Analyzing Prospective Mathematics Teachers' Mathematizing Through Their Solution Approaches to a Modeling Problem

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Mathematization is a critical competency in the modeling process. This study aims to elicit prospective mathematics teachers' (PMTs) mathematizing by analyzing their solution approaches to a modeling problem related to The Bales of Straw Problem. We conducted this study with 75 fourth-year PMTs enrolled in a teacher training program in elementary mathematics education, referring to children aged 10–13 years. The primary data sources were the PMTs' written and verbal explanations and drawings. The study results revealed that PMTs used different solution approaches: trigonometric, estimation of measurements, area/perimeter, and similarity of triangles in the modeling task. They mostly solved the problem using trigonometric ratios and estimation of measurement approaches. The solution approaches included different levels of mathematization. Although the mathematical modeling task required PMTs to develop an estimated numerical result, about sixty percent of the solutions had numerical results. More than half of the numerical results were implausible. We suggest that examining successful and unsuccessful solution attempts can help PMTs recognize the specific roles of mathematization competency in modeling processes.

Keywords: Mathematical Modeling; Mathematization; Prospective Teachers; Solution Approach

Introduction

Mathematical modeling is essential for establishing strong bridges between the real and mathematical worlds. Mathematical modeling is one way to make mathematics meaningful for students (Blum & Ferri, 2009). Mathematical modeling is valuable in many professions, including engineering, finance, computer science, and data analysis. By teaching students how to mathematize real-world problems, educators prepare them for future careers that require analytical thinking and problem-solving skills. Hence, researchers suggest that mathematical modeling should be part of the school curriculum and that learners should learn how to model mathematical situations in the real world (Blum & Ferri, 2009; Kaiser, 2020). While performing a mathematical modeling task, students learn to use and organize their knowledge helpfully and to mathematize real-life situations (English & Sriraman, 2010). In this way, they can solve both realistic and complex problems. As students entering the modeling cycles have the opportunity to test, question, and improve their existing thinking styles, they also have the opportunity to develop or rebuild their understanding of the mathematical concepts they encounter in the modeling process (Johnson & Lesh, 2003).

In recent years, the measurability of teacher competencies for mathematical modeling has gained importance because teachers have a significant role in helping students gain modeling competencies (e.g., Anhalt et al., 2018; Borromeo Ferri, 2014). Teachers have many responsibilities, from choosing appropriate modeling tasks to guiding their students through the modeling process with necessary feedback (Borromeo Ferri, 2018). In addition, teachers need to identify the difficulties students encounter while implementing a modeling task (Borromeo Ferri, 2014). Teachers must have a strong knowledge of the modeling process and model-building tasks (Borromeo Ferri & Blum, 2010). For example, teachers are expected to master different solutions to a modeling task because students can develop different mathematical models for the solution and use different solution approaches in modeling tasks (Borromeo Ferri, 2018; Schukajlow & Krug, 2014). During the modeling task, the teacher should be familiar with the students' possible approaches to mathematical solutions to guide the students effectively. In this sense, prospective teachers need to experience solving modeling tasks in their courses during their university education before teaching modeling in schools (e.g., Cevikbas et al., 2022).

Due to the ongoing debate on the measurement of modeling competencies, researchers distinguish between general competencies (holistic approach) and sub-competencies of modeling (atomistic approach) (Blomhøj & Jensen, 2003; Cevikbas et al., 2022). The holistic approach depends on a full-scale modeling process, with the learners engaging in all phases of the modeling process (Niss & Højgaard, 2011). In the atomistic approach, on the other hand, learners focus on particular phases of the modeling process, especially the processes of mathematizing and analyzing models mathematically, because these phases are seen as especially demanding (Cevikbas et al., 2022). According to Grigoras et al. (2011), mathematization is the most critical sub-competency in the modeling process for students to do mathematical modeling independently. Mathematizing in mathematical modeling is the process of translating real-world phenomena or problems into mathematical language and structures. (Borromeo Ferri, 2006). It is a fundamental step in the mathematical modeling process, where the complexities of the real world are represented using mathematical symbols, equations, and relationships (Grigoras et al., 2011; Yilmaz & Tekin-Dede, 2016). The primary objective of mathematization is to provide a logical, verifiable, and rational approach to dealing with real-life issues using mathematical knowledge and techniques (Grigoras, 2010). For this reason, it is suggested that teachers should integrate modeling tasks into mathematics lessons to support students' mathematizing competency (Blomhøj & Kjeldsen, 2006; Maaβ, 2006).

Teachers' competency in mathematizing within the context of mathematical modeling is critical for effective instruction and student learning (Kaiser, 2020). At this point, teachers need a deep understanding of mathematical concepts and their applications to real-world problems. By developing their understanding of mathematical concepts, ability to identify relevant variables and relationships, flexibility and creativity in problem-solving, and integration of interdisciplinary knowledge, teachers can facilitate meaningful learning experiences that prepare students for success in mathematics and beyond (Jablonski, 2023; Yilmaz & Tekin-Dede, 2016). However, in previous studies, prospective teachers' modeling competencies were generally considered holistic (e.g., Eraslan, 2012; Tekin-Dede & Yilmaz, 2013). Besides, in the current study, we focused on prospective mathematics teachers' (PMTs') solution approaches for explaining how the mathematizing process emerges in a modeling problem through the following research questions:

1. What solution approaches do PMTs use in the mathematizing process for a modeling problem?

2. How is the success in solving the task related to the solution approach PMTs used?

Theoretical Background

The Different Perspectives on Mathematical Modeling

In mathematics education, Kaiser and Sriraman (2006) classified mathematical modeling perspectives into five categories: realistic modeling, educational modeling, models and modeling perspective, socio-critical modeling, and epistemological modeling. These perspectives are not mutually exclusive (Kaiser et al., 2007). There are similarities and differences among these perspectives in terms of goals, definitions, task design, and research focus. Abassian et al. (2020) summarized the main features of these perspectives in Table 1.

Table 1.
Summary of the main features of modeling perspectives (Adapted version of Abassian et al., 2020, p. 56)

Perspective	Realistic	Educational	Model and modeling*	Socio-critical	Epistemology
Goals	Develop skills to model and understand authentic, real-world scenarios	Develop skills to model a real-world scenario and understand mathematics.	Develop a deep understanding of mathematics through a modeling context	Develop mathematical modeling skills to make decisions in society	Develop formal mathematical reasoning.
Definition of a mathematical model	Mathematical objects (graphs, equations, etc.) explain the given real-world scenario	Mathematical objects that have a relationship to the given real-world scenario.	A conceptual system that maps the structural characteristics of a relevant system.	Mathematical representation of a relevant scenario.	The result of an activity based on situations and mathematical concepts.
Design of task	Authentic, messy, real-life tasks that require the use of the modeling cycle.	Authentic tasks that can be simplified to reveal specific mathematical goals.	MEAs designed to develop a specific mathematical concept. Must be in the context of a real-world problem. Must meet 6 guiding principles.	Tasks are in a societal context but should also focus on the development of specific mathematics concepts.	No set requirements.
Research focus	Mathematical modeling competencies	Mathematics in mathematical modeling and mathematical modeling in curricula.	The use of MEAs to teach mathematics	Learners' use of mathematics to understand society critically.	Teaching and learning of specific mathematical concepts.

*MEAs refer to model-eliciting activities that include real-world tasks that need to be set in the real world but not necessarily be a problem or situation that the modelers encounter in their daily lives.

For example, the objectives of mathematical modeling are grouped into two main categories: (i) promoting the acquisition of mathematical knowledge and (ii) investigating nonmathematical situations by employing mathematics as a tool. Niss et al. (2007) characterize the initial category as "Modelling for the learning of mathematics" (p. 6), while Julie and Mudaly (2007) define it as "Modelling as a vehicle" (p. 504). The ultimate objective of this category is to facilitate the learner's acquisition of a more profound comprehension of mathematics by employing contextual scenarios. However, the second category, “Modelling as content, " asserts that Modeling is the learning goal, without a particular emphasis on the mathematical content. To be more concrete, in a realistic perspective, while the main focus is to develop learners’ mathematical modeling competencies, the epistemology perspective focuses on teaching and learning particular mathematical concepts.

Our research aligns with the realistic modeling perspective, which emphasizes mathematical modeling competencies as learners engage in a cyclical, multi-step procedure while engaging in realistic, authentic, and messy tasks. This perspective focuses on solving real-world problems rather than the development of mathematical content (Blomhøj, 2009; Kaiser et al., 2011). However, researchers do not use a universally accepted cycle. For example, Figure 1 demonstrates mathematical modeling phases and sub-competencies. While the phases result in products of the modeling cycle (such as a situational model, mathematical results, and real results), the sub-processes provide information about the learner's actions during the modeling process (see the right side of the diagram in Figure 1). In the sub-competencies, learners construct their mental models of the given problem and formulate an understanding of the problem (*constructing*). Then, learners identify relevant and irrelevant information by making the necessary assumptions (*simplifying*). Learners produce mathematical representations of real-world situations and solve them to reach mathematical results (*mathematizing and working mathematically*). Learners interpret mathematical results in a real-world context (*interpreting*). Finally, students determine the relevance of the mathematical results in the real world, a process known as "validating". In the Realistic Modeling perspective, the main focus is on modeling competencies (Blomhøj, 2009).

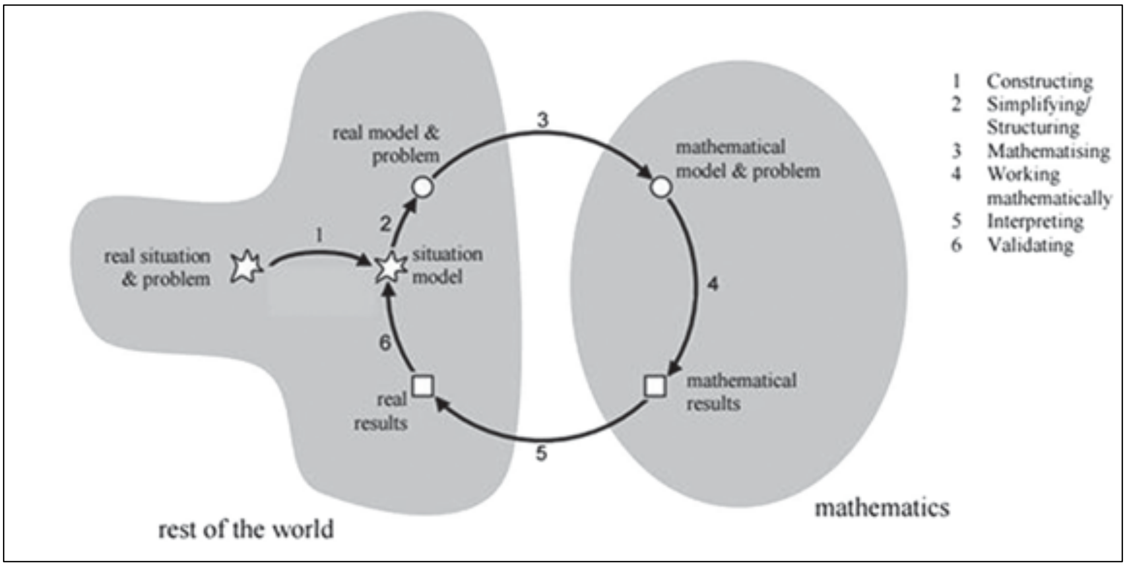


Figure 1. The mathematical modeling cycle (Blum & Leiß, 2007)

Researchers have provided different conceptualizations for the skills and abilities related to mathematical modeling. Some researchers propose a comprehensive understanding of a multifaceted skill known as modeling competency (Blomhøj & Jensen, 2003; Kaiser & Brand, 2015). For example, modeling competency is "to autonomously and insightfully carry through all aspects of a mathematical modeling process in a certain context" (Blomhøj & Jensen, 2003, p. 126). Niss et al. (2007) defined modeling competency as “the ability of an individual to perform certain appropriate actions in problem situations where these actions are required or desirable” (p. 12). Besides, some scholars advocate for a more analytical understanding of particular sub-competencies associated with different phases of the modeling process (Kaiser, 2007; Maaβ, 2006; Schukajlow et al., 2018). The sub-competencies encompass the ability to do tasks at each phase of the modeling process and effectively manage the transitions between and across these stages (Stillman et al., 2007).

As a sub-competency, mathematizing, or the conversion of reality-related settings into mathematical structures, concepts, or models, occurs not just in complicated modeling processes but also in much simpler tasks (Buchholtz, 2017; Freudenthal, 1983). According to Lesh and Doerr (2003), “model-eliciting activities usually involve mathematizing—by quantifying, dimensionalizing, coordinatizing, categorizing, algebratizing, and systematizing relevant objects, relationships, actions, patterns, and regularities” (p. 5). Moreover, the following questions are related to the existence of mathematization (Maaß, 2006): “Did the student identify relevant variables correctly? Did the student idealize or simplify the conditions and assumptions? Did the student identify a principal variable to be analyzed? Did the student successfully analyze the principal variable and arrive at appropriate mathematical conclusions?” Mathematizing is thus viewed as a significant part of mathematical modeling in educational standards and international assessment studies. For example, according to the PISA 2012 draft mathematics framework (OECD, 2010), mathematization is a high-level competence because learners must identify and specify several assumptions, variables, and connections. Mathematizing provides a better understanding and examination of learners' mathematical attempts at solving modeling problems (Grigoras et al., 2011). In this study, to characterize the mathematizing process, we interpret the prospective teachers’ solutions in terms of structural aspects (i.e., the objects, relations, actions, patterns, regularities, assumptions, etc.) in the real world and the expression of this structure in a mathematical model (e.g., Buchholtz, 2017; Botha & van Putten, 2018; Djepaxhija et al., 2017; Greefrath et al., 2018; Grigoras et al., 2011; Jankvist & Niss, 2020; Ledezma et al., 2023; Stillman & Brown, 2014; Vos, 2013).

Ongoing Debate on the Measurement of Modeling Competencies

One important objective in teaching Modeling is to develop and stimulate an individual's Modeling competency. To do this, assessing learners’ mathematical modeling competencies is crucial. For this reason, the issue of measuring modeling competencies is a broad area of research. In the 1980s, scholars from Germany, England, and the Netherlands initiated investigations on the assessment of mathematical modeling. Kaiser-Meβmer (1986) defined modeling sub-abilities that facilitate students’ engagement in different phases of the modeling cycle. Later, Stillman (1998) characterized these abilities as *mathematical modeling competencies.* Beginning in the 1990s, studies focused on assessing learners’ modeling competencies using a *formative assessment* tool (Lesh et al., 2000; Niss, 1993) across all phases of the modeling process. On the other hand, research also addressed *the summative assessment* of mathematical modeling through the written reports of national testing programs at the upper secondary levels (e.g., Netherlands (Drijvers et al., 2019) and Sweden (Frejd, 2013)).

Different conceptualizations of mathematical modeling competency result in distinct approaches to assessment, such as holistic, atomistic, and hybrid (Blomhøj & Jensen, 2003; Turner et al., 2022). The holistic approach requires a full-scale modeling process in which learners work through all phases of the modeling process. In other words, "if the aim is to assess students' ability to complete a modeling process (often called the general modeling competency), it is preferable to use holistic tasks" (Hankeln et al., 2019, p. 146). In this assessment approach, it is possible to evaluate students' performance based on the complexity and completeness of solutions (Llinares & Roig, 2008). However, while a holistic approach can look at all parts of a student's work, it also takes time and effort to mathematize and analyze it (Cevikbas et al., 2022; Hidayat et al., 2022). The atomistic approach uses a collection of separate and independent modeling sub-competencies often associated with modeling cycle phases (Kaiser & Brand, 2015). The atomistic approach focuses mainly on "the processes of mathematizing and of analyzing models mathematically" because these are particularly challenging for learners (Cevikbas et al., 2022). Proponents of this method acknowledge that the modeling process takes significant time (Kaiser, 2017).

Additionally, if students encounter difficulties with the initial components, they may be less likely to show proficiency in later stages. Researchers believe that using an atomistic approach in mathematics instruction promotes mathematical learning (Frejd & Bergsten, 2018; Hidayat et al., 2022). Moreover, it is possible to prepare students for more extensive modeling tasks by concentrating on strategic modeling competencies in the atomistic approach (Kotze, 2018). Some scholars used the atomistic approach to measure students' mathematical modeling (e.g., Durandt et al., 2021; Zöttl et al., 2011). They argue that because students may encounter difficulties in many parts of the solution process when performing modeling tasks, a reduction of the task in terms of the complexity and level of processing demands in the atomistic approach can be beneficial in promoting or accurately diagnosing the partial competencies of modeling (Wess et al., 2021).

Recent research has also attempted to combine atomistic and holistic approaches to assess mathematical modeling competency (Djepaxhija et al., 2017; Fu & Xie, 2013; Hidayat et al., 2022; Turner et al., 2022). According to Schukajlow et al. (2018), "these two different kinds of approaches are not mutually exclusive but ideally complementary" (p. 8). Besides, Kaiser and Brand (2015) note that while a holistic approach appeared to be more helpful in fostering, interpreting, and validating capacities, an atomistic approach better promoted the development of working mathematically. Since we focus on mathematizing competency in this study, we utilize an atomistic approach while examining prospective teachers' solutions to a Modeling task.

Methodology

A qualitative case study design was used to examine prospective teachers' mathematization through their solution approaches in depth (Yin, 2003).

Participants

We conducted this study with 75 fourth-year prospective mathematics teachers (PMTs) enrolled in a training program in Elementary Mathematics Education (EME), which pertains to education for children aged 10–13. The participants were 53 females and 22 males enrolled in a 4-year teacher education program at a state university in Turkey. Their ages ranged from 21 to 23. They took many of the important teaching courses related to mathematics education (e.g., Teaching Numbers, Teaching Algebra, Teaching Geometry and Measurement, Teaching Probability and Statistics) and have completed all the mathematics field courses (e.g., Analysis I-II-III, Geometry, Analytical Geometry). Data were collected from the participants who enrolled in a compulsory education course entitled "Mathematical Modelling for Teachers." This course aimed to develop prospective mathematics teachers' knowledge of mathematical modeling and foster their Modeling skills by executing various modeling tasks.

The participants knew the nature of mathematical modeling because they experienced a mathematical modeling process through the Summer Job Problem (see Lesh & Lehrer, 2000) in the third year of their undergraduate program. This problem was used in the Teaching Probability and Statistics course to understand how statistical reasoning is used in the modeling process. They solved the Summer Job Problem through both individual and group work. They solved the summer job problem individually over 15 days as homework. Afterward, they discussed their solutions in groups of 3, which the researcher created by considering the differences in the answers, in a time interval between 40 minutes and 2 hours. They provided both individual and group reports to explain their solution process. Hence, PMTs had experience in mathematical modeling before collecting data in this study.

Data collection procedures

This study used the Bales of Straw Problem (Borromeo Ferri, 2007) to examine PMTs' mathematizing in their written solutions (see Figure 2). We selected this problem for various reasons. The first reason was that we preferred this problem because most participants lived in rural areas. The second reason was that PMTs had the mathematical and geometric knowledge required for the activity. Third, the Bales of Straw Problem has been used in related literature for almost all learning levels (Greefrath et al., 2018; Ledezma et al., 2023; Stillman et al., 2020; Tekin-Dede, 2019). The final reason for selecting this problem was the complexity involved in solving the problem. Additionally, it offers a chance to understand the mathematization process from the written and visual data in the solutions (Greefrath et al., 2018). This way, it was possible to examine PMTs' mathematizing processes through their solutions.

The data was collected during the second week of the Mathematical Modeling for Teachers course. In the first week, the first author introduced the general aspects of mathematical modeling. The lesson in the second week started with the presentation of the bales of straw problem to the participants. The first author provided the problem sheet and blank papers to each participant. The problem was posed for them to solve individually. We asked them to write down all their procedures and ideas. We also asked them not to delete anything they wrote. In particular, we recommended not using erasers or ballpoint pens. After PMTs completed their solutions, they gave their solution papers to the researcher. PMTs solved the problem in approximately 40 minutes. After examining all the solutions, the first author interviewed the participants whose solutions were unclear or not understandable. In this way, the researcher interviewed six participants. In the mini-interviews, the researcher asked how they presented a solution to the problem, what the calculations meant, and how they found the height numerically. This way, all written and visual data became clear and understandable for researchers.

Bales of straw are piled up in this way. There are 5 straw bales on the bottom line. In the next, then 4, 3, then 2, and on the top, 1 bale. Try to find out how high this mountain of bales of straw is. Write your thoughts in detail.

Figure 2. Bales of Straw Problem (adapted from Borromeo Ferri, 2007, p. 2084)

Data analysis

As data sources, we used PMTs' written explanations, drawings, and verbal explanations in mini-interviews. After we numbered PMTs' solutions from PMT1 to PMT75, we scanned all the written solutions. We created an Excel document (see Figure 3). If the participant had made two solutions for the mathematical task, we put each solution on a separate line in the Excel document.

We analyzed the data using inductive and deductive approaches, drawing from qualitative research methods. We carefully examined all written solutions to code PMTs' solution approaches used in the mathematization process. We grouped solution approaches into four categories based on data and related literature. Ledezma et al. (2023) grouped students' mathematical models in solutions to the Straw Bale Problem into two categories: trigonometric ratios (Type A strategy) and estimation of measurements (Type B strategy). In our study, we coded our data based on these strategies.

Moreover, two approaches emerged from the data: the area-perimeter approach and the similarity of triangles approach. The findings show detailed examples of each solution approach. We coded which PMT exhibited which solution approach in the Excel document (see Figure 3).

		Solution approach				Results		
Participants	Gender	Estimation	Trigonometri	Area-Perimete	Similarity	Numeric	Algebra	Plausibility
PMT1	F		1				1	
PMT2	M		1				1	
PMT3	M	1					1	

Figure 3. Coding of the data in the Excel

In the mathematical Modeling task, prospective teachers were expected to estimate the height of the straw bale numerically. Like Ledezma et al.'s (2023) study, we coded the solutions as numerical, algebraic, or both numerical and algebraic. If a PMT reached a result that included only algebraic expressions to represent the height of the mountain, using a function of the radius or diameter of each bale as a variable and did not provide a real numerical quantity, we coded it as an algebraic result. If a PMT reached a result that included only expressions or calculations on numerical quantities to represent the height of the mountain and did not provide algebraic expressions, we coded it as a numerical result. Finally, if a solution included numerical and algebraic expressions for the height of the mountain, we coded it as both a numerical and algebraic result. Hence, we examined the possible relations between solution approaches and the types of results obtained.

The numerical results may not always be plausible in the real world. We expect PMTs to estimate a numerical quantity for the height of the mountain of bales of straw. For this reason, the solution must make sense in the real world (Ledezma et al., 2023). We examined the numerical results in terms of plausibility. A plausible result would vary depending on the relationship between the diameter of a bale and the human's height in the picture. We used two criteria to evaluate the plausibility of numerical results: (i) the results must exist in reality (e.g., the height cannot be a negative value) and (ii) the results must be reasonable according to the picture (e.g., a height of 100 m or 30 cm would not be reasonable when comparing a bale and the woman's height). When the woman in the image is associated with the bale, we evaluated that a reasonable value in real life can be between 6 m and 9 m (Ledezma et al., 2023). Therefore, we coded results within this range as plausible. We coded very small or very large values as not plausible (see Table 5 **presents the plausibility of numerical** results regarding the height of the mountain of bales of straw. PMTs achieved plausible numerical results in 47% of the solutions. However, more than half of these numerical solutions were unreasonable for various reasons.

Table 5). Figure 3 displays the coding components of the data in Excel. Three mathematics educators analyzed the data separately. Various encodings were studied. After reaching a consensus among the coders, a fourth mathematics educator examined all the data regarding all the components in Figure 3. Then, four researchers came together and reached a 100% consensus for each component in coding.

Findings

Table 2 shows the frequencies and percentages of solution approaches PMTs use for The Bales of Straw Problem. According to Table 2, PMTs' solutions included four different approaches: (i) trigonometric ratios; (ii) estimation of measurements; (iii) area-perimeter; and (iv) similarity of triangles approach. Since some prospective teachers offered more than one solution, we identified 79 solutions.

Only one solution used the similarity of triangles approach, whereas 67% used the trigonometric ratios approach. In the modeling problem, we expected the PMTs to numerically estimate the height of the mountain of bales of straw. However, 38% of the solutions included only algebraic expressions for the mountain height of bales of straw. While the PMTs using the trigonometric ratios solution approach tended to leave their solutions as algebraic expressions, the PMTs who used the estimation of measurement approach mostly provided numerical results. The details of the solution approaches are presented in the following headings.

Table 2.
Mathematical Solution Approaches

Solution approaches	Solution ways			Total (%)
	Numerical	Algebraic	Both	
Trigonometric ratios (TR)	8	25	20	53 (67)
Estimation of measurements (EM)	16	3	0	19 (24)
Area–Perimeter (AP)	2	1	3	6 (8)
Similarity of triangles (ST)		1	0	1 (1)
Total (%)	26 (33)	30 (38)	23 (29)	79 (100)

Prospective Teachers' Solution Approaches in the Mathematization Process

Trigonometric ratios approach. PMTs who used the TR approach tried to obtain the height of the mountain of bales by using the angle-side relations in a right triangle. Their common assumption was that the bales of straw consisted of identical circles. This approach transferred the straw bales in the photograph into identical circles within a simplified mathematical model. We observed three different ways of finding the height of the mountain of bales (Table 3).

Table 3.
Types of Trigonometric Ratios (TR) Solutions

Types of TR solutions	Frequency			*Total*
	Algebraic	Numerical	Both	
TS1 (see Figure 4a&b)	17	3	18	38
TS2 (see Figure 4c&d)	7	2	3	12
TS3 (see Figure 4e)	3	0	0	3
Total	27	5	21	53

Note. TS refers to a trigonometric solution.

Table 3 shows that thirty-eight solutions grouped under the TR approach were included in the category of TS1. In these solutions, PMTs calculated the height of the equilateral triangle they formed by joining the centers of the circles at the corners of the triangle (Figure 4a). Moreover, in TS1, some PMTs drew a right triangle into the bale and used trigonometric ratios based on the angle-side relations of an equilateral triangle (Figure 4b).

In both solutions, PMTs arrived at the algebraic expression for the height of the mountain of bales, r being the radius of a straw bale. However, nearly half of the PMTs who noticed this expression did not notice the woman in the photograph. For this reason, they completed their solutions algebraically rather than reaching an estimated height value for the mountain of bales (see Table 2). As previously mentioned, they expected a value between 6 and 9 m by associating the height of a straw bale with the woman's height in the photograph. In this sense, numerical solutions grouped in TS1 are generally acceptable for the height of the mountain of bales. PMTs who could not reach the appropriate numerical value in the TS1 category mostly found a value of between 5 and 6 m. This showed that they could not establish a realistic numerical relationship between the bale size and the size of the human figure. A few PMTs have reached a very low height value (e.g., 2.68 m) by ignoring the person in the photograph, claiming that the bales will have a diameter of around 30 cm in real life. In those responses, PMTs applied extra-mathematical knowledge shaped by their experience and familiarity with real-life situations.

According to

Table 3 3, 12 solutions within the approach of trigonometric ratios are grouped in the TS2 category (e.g., Figure 4c and Figure 4d). In this solution, the PMTs assumed that the straw bales were equal circles and drew a large triangle that would be tangent to the bales from the outside. They found the height of this triangle to be $5\sqrt{3}r$, as in Figure 4b, using the Pythagorean Theorem by associating the height of the circles with the radius (r). The main problem with this solution was that it took the hypotenuse length of the right triangle as 10r. When a triangle is drawn to be externally tangent to the circles, the length of the hypotenuse becomes more than 10r, and the length of the perpendicular base becomes more than 5r (see Figure 4c). The height value of $5\sqrt{3}r$ found is smaller than that of $4\sqrt{3}r + 2r$ in Figure 4a. Although the tendency to find algebraic results in TS2 is intense, the height of the mountain of bales PMTs found was generally close to the plausible results in real life (e.g., between 5 and 6 m) but below the range of 6 and 9 m. For example, PMT7 accepted the hypotenuse 11r and the base 6r in a right triangle. He then achieved a height of $\sqrt{85}$r shown in Figure 4d. As a result, he assumed the leg length of the woman in the photograph to be 50 cm and found the height of the mountain to be between 4.50 and 4.65 m. This value is below the expected numerical value (6 and 9 m). The PMT measured the adult person in the photograph at around 90–100 cm height, which is shorter than the average human height of 170 cm.

Table 3 shows three solutions, including the trigonometric ratios approach, which were grouped in the TS3 category. In these solutions, PMTs drew the triangle neither tangent to the outside of the mountain of bales of straw nor passing through the centers of the circles. However, they assumed the hypotenuse is 10r, as in Figure 4e. However, under these conditions, the length of the hypotenuse becomes more than 10r. Since the PMTs who made this solution provided only algebraic results, we could not examine the numerical plausibility of the solutions.

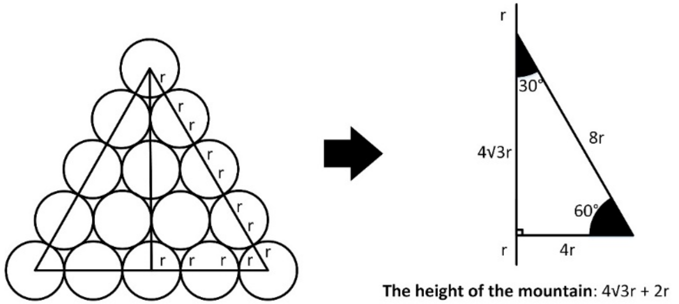
Estimation of measurements approach. According to Table 2, 24% of the solutions (n = 19) included an estimation of measurements approach for the height of the mountain of bales. Sixteen of these solutions reached a numerical value for height. In ten solutions, PMTs assigned an undetermined height (2r) to each bale and found the total height of the mountain as the number of rows and the height of each bale (10r) (Figure 5a). This scenario demonstrated that numerous PMTs overlooked the overlap resulting from the arrangement of the straw bales and the influence of pressure. Sample explanations of the PMTs who claimed that the height is 10r, as shown in Figure 5a, are presented in Table 4.

Table 4.
Examples of PMTs' Explanations

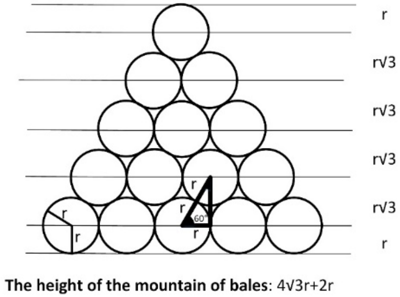
The use of the human figure	PMTs' written explanations in the solutions
No reference to a human figure	PMT6: If we call the diameter of a bale 1 meter, the height of the mountain will be 5 m.
	PMT22: The maximum diameter of a bale that a person can carry is 100 cm. Since there are 5 rows, the height of the mountain is 5 m.
	PMT29: The bales I saw in the fields were usually 20 cm or 30 cm in diameter. If we say that the diameter of a bale is 30 cm, the height of the mountain will be 3 m.
Reference to the human figure	PMT36: Let the leg of the man in the picture be 1 m. Suppose the diameter of a bale is equal to the leg length. Then the height will be 10 m.
	PMT53: A bale is the same height as a man in the photograph. The average height for a man is 174 cm. 5 bales go up the mountain. 5 x 174 cm equals 8.7 m.
	PMT72: This bale looks as tall as a human if a person stands up. Let's assume the diameter of the bale is equal to the height of a human. A woman is 164 cm tall and a man is 177 cm tall on average in Turkey. Then the mountain is between 8.20 m and 8.85 m.

Note. In these solutions, PMTs found the height of the mountain of bales of straw as in

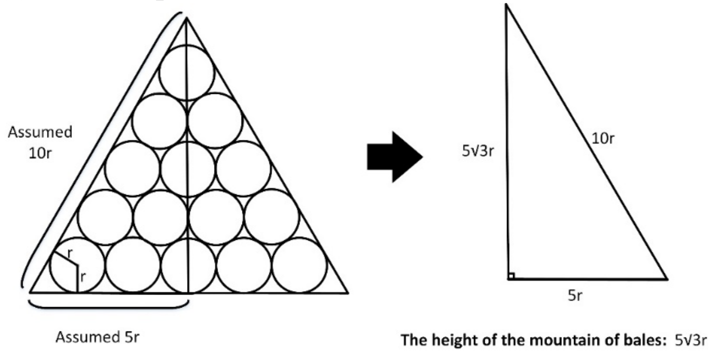
(a) An example of TS1



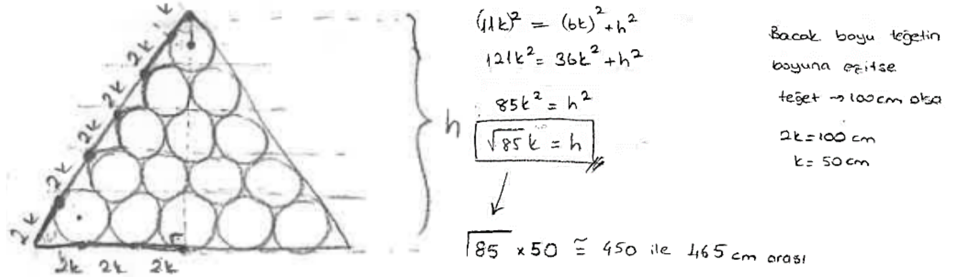
(b) An example of TS1



(c) An example of TS2



(d) An example of TS2



Translation: I assumed that the height of a leg is equal to the tangent line segment (2k). 2k is also equal to the diameter of a circle. I think that 2k = 100 cm.

(e) An example of TS3

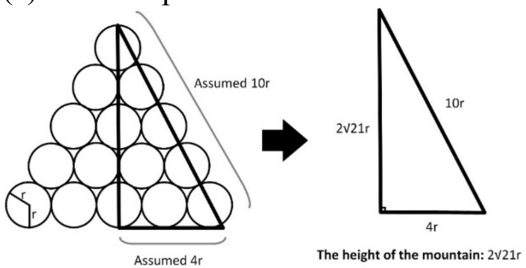


Figure 4. Types of trigonometric ratios solution approaches

According to Table 4, some PMTs assigned random values (PMT6) or benefited from their life experiences (PMT22 and PMT29) instead of the height of the human figure in the modeling problem while reaching the numerical result. Therefore, the values given in these answers were well below the expected 6–9 m for the height of the mountain of bales of straw. On the other hand, PMTs that considered the human figure in the problem neglected the overlap and pressure of the bales but generally achieved more reasonable numerical results (e.g., PMT72 and PMT53).

In nine solutions grouped under the estimation of the measurement approach, the PMTs focused on overlapping the rows. They made various assumptions. Interestingly, nearly all PMTs, upon noticing the placement of the bales in the top row between those in the lower row, considered the human element in the problem. Figure 5b presents one of these solutions. PMT40 sliced each circle into four equally spaced parts. The distance between each section is considered equal to half the radius. PMT40 calculated the bale height as 16 slices by counting the slices. Thus, she found the bale height equal to 4 diameters. Then, she accepted the person in the mountain of bales as equal to the diameter of 180 cm, arriving at a numerically plausible result. PMT70 examined the sitting condition of the bales on each other by taking the diameter of a bale as R, as in Figure 5c. PMT70 stated that the height of the mountain of bales would be less than 5R. Similarly, PMT37 also stated that the height would be around 4R by taking the woman's height as 170 cm (4 x 170 = 680 cm).

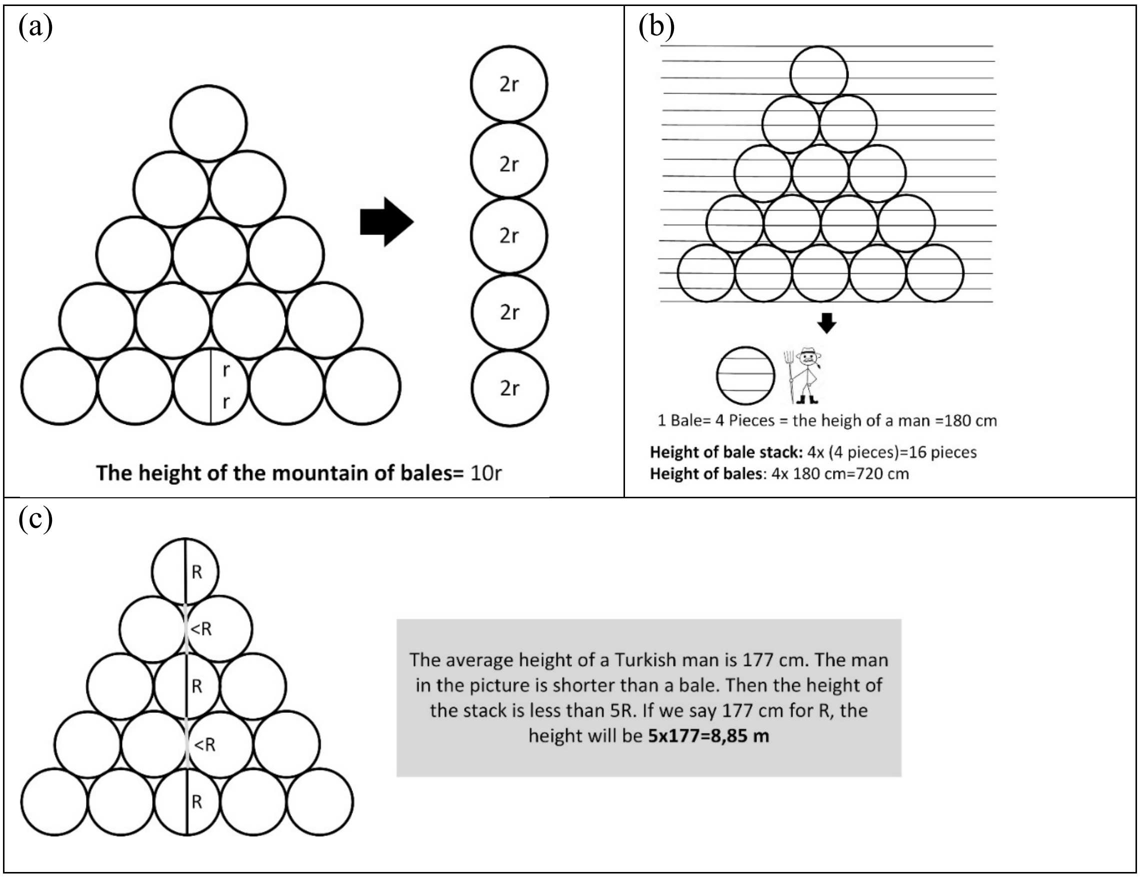


Figure 5. PMTs' solution examples related to the estimation of measurements

The area–perimeter approach. For the solution of the modeling problem, six PMTs used the area-perimeter approach (see Table 2). Only one of these PMTs (PMT5) tried to find a circle's circumference and a triangle's height. In this solution, PMT5 assumed that the straw bales were equal circles and drew a large triangle tangent to the bales from the outside. Accordingly, he claimed that half of the circumferences of five circles form one side of the triangle in Figure 6a. Accordingly, he took the side of the triangle to be 15r, which is the sum of half the circumferences of five bales. However, if PMT5 had done a fine mathematical calculation from the tangent points of the circles to the triangle, he could have found that 15r was more for one side. Due to the high value, the height of the mountain of bales was above 6 and 9 m. Besides, five PMTs calculated the length of the mountain of bales by considering the total area of the circles they assumed congruent. These PMTs found the mountain's height equation by equating the sum of the circles' areas with the triangle's area. Figure 6b presents an example of one solution. In this example, PMT62 took the base of the triangular pile from 10r after saying r to the radius of a bale. PMT62 found the area of 15 straw bales to be 45r (pi = 3). Then, she concluded that the height could be 9 m when he equalized this value to the area of the triangle $(10r\ x\ h)/2$. This solution approach contains mathematical errors because the triangle base is more than 10r. In this solution approach, it is mathematically incorrect to equate the total area of the circles to the area of the triangular region because the areas between the bales are ignored when considering the total area of the bales. When the triangular region is formed to pass through the center of the circles (as in Figure 4a), the regions outside the triangle passing through the centers of the circles can be neglected.

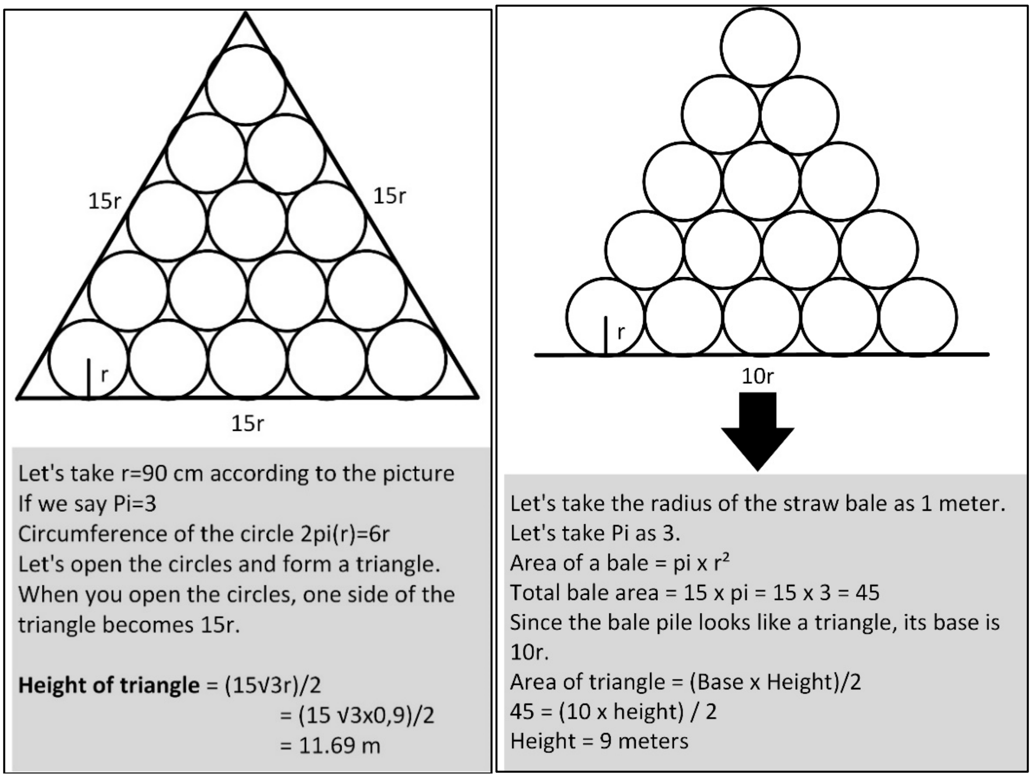


Figure 6. PMTs' solution examples based on (a) perimeter and (b) area of the triangle

Similarity of triangles approach. Only PMT38 solved the problem using the similarity approach. She (PMT38) drew a large triangle, which accepts the line passing through the center of the circles at the bottom of the bale pile as the base (see Figure 7a). The base length of the large triangle is 10r. She drew a small triangle on the top of the bale pile. The base of the small triangle runs through the center of the top bale and is parallel to the base of the large triangle. The base length of the small triangle is 2r, and the height is r. According to PMT38, ABC and ADE triangles are similar (see Figure 7b). However, AKL and ADE triangles were similar. The PMT has reached a result of 6r for the bale pile by establishing a similarity relationship over the wrong triangles (Figure 7a). The prospective teacher left the result algebraic, but assigning a reasonable numerical value to r (e.g., 85 cm) would have resulted in a lower height value than expected (between 7 and 9 m) due to the prospective teacher's incorrect assumption of triangle similarity.

(a) PMT38's original solution | (b) PMT38's reorganized drawing

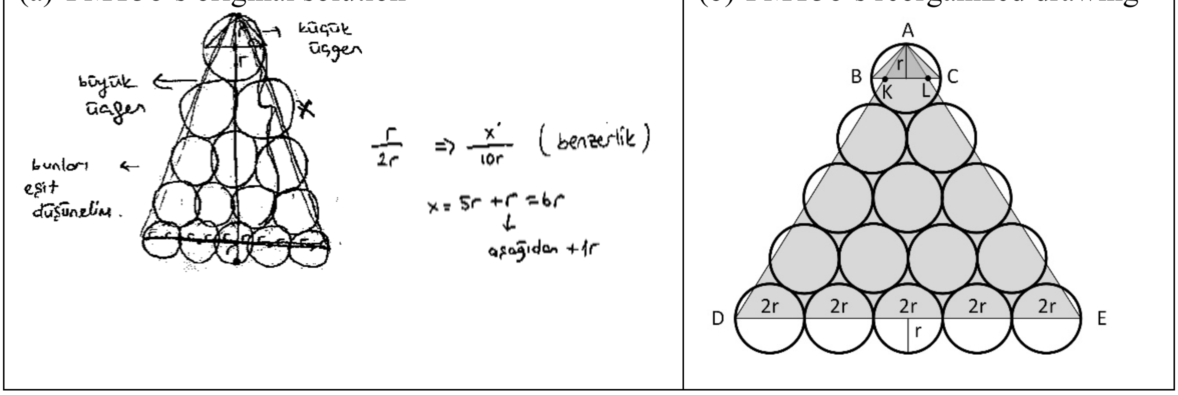


Figure 7. A solution example for the similarity of triangles approach

The Relationship between Solution Approaches and Success in Solving the Task

We can mathematically calculate the height of the mountain of bales as $4\sqrt{3}\,r+2r \cong 8.9r$ by assuming that each bale is an equal circle with an r radius. Considering the height loss resulting from the bales' pressure on each other, we find that the height is slightly less than 8.9r. Considering that the height of a bale (2r) is approximately as tall as a human (e.g., 170 cm), the height of the mountain is around $8.9\,x\,\frac{170}{2} \cong 7.5\,m$. Given a person's height of 150–190 cm, we can estimate their height to be between 6 and 9 m based on the bale diameter–person height comparison (Figure 8). Therefore, values in this range are considered plausible.

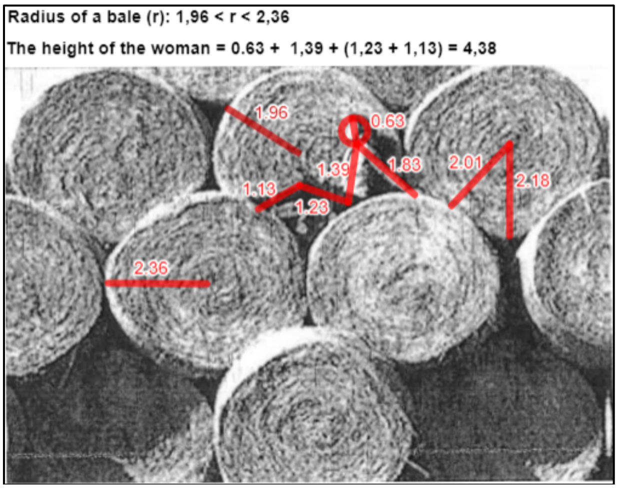


Figure 8. Numerical calculations in GeoGebra

Table 5 presents the plausibility of numerical results regarding the height of the mountain of bales of straw. PMTs achieved plausible numerical results in 47% of the solutions. However, more than half of these numerical solutions were unreasonable for various reasons.

Table 5.
The value ranges of numerical results in solutions

Solution approaches	Ranges of numerical results				Total
	$h < 3$	$3 \leq h < 6$	$6 \leq h < 9$	$h \geq 9$	
Trigonometric ratios	4	9	15	1	29
Estimation of measurements	2	5	6	1	14
Area–Perimeter	1		1	3	5
Similarity of triangles			1		1
Total	7 (14)	14 (28)	23 (47)	5 (10)	49 (100)

Note. h represents the height of the mountain of bales of straw.

The implausible numerical results were related to three conditions. In ten numerical solutions, PMTs did not associate the bale diameter with the human figure in the photograph. Besides, in sixteen implausible numerical solutions, PMTs associated the bale diameter with the human figure in the photograph. However, they made either low-value or high-value assignments. As a result, they provided low-value assignments. They considered the bale diameter 20–30 cm based on the bale that a person would carry or their real-life experiences. We observed that 10 PMTs who found the height of the mountain of bales between 3 and 6 m considered the human figure but took the bale radius (e.g., 55 cm) or human height (e.g., 140 cm) as a much smaller number than expected. Finally, in the five solutions, we observed that the PMTs assigned a very high value (e.g., 1 m) to the bale radius despite considering the human figure. However, when the bale radius is 1 meter, the human height should be between 1.90 m and 2.25 m. However, the photograph does not support this assumption. In addition, the solutions with the area approach generally included high-value assignments. One reason for this was that they took the bale radius larger, and another reason was that in this approach, they got a longer height value by taking the base length of the triangle shorter than it should be.

Discussion

This study aimed to elicit prospective mathematics teachers' mathematization by examining their solution approaches while solving a modeling problem related to the bales of straw. Examination of the solution approaches in the modeling process provided an in-depth understanding of the process of PMTs' mathematization. The results showed that PMTs used different mathematical solution approaches. These solutions were (i) the trigonometric ratios (TR), (ii) the estimation of measurements (EM), (iii) the area-perimeter calculation (AP), and the similarity of triangles (ST). As expected, while the most common approach in the solutions was trigonometric ratios, the least common approach was triangle similarity. Examining PMTs' successful and unsuccessful solution attempts in the modeling process offers an opportunity to gain deeper knowledge about the process of mathematization (Schaap et al., 2011).

We recognized some similar situations in all approaches to mathematization. For example, to visualize the mountain of bales of straw, all participants made the common assumption that bales are identical circles. Almost all PMTs presented a diagram with the five rows to visually represent the situation. However, the results also revealed that PMTs' mathematization attempts varied according to the approaches they used while solving the modeling problem. PMTs who used the TR approach mostly used mathematical notations and algebraic expressions and utilized known mathematical rules (e.g., properties of an equilateral triangle) and theorems (e.g., the Pythagorean theorem). They constructed different mathematical models of right triangles by utilizing the radius of circles (e.g., TS1, TS2, and TS3).

On the other hand, PMTs who used the EM approach while solving the problem focused on numerical quantities (e.g., the multiple addition of heights) rather than algebraic expressions and notations. They mostly started to solve problems with numerical assumptions about the woman's height in the picture and the circle's radius by using visual comparisons to perform the necessary calculations to arrive at the height of the mountain of bales. These results supported the idea that these two solution approaches (TR and EM) are complementary (Ledezma et al., 2023). Therefore, the awareness of numerical assumptions and norms in the EM approach can support the algebraic and mathematical structure of the TR approach.

Examining PMTs' solution approaches can be useful in capturing the complexity of mathematization since there is no agreement or paradigm on how to analyze mathematical activity. Our results indicated that PMTs' solution approaches showed different levels of mathematizing (Grigoras et al., 2011). Moreover, a detailed analysis of PMTs' solution approaches to the bales of straw problem provided information about some mathematical misconceptions or misunderstands, at least in the mathematization process (e.g., Jankvist & Niss, 2020), related to geometric concepts. In particular, their representations and calculations helped to show their invalid assumptions about mathematical representations and algebraic expressions (Delice & Kertil, 2015; Yilmaz & Tekin-Dede, 2016). Although they knew the rules and theorems necessary to solve the bales of straw problem, most of the PMTs constructed inappropriate diagrams to visualize the mountain of bales. We identified these problematic situations across all approaches. This result showed that prospective teachers needed help in mathematizing in terms of transferring the real model into mathematics through measuring, calculating, and estimating in order to put them into a meaningful relationship.

We found a relationship between solution approaches and the types of results (numerical, algebraic, or both). While the solutions, including the TR approach, mostly resulted in algebraic expressions, the solutions, including the AP and EM approaches, resulted in numerical quantities for the height of the mountain of bales of straw. The characteristics of the solution approaches may explain this. PMTs who used the TR solution approach gave more importance to mathematics (e.g., rules, theorems, algebraic expressions) than the realistic situation (Tekin-Dede, 2019). However, PMTs using the EM approach made assumptions to arrive at a numerical result rather than purely algebraic calculations. The common point in both approaches was that the number of PMTs who evaluated the plausibility of the obtained numerical result in the real world was rare.

We observed several possible reasons for unsuccessful attempts at mathematization, as follows: (i) a lack of relevant mathematical knowledge, (ii) an inability to use mathematical knowledge appropriately in modeling, (iii) the production of incorrect or partial mathematical representations that capture the modeling situation, and (iv) the failure to control the validity of their results in real-life phenomena. Our results are similar to those of

Schaap et al.'s (2011) and Stillman and Brown's (2014) studies, in which they found that learners with limited experience in modeling have difficulties with mathematizing due to impeding formulations of problem statements. We suggest that it is necessary to facilitate PMTs' mathematization in modeling situations.

Recommendation for Future Research

This article analyzes potential solutions and difficulties in the mathematical modeling task process. Based on the results of this study, teachers can also utilize the task diagnostically to understand their students' solution approaches. Our study also had limitations. For example, we examined PMTs' mathematization through their solution approaches to a single-task implementation. We cannot generalize our findings to outcomes from different task implementations. However, future studies can compare mathematization processes by using different modeling problems, such as mathematical language, representations, mathematical definitions, rules, and theorems. In the current study, we presented PMTs' mathematization through their written solutions (due to a lack of space). However, we conducted a classroom discussion on different solution approaches from the individual solutions of the problem. Further studies can examine PMTs' mathematization processes in different methodological designs (e.g., individual-group-classroom, group-classroom, individual-classroom, only groups) (Ledezma et al., 2023; Tekin-Dede, 2019).

Concluding Remarks

In our study, PMTs' different solution approaches indicated they were equipped with better mathematical knowledge. In this sense, focusing specifically on the process of mathematization has been crucial in demonstrating PMTs' mathematical knowledge and skills. However, despite the richness of PMTs' mathematical solutions, some PMTs' assumptions and associations between real-life and mathematical worlds were problematic. This situation may be related to the Turkish educational system, where students are generally accustomed to solving problems with operational characteristics rather than open-ended problems. This habit may lead them to be more active in vertical mathematization (refers to an internal movement in the world of mathematics) processes rather than horizontal mathematization (refers to the transition between real-world scenarios and mathematical representations, and vice versa) processes in a problem (Yvain-Prébiski, 2021).

Similarly, in this study, PMTs transitioned to horizontal mathematization processes by performing mathematical operations as soon as they read the problem rather than focusing on the interactions between mathematics and real-life situations. Thus, our results indicate the need for a detailed examination of the dialectic relations between vertical and horizontal mathematizations in mathematical modeling in order to get a better understanding of PMTs' mathematization processes. Hence, by examining teachers' practices, challenges, and strategies related to mathematizing, researchers can identify effective instructional approaches and areas for professional development. Findings from such studies can inform teacher education programs by highlighting the importance of integrating mathematizing skills into teacher preparation courses. Educators can incorporate specific strategies and activities to support pre-service and in-service teachers in developing their mathematizing competencies. Moreover, in this study, we recognized that visualization ability is crucial in choosing an appropriate picture for the scenario in the modeling problem and extracting the mathematical meaning from it (Delice & Kertil, 2015). Our results suggest that the impact of visualization ability on the modeling process still requires further investigation.

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