th international conference for psychology of mathematics education* (pp. 263-270). Portugal: University of Lisbon.
Meng, C.C. (2009). Enhancing students' geometric thinking through phase based instruction using geometer's sketchpad: A case study. *Journal Pendidik dan Pendidikan, 24,* 89-107.
Miles, M.B. & Huberman, A.M. (1994). *An expanded sourcebook: qualitative data analysis (2nd Editon).* Thousand Oaks, CA: Sage.
Mistretta, R. M. (2000). Enhancing geometric reasoning. *Adolescence*, *35*(138), 365–379.
Monaghan, F. (2000). What difference does it make? Children's views of the differences between some quadrilaterals. *Educational Studies in Mathematics*, *42*(2), 179-196.
National Council of Teachers of Mathematics [NCTM] (2000). *Principles and standards for school mathematics*. Reston, VA.
Niyukuri, F., Nzotungicimpaye, J., & Ntahomvukiye, C. (2020). Pre-service Teachers' Secondary School Experiences in Learning Geometry and their Confidence to Teach it. *Eurasia Journal of Mathematics, Science and Technology Education*, *16*(8), em1871.
Putnam, R.T., Heaton, R.M., Prawat, R.S, & Remillard, J.T. (1992). Teaching mathematics for understanding: discussing case studies of four fifth-grade teachers. *The Elementary School Journal*, 93(2), 213–228.
Russell, D. (2018). What is geometry? Retrieved from https://www.thoughtco.com/what-is-geometry-2312332
Schoenfeld, A.H. (1992). Learning to think mathematically: problem solving, metacognition, and sense-making in mathematics. *In D. Grouws (Ed.), Handbook for research on mathematics teaching and learning,* New York: Macmillan, pp. 334-370.
Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.
Smith, M. (2000). Redefining success in mathematics teaching and learning. *Mathematics Teaching in the Middle School, 5*(6), 378-389.
Sunzuma, G., & Maharaj, A. (2019). In-service teachers' geometry content knowledge: implications for how geometry is taught in teacher training institutions. *International Electronic Journal of Mathematics Education, 14*(3), 633-646.
Swafford, J. O., Jones, G. A., & Thornton, C. A. (1997). Increased knowledge in geometry and instructional practice. *Journal for Research in Mathematics Education, 28*(4), 467-483.
Ulusoy, F. (2019). Early-years prospective teachers' definitions, examples and nonexamples of cylinder and prism. *International Journal for Mathematics Teaching and Learning, 20*(2), 149-169.

Ulusoy, F. (2020). Prospective early childhood and elementary school mathematics teachers' concept images and concept definitions of triangles. *International Journal of Science and Mathematics Education*, https://doi.org/10.1007/s10763-020-10105-6

Weber, K., Porter, M. & Housman, D. (2008). Worked examples and conceptual usage in understanding mathematical concepts and proofs*., In M. P. Carlson and C. Rasmussen (Eds.), Making the connection: Research and teaching in undergraduate mathematics* (pp. 245-252), Washington, DC: Mathematical Association of America

Van de Walle, J., Karp, K.S., & Bay-Williams, J.M. (2008). *Elementary and middle school mathematics: Teaching developmentally (7th Ed.).* New York, NY: Pearson Education.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology, 14*(3), 293-305.

Yemen-Karpuzcu, S. & Işıksal-Bostan, M. (2013). Geometrik cisimler: silindir, prizma, koni, piramit ve kürenin matematiksel anlamı. İ.Ö. Zembat, M. F. Özmantar, E. Bingölbali, Şandır, H. & A. Delice (Ed.), *Tanımları ve tarihsel gelişimleriyle matematiksel kavramlar* (p. 278-279). Ankara: Pegem Akademi.

Yıldızlı, H. & Sarı, M. H. (2017). The investigation of primary teachers' content knowledge on GOs. *Journal of Uludag University of Faculty of Education, 30*(2), 601-636.

Yi, M., Flores, R., & Wang, J. (2020). Examining the influence of Van Hiele theory-based instructional activities on elementary preservice teachers' geometry knowledge for teaching 2-D shapes. *Teaching and Teacher Education, 91,* 103038.

Yin, R.K. (2011). *Qualitative research from start to finish. New York: The Guilford Press*.