Improving Pre-service Teachers' Understanding of Fractions: The Case of Nina

Melike Kara Atas
Towson University
mkara@towson.edu

Pre-service teachers (PSTs) often encounter challenges grasping the conceptual understanding of fractions, which can affect their readiness to teach this topic. This study investigates a novel approach to enhancing PSTs' comprehension of fractions by focusing on the fraction-as-a-measure meaning within a measurement context. Through a modified teaching experiment utilizing hypothetical learning trajectories, this research explores the progression of one PST, Nina, and her understanding of fraction concepts. Specifically, it examines the effectiveness of instructional task sequences in promoting conceptual understanding and addressing challenges and difficulties in fraction concepts and reverse reasoning. Results reveal Nina's transitions in understanding fraction concepts through task sequences, with a focus on three sequences: Generating Quantities Using Partitioning and Iterating, Comparing Fractions, and Recursive Partitioning. Findings illustrate PSTs' challenges and the need for carefully engineered instructional support. Integration of a measurement context and the *Fraction Bars* application proves beneficial in enhancing PSTs' conceptual understanding and reasoning skills. Further research is recommended to explore the sustained impacts of such instructional interventions on PSTs' conceptual development and variations in task sequences across diverse student populations.

Introduction

Pre-service teachers (PSTs) often struggle understanding fraction concepts and show a lack of readiness for teaching fractions (Lamon, 2007; Olanoff, Lo, & Tobias, 2014, Simon & Blume, 1994). PSTs' weak knowledge for fractions and teaching fractions has been documented in several studies (e.g., Behr, Khoury, Harel, Post, & Lesh, 1997; Jansen & Hohensee, 2016; Ma, 2010; Toluk-Uçar, 2009). One cause for this problem is due to the complexity of learning fractions (Smith, 2002). Teaching fractions adequately, especially to students who encounter difficulties, requires a strong fractions knowledge (Ward & Thomas, 2006). Teachers need a level of understanding that allow them to explain not just how to perform procedures, but also why those procedures work (e.g., Ball, Thames, & Phelps, 2008) and address students' difficulties (Ward & Thomas, 2006). Curriculum standards such as Common Core State Standards for Mathematics (CCSS-M) require teachers to have a deep conceptual understanding of fractions (CCSSI, 2010). Similarly, Standards for Preparing Teachers of Mathematics (Association of Mathematics Teacher Educators, 2017, *Standard C.1*) emphasizes beginning teachers' "proficiency with fractions" as well as well-preparedness in reading, analyzing, interpreting, and enacting mathematics curricula and understanding content trajectories. Prior research has proven that we need better instruction for PSTs and further research on PSTs' mathematical knowledge for teaching fractions (Olanoff et al., 2014) to improve fractions instruction in the classroom.

The purpose of this study is to use a novel approach to investigate and improve PSTs' understanding of fractions. This new approach is to teach PSTs fractions in a measurement context by focusing on the fraction-as-a-measure meaning. Among several meanings of fractions (e.g., part-whole, division, and ratio) (CCSSI, 2010; Norton & Boyce, 2013), the fraction-as-a-measure meaning proved to be promising in promoting conceptual understanding (e.g., Lamon, 2007).

This study aimed to explore the following research questions: *How can PSTs develop, and how can we promote development of conceptual understanding of fraction topics using the fraction-as-a-measure meaning? What type of task sequences can be used for developing PSTs' understanding of fraction concepts in a measurement context?* To investigate these research questions, instructional task sequences were designed. Clinical interviews were conducted with a PST using these instructional tasks via a one-on-one modified teaching experiment methodology. The product of ongoing and retrospective data analyses included accounts of the progression of this PST and suggestions for modifications to the task sequences.

The initial section of this report introduces the theoretical framework and its pertinent constructs and related literature. Subsequently, the methods section delineates the methodological components employed in the study. Following this, the results and discussion section examines the findings of the data analysis. The final section offers insights gleaned from the data analysis, elucidating its implications.

Theoretical Background and Literature Review

Learning Through Activity Research Framework and Relevant Constructs

The Learning Through Activity (LTA) theoretical framework is an integrated theory of conceptual learning and instructional design (Simon, Kara, Placa, & Avitzur, 2018). It proposes a method to specifically build on students' existing knowledge to promote conceptual understanding with carefully engineered tasks. LTA research aims to clarify the progression of students from one conceptual stage to another and to identify strategies to facilitate this progression. Within the LTA framework, a *mathematical concept* refers to a researcher's articulation of the intended or inferred knowledge of students regarding the logical necessity inherent in a specific mathematical relationship (Simon et al., 2018). It is a construct employed by researchers to accurately characterize student knowledge, aiming to specify the precise understanding targeted by a particular instructional design, rather than solely reflecting students' own verbal expressions of their understanding.

The process that generates a mathematical concept is *reflective abstraction*, which is distinct from empirical learning processes (Simon et al., 2018). While empirical learning processes lead to pattern recognition, mathematical concepts arise from reflective abstraction, enabling individuals to understand the logical necessity of mathematical relationships. Reflective abstraction involves the coordination of concepts, rather than isolated actions, with learners developing higher-level concepts through the coordination of existing ones. This process begins by setting new goals for tasks and sequentially executing available actions. Eventually, this leads to the anticipation of outcomes without the need for sequential actions. The result of this process is that new, higher-level concepts are formed. The underlying premise of the LTA framework is that if a concept is considered a result of reflective abstraction—meaning it is an abstraction derived from a specific activity—then it becomes feasible to design a sequence of tasks that elicit that activity and promote abstraction from it.

According to Simon, Kara, Placa, and Sandir (2016), *reversibility* denotes the permanent capability to return to the initial state of a given operation, particularly in logical thought.

The researchers distinguish between reversible and reverse concepts, where reversible concepts are constructed from original concepts and serve as higher-level constructs utilizing the original concept as a foundation. This reflective abstraction process involves forming new concepts by reflecting on previous ones, leading to the construction of reversible concepts crucial for conceptual understanding (Simon et al., 2018).

Hypothetical learning trajectories (HLTs) are helpful constructs to investigate conceptual learning. An HLT outlines a plan for facilitating the learning of a specific concept or a set of interconnected concepts. It consists of three components: (1) a learning goal (articulated concept), (2) a sequence of mathematical tasks designed to promote abstractions, and (3) a hypothesized learning process (Simon, 1995). An HLT proposes a hypothesis regarding the sequence of steps learners will take as they engage with a series of tasks, ultimately leading to the specified learning goal. Simon, Placa, Kara, and Avitzur (2018) developed hypothetical learning trajectories (HLTs) that aimed to foster fraction concepts in a measurement context. The current study used these HLTs to investigate PSTs' understanding of fractions.

The LTA instructional design approach was employed to develop instruction aimed at fostering conceptual learning and facilitating the collection and analysis of data throughout the learning process, rather than solely focusing on the outcome of learning. Additionally, the utilization of the LTA framework aimed to generate effective data for the development of empirically-based HLTs for specific mathematical concepts. Existing HLTs for fractions informed the design steps of the study, including specifying PSTs' prior understanding, articulating learning goals, and designing task sequences to promote abstraction of the concept. These components and HLT task sequences were adapted and refined based on the progress of the PSTs. Both the task sequences and the data were analyzed with regards to the reversibility construct.

Meanings of Fractions

The meanings or interpretations of fractions vary, including the part-whole, measurement, division, and ratio concepts, as described by the CCSSI (2010) and Norton & Boyce (2013). It is important for instructional materials to address these meanings comprehensively. For example, in the part-whole interpretation, 4/5 represents a whole divided into five equal parts, with four of those parts shaded. While the part-whole meaning aids in building fraction concepts in early grades, it may not suffice for advanced topics and fails to convey the concept that "Fractions are numbers." Conversely, in the measure interpretation, 4/5 represents four iterations of a length (1/5) that, when iterated five times, results in a length of one unit. Consequently, the measure meaning focuses on the length or area of a bar, leading to the consideration of a fraction as a number.

Some curricular approaches, such as the Elkonin-Davydov (E-D) curriculum in Russia, which has been adapted in the United States by several researchers, demonstrate the effectiveness of the fraction-as-a-measure meaning. In the E-D approach, fractions are introduced through problems in a measurement context, addressing scenarios where the unit does not measure the quantity an integer number of times. This measurement context has shown promise in fractions instruction (Lamon, 2007). Simon, Placa, Avitzur, and Kara, (2018) proposed that developing the concept of a fraction-as-a-measure has the potential to alleviate students' difficulties in learning fractions. Their empirical study yielded twelve hypothetical learning trajectories (HLTs) aimed at fostering fraction concepts within a measurement context, with the fraction-as-a-measure meaning proving effective in promoting conceptual understanding.

Prior Research on PSTs' Understanding of Fractions

Fractions understanding of school students has been extensively studied for several decades, but research on PSTs' fraction knowledge has been limited. To identify existing knowledge and gaps, Olanoff, Lo, and Tobias (2014) reviewed 43 studies on PSTs' understanding of fractions from pre-1998 to 2012. Below is a summary of their findings:

- Research prior to 1998 mainly focused on PSTs' challenges in relating multiplication and division to fractions. The findings indicated that PSTs are generally comfortable with fraction algorithms, but struggle with explaining reasoning or creating word problems, and often misapply whole number rules to fractions.
- Later studies continued to address fraction operations, highlighting PSTs' difficulties with word problems despite their ability to evaluate division expressions and recognize common mistakes. Overall, PSTs often relied on traditional, teacher-focused approaches, and various misconceptions in fraction multiplication were identified.
- Recent studies have shifted towards a more comprehensive understanding that includes both fraction concepts (e.g., comparison, equivalence) and operations. These studies have revealed PSTs also have difficulties with number sense, number lines, part-whole concepts, fraction diagrams/models, and word problems. PSTs' understanding remains predominantly procedural, with struggles in representing and explaining fractions conceptually. PSTs frequently use standard strategies for fraction comparison and face difficulties with fraction conversion and division. The focus on part-whole meaning limits PSTs' handling of complex fraction concepts. There were also studies that highlighted improvements through targeted tasks and manipulatives.

Olanoff et al. concluded that there is limited research on effective methods to improve PSTs' understanding, and future studies should explore the use of manipulatives, the role of language in fractions, PSTs' difficulties with number lines, and international comparisons.

Other studies not included in the review also indicated similar findings regarding PSTs' knowledge gaps, challenges with representations, and explanations (e.g., Huang, Li, & Lin, 2009; Lee, 2017; Rosli, Han, Capraro, & Capraro, 2013). For example, Huang, Li, and Lin (2009) observed that PSTs' conceptual understanding was less developed than their procedural skills. Similarly, Rosli Han, Capraro, and Capraro (2013) found that PSTs struggled to provide appropriate models or represent their solutions, particularly in the multiplication of mixed numbers. Further, Rosli et al. found that while PSTs could compare fractions using fraction strips, they often used different wholes for non-contextual versus contextual problems.

Lovin, Stevens, Siegfried, Wilkins, and Norton (2018) aimed to expand knowledge about PSTs' understanding of fractions by applying and validating a hierarchy of fraction schemes and operations previously used with upper elementary and middle school students. The study confirmed that the hierarchy was valid for PSTs, with lower-level schemes (e.g., part-whole, partitive) serving as prerequisites for higher-level schemes (e.g., reversible, iterative). However, while the majority of PSTs had constructed lower-level schemes, fewer had developed the more advanced ones. Additionally, the study highlighted challenges in assessing PSTs' procedural knowledge and their ability to manage complex fraction concepts. The researchers suggested that the focus on part-whole reasoning in K-12 education may contribute to the difficulties PSTs face in developing more advanced fraction schemes. One insight offered was to adjust the focus of instruction from part-whole meaning to language that encourages iterative thinking (e.g., describing the fraction 3/5 as 3 equal-sized parts, each of which is 1/5 of the whole). Similar to Olanoff, Lo, and Tobias (2014) the researchers concluded that further research is needed into how PSTs develop their knowledge of fractions so that findings could inform ways to improve PSTs' understanding of fractions.

In summary, to address the gap in the literature, further research is needed to focus on PSTs' understanding of fractions within other meanings of fractions, such as the fraction-as-a-measure meaning. This area of study is particularly important given the limited existing research in this area and the observed challenges PSTs face in conceptualizing and explaining fractions beyond procedural knowledge.

Methodology

This study aimed to examine the change in mathematical thinking in the context of instructional interventions (i.e., tasks). This change was expected to occur during the observation period as a result of the learner's interaction within the context of the mathematical tasks and without any direct guidance or influence from the interviewer. To minimize the effects of confounding factors (such as interaction with another student), a modified single-subject teaching experiment methodology was employed (Simon et al., 2010; Simon, 2018). In these modified teaching experiments, researchers create an initial HLT and, later, progressively develop models and instructional designs. For this study, the existing HLTs created by Simon, Placa, Kara, and Avitzur (2018) served as a starting point. The researcher further modified the initial HLTs according to the learner's prior understanding, and then proceeded to implement appropriate tasks during data collection interview sessions. Through ongoing and retrospective analysis, revised and validated HLT task sequences were constructed.

The participant of the study was recruited from elementary education majors (i.e., PSTs) at a university in the mid-Atlantic region of the U.S. The primary criterion for participant selection was to have successfully completed the mandatory mathematics content courses, specifically covering fraction topics within the elementary education program, with a minimum grade of B (i.e., 75%). One undergraduate student, Nina (pseudonym), was selected using a convenient sampling technique. Nina was a sophomore at the time of the study and had not completed any teaching-mathematics methods courses.

Five one-hour interview sessions were conducted with Nina via online video conferencing (Zoom Video Communications, 2023) during a two-month time frame in the summer of 2023. The researcher conducted all of the interviews within the study. The interviews were audio and video recorded, including screen recordings of the participant's work during the time she shared her screen. Subsequently, Nina designed a fraction lesson (approximately one hour long), rehearsed, and revised her lesson. The research questions of this paper are addressed using only the interview data, which include Nina's work and interaction with the virtual manipulative, screen and camera recordings, as well as verbatim transcripts of the interviews.

Interview Tasks

Instructional task sequences were created based on existing HLTs for fractions. Initially, the interview tasks focused on building the meaning of fraction-as-a-measure, and later progressed to more sophisticated fraction concepts. A modified version of the *Fraction Bars* computer application was used during the interviews (Kaput Center for Research and Innovation in STEM Education, 2015; see also Kara, 2023). This application enables users to draw bars, partition them, extract specific segments, and iterate the parts. A screenshot of the program is included in Figure 1.

The task sequences implemented during the interviews addressed the following topics related to several concepts of fractions:

- Generating quantities using partitioning and iterating: Whole numbers and fractions,
- Comparing fractions: Comparing fractions with the same denominator, comparing unit fractions, comparing fractions with the same numerator,
- Fractions greater than one (i.e., improper fractions),
- Relation of mixed numbers and fractions greater than one,
- Recursive partitioning, and
- Equivalent fractions.

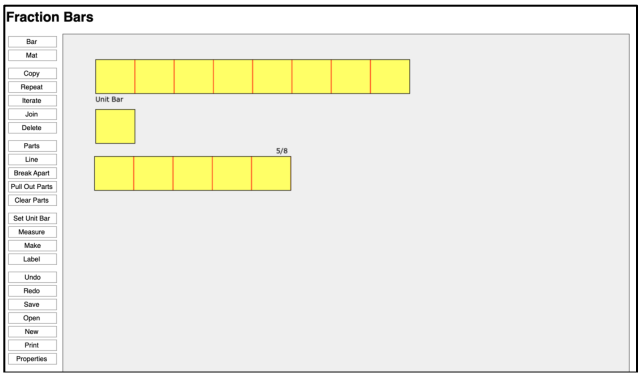


Figure 1. Modified version of the Fraction Bars application

The first task sequence involved familiarizing the PST with the Fraction Bars program and the fraction-as-a-measure meaning. Before the second and third task sequences, an initial assessment was conducted, consisting of 2–4 verbally asked questions. The initial tasks of each sequence focused on the PST using the Fraction Bars program. The subsequent tasks centered on having the PST imagine the actions she would take to solve the tasks, utilizing the context of Fraction Bars without requiring their actual use (e.g., "Imagine that you had a bar that is one unit long… What would you do? Use the bars only if you need."). The final tasks of each sequence featured larger terms for the fractions in abstract form and intentionally lacked any specific context.

Data Analysis

Two types of data analysis were conducted: ongoing and retrospective. Ongoing analysis occurred between the cycles of interviews to assist the researcher in developing an understanding of the PST's learning of the concept in question and how to promote that learning. More extensive retrospective data analysis took place after the data collection was completed. The goal of the retrospective analysis was to create an integrated model of the PST's learning and instructional interventions that can facilitate learning. This analysis was comprised of three levels, with the results of each level informing the next level of analysis (Simon, 2018). The first level of analysis focused on making small and local inferences. In the second level of analysis, the researcher made decisions about the consistency of the PST's way of operating with a particular HLT. Finally, during the third level of analysis, the researcher revised tasks and hypothesized learning processes, accordingly, modifying the HLTs for the PST. The questions guiding each analysis are detailed in Table 1. Through data analysis, accounts of the PST's progression in fraction concepts were created. The constant comparative analysis technique (Merriam, 1998) was utilized to analyze these qualitative data sets.

Table 1
Data Analysis Questions

Analysis	Questions
Ongoing Analysis	How does the sequence of tasks engage a PST in reasoning about the fraction concepts (within the context of Fraction Bars)? What is the nature of a PST's reasoning in each task or sequence of tasks? How is a PST's thinking changed, modified and refined during and after a sequence of tasks? How does this thinking connect to understanding that particular fraction concept?
First Level Retrospective Analysis	What was the PST doing and why was the PST doing it at this point in the session? What did the PST mean by what she said at this point? What was the PST thinking at this point? What might this show about the PST's understanding at this point? What was the effect of the particular task (or change in task)?
Second Level Retrospective Analysis	What understandings did the PST exhibit? How can we explain the transition from one understanding to the next? How did the task sequence afford and constrain their understanding?
Third Level Retrospective Analysis	What is the overall understanding of the PST? How are the instructional goals of the tasks aligned with the outcomes in the PST's learning? What are the revisions needed for the instructional tasks and hypothesized learning process?

Note: The questions are adapted from Simon (2018).

Results and Discussion

The results reported in this article focus on analyses revealing the transitions in the PST's understanding of fraction topics, presented within the context of task sequences and examining revisions made through the PST's interaction with the tasks. Due to this emphasis, although six task sequences were covered during the interviews, this article specifically highlights three of them: 1) Generating quantities using partitioning and iterating, 2) Comparing fractions, and 3) Recursive partitioning. For each task sequence, the data analysis results are presented first, followed by a discussion of these findings.

Task Sequence 1: Generating Quantities Using Partitioning and Iterating

To build upon Nina's existing knowledge, the first task sequence started with whole number tasks involving partitioning and iterating. Tasks within the whole number sequence centered on actions of partitioning and iterating to either determine the size of the resulting quantity or to create another quantity (e.g., "The bar on the screen is 1 unit long. Iterate the bar 4 times. How long is the new bar?"). These tasks served as both an assessment of prior knowledge and assistance for Nina in using the Fraction Bars program while working with whole numbers through partitioning and iterating. These tasks varied in terms of the use of Fraction Bars, as the rationale behind incorporating physical objects was to signify a transition from concrete to abstract use, indicating more sophisticated cognitive processes. Nina successfully completed the task sequence, demonstrating her readiness to transition to tasks involving fractional quantities.

The task sequence with fractions followed a structure similar to that of the whole number tasks. The tasks were organized sequentially, either prompting the creation of a unit fraction or proper fraction from a unit, or determining the quantity when a unit fraction was iterated a certain number of times. Subsequent tasks involved a reverse reasoning of these actions. Table 2 illustrates example tasks from the task sequence and a summarized analysis of Nina's corresponding responses.

Table 2
Task Sequence 1 – Generating Quantities and Nina's Summarized Reasoning

Tasks	Nina's Reasoning
1. The bar on the screen is one unit long. Make a bar that is 5/8 of a unit long.	Solved the tasks correctly and provided a clear explanation of her reasoning.
2. The bar on the screen is 1/4 of a unit long. Iterate the bar 3 times. How long is the new bar?	Solved the tasks correctly and provided a clear explanation of her reasoning.
3. The bar on the screen is 1/7 of a unit long. Make a bar that is one unit long.	Partitioned the bar into seven pieces, pulled out one piece. Later, correctly solved similar tasks.
4. The bar on the screen is 4/9 of a unit long. Make a bar that is one unit long.	Iterated the bar five times.
5. The bar on the screen is 2/3 of a unit long. Make a bar that is one unit long.	Iterated the bar once.

6. The bar on the screen is 4/5 of a unit long. Make a bar that is one unit long.	Iterated the bar once. Later, suggested dividing the bar into four pieces but then did not continue.

Nina successfully completed Tasks 1 and 2, as exemplified in Table 2. These tasks involved creating a fractional quantity from a bar that is either one unit long or a unit fraction amount. However, in Task 3, which was a reverse task of Task 1, when asked to create the unit from a unit fraction, instead of iterating the unit fraction, she partitioned the bar into the number indicated in the denominator. Upon being asked about the unit of the original bar, Nina recognized her mistake and correctly iterated the given bar to obtain the whole unit. Nina's initial response to Task 3 indicated a potential struggle for reverse tasks.

Next, when asked to create the unit from a bar with a non-unit fraction, such as *a/b*, in Tasks 4 through 6, Nina consistently iterated the bar *b-a* times to make the bar reach the size of the whole unit. (A correct approach to these problems involves partitioning the given bar with a measure of *a/b* into *a* pieces, pulling out one piece, and iterating it *b* times. In these tasks, the given bar should be perceived as the result of partitioning a unit bar into *b* parts and then iterating one of the parts *a* times.) While she correctly predicted that when given a bar representing a proper fraction, the unit bar must be greater than the given bar, she overall did not provide correct reasoning in her solutions.

Nina demonstrated proficiency in resolving tasks involving forward or original reasoning using partitioning and iterating. However, she faced difficulty when attempting to revert a bar, such as 2/3, back to the unit. Nina's struggle highlighted a misalignment between the original task sequence and her existing understanding, indicating that the original task sequence was not effectively building upon her current knowledge. The initial attempt to address this misalignment involved including tasks in which Nina created *a/b* from the unit bar. Later, with her solution still on the screen, she was asked to discuss how she would recreate the unit if the bar she had created was given. Nina could articulate how to go backward to the unit when the task solution was already on her screen, but she continued to struggle when presented with an unmarked bar that was *a/b* of a unit and asked to recreate the unit. She was uncertain about what number to partition or iterate in such cases. This initial attempt did not address Nina's difficulty in the reverse tasks, which led to further analysis of the task sequence.

Figure 2 served as an analysis chart for the task sequence that was used during ongoing analysis. Tasks 1 and 2 are represented with arrows pointing to the right, indicating forward reasoning, with check marks above indicating Nina's correct solutions. Task 3, requiring reverse reasoning, is represented with an arrow from a unit fraction to the unit pointing left. Although Nina initially struggled, she later solved it correctly, which is why there is an "X" mark followed by a check mark. Tasks 4 through 6 are represented with the bottom arrow pointing left. There is an "X" marked above that arrow, indicating Nina's unsuccessful attempts to solve those tasks.

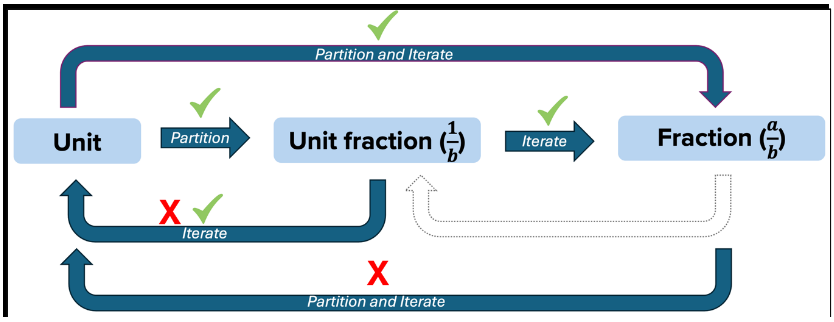


Figure 2. Analysis chart for the Task Sequence 1 – Generating quantities

The challenge during the ongoing analysis was how to devise tasks to include in the sequence so that upon its completion, Nina could handle reverse tasks without struggle. It was hypothesized that including tasks focusing on creating a unit fraction from a bar with a measure of *a/b* could aid Nina in developing reverse reasoning. These tasks would involve a partitioning action while considering the size of the bar as a collection of unit fractions. Such tasks are represented with dashed arrows in the chart, indicating they were not initially included in the original sequence. As a modification to the original sequence, these reverse tasks, which involve making a unit fraction from a non-unit proper fraction, were included in the task sequence of the interviews. Nina completed these tasks successfully without any difficulty. She was then presented with tasks focused on creating a unit from a non-unit fraction, which she also completed successfully, confirming the hypothesis.

Task Sequence 2: Comparing Fractions

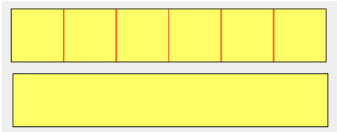
During the initial assessment, Nina's answers for fraction comparisons were correct; however, her explanations were not complete. For example, when comparing 7/8 and 5/6, Nina stated that both fractions are "one away from being whole", but she did not elaborate on this thinking further. She later stated "…7/8 is larger… because 7 and 8 is larger than 5 and 6." Although Nina initiated using a benchmark strategy, likely learned from previous instruction, her explanation was neither clear nor complete. When comparing fractions with the same numerator, she employed the fraction-as-division meaning in her explanation. For instance, when comparing 223/12 and 223/9, she said, "9 is smaller than 12. So, it's going to take a larger number to get to 223 with 9 than it is [with] 12." Nina's explanation included some conceptual components; however, her explanation would not align with the reasoning of elementary students or those unfamiliar with the concept.

After the initial assessment, Nina was presented with a task sequence that again involved the fraction-as-a-measure meaning within the context of measurement. Nina successfully completed the tasks that involved comparing fractions with the same denominator. Next, task sequences for comparing unit fractions and then fractions with the same numerator were introduced (see Figure 3 for the task sequence).

Comparing unit fractions $\frac{1}{a}$ and $\frac{1}{b}$

Task 1. This bar is one unit long. Using the "LINE" button, estimate where to cut the bar to create a piece that is $\frac{1}{5}$ of a unit long. Check your estimate. If it is too big or too small, create a new estimate and check it. Continue until your estimate is accurate.

Task 2. [Have the students create the bars below. Two congruent bars are created. The first bar is partitioned into sixths.] Using "LINE" button, estimate where you would cut the second bar to create a piece that is $\frac{1}{5}$ of a unit. Check to see if your estimate is accurate. If your estimate is not close, make a second estimate and see if it is closer. Is the part you estimated for $\frac{1}{5}$ larger or smaller than the $\frac{1}{6}$ of a unit marked on the unit bar? Explain why.



Task 3. Which is larger, $\frac{1}{13}$ or $\frac{1}{9}$? Explain.

Comparing fractions with common numerator

Task 4. Here is a 1-unit bar. Imagine you create two bars that are $\frac{1}{5}$ of a unit and $\frac{1}{3}$ of a unit. Which one would be longer? $\frac{1}{5}$ or $\frac{1}{3}$? How do you know?

Imagine that you create two bars that are $\frac{2}{5}$ and $\frac{2}{3}$ of a unit. which would be longer$\frac{2}{5}$ or $\frac{2}{3}$? (Make these bars only if you are unsure.).

Task 5. Which is larger $\frac{5}{6}$ or $\frac{5}{8}$? Explain. Use Fraction Bars only if needed.

Task 6. Which is larger $\frac{155}{98}$ or $\frac{155}{87}$? How do you know?

Figure 3. Task Sequence 2 – Comparing fractions

Nina successfully completed Tasks 1 and 2 using Fraction Bars, as the task required. During Task 3, where no context or physical representation was provided, Nina correctly answered 1/9 is greater than 1/13 and explained her response in the bar context without actually using the Fraction Bars program:

> One-ninth will be bigger. Because when you have two bars, one on the top, one on the bottom, and one has thirteen parts and the other, the one on the bottom, has nine, there is going to be fewer bars in the one-ninth unit than there is in the one-thirteenth unit. So, if there is fewer bars… that means that one-ninth is going to be larger than one-thirteenth.

Nina was able to explain her reasoning clearly and integrated the use of bars in her justification. This is particularly significant because she was able to use the fraction-as-a-measure meaning within the measurement context without any prompt.

Next, we transitioned to the task sequence for comparing non-unit fractions with common numerators. When asked to compare 2/3 and 2/5 in Task 4, Nina answered correctly that 2/3 was greater, but her explanation without the use of the bars was not clear. She said, "Two-fifths is smaller on the scale of one-fifth, then, and two-thirds is at a higher point… I don't know how to explain it." When suggested to use Fraction Bars, she created two equal-sized bars, partitioned the first into 5 pieces and the second bar into 3 pieces (Figure 4), and explained,

> So, this is two-fifths. This is two-thirds. You can tell that the two-fifths is smaller than the two-thirds. Because two in this unit [bottom bar] is larger than the two in this unit [top bar]. Because this only has three [bottom bar], but this has five [top bar].

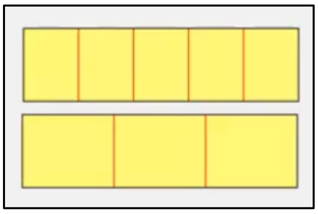


Figure 4. Nina's drawing on the Fraction Bars application for Task 4 of Comparing Fractions sequence

While using the bars, Nina articulated a clear reasoning, unlike her initial response. For the remaining tasks, Nina provided justifications for her answers using the fraction-as-a-measure meaning within the measurement context, even though the tasks were decontextualized.

The task sequences for comparing fractions aligned with and built on Nina's existing understanding; therefore, no modifications were made to the original tasks during the interviews. Nina's justifications clearly indicated that the sequence helped her improve her explanations and justifications and motivated her to use the fraction-as-a-measure meaning in a measurement context.

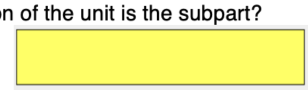
Task Sequence 3: Recursive Partitioning

The next task sequence focuses on recursive partitioning with the following learning goal:

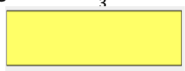
> Students will abstract that $1/b$ of $1/a = 1/ab$. That is, because $1/a$ iterates a times to the unit, and $1/\underline{b}$ of $1/a$ iterates b times to $1/a$, $1/b$ of $1/a$ iterates ab times to the unit. Thus, $1/b$ of $1/a = 1/ab$. Students will develop the concept and reverse concepts so that they can find any of the three quantities given the other two. Further, they will develop the reverse concept needed to make $1/a$ from $1/ab$. (Simon, Placa, Kara, & Avitzur, 2018, p. 195)

During the initial assessment, Nina demonstrated unfamiliarity with recursive partitioning concepts. For instance, she mentioned not knowing what 1/4 of 1/6 was. When presented with a bar and asked, "This bar is 1/6 of a unit. Partition it into 8 parts and pull out a subpart. What fraction of the unit is the subpart?" Nina completed the steps but incorrectly answered, "If it's starting to be one-sixth, and then you divided that into eight pieces...and, one of those new eight pieces would be one-fourteenth." Nina mistakenly added 6 and 8 instead of multiplying these numbers, revealing her struggle with multiplicative thinking, where each sixth of the unit is partitioned into eights. This was one of several instances where she exhibited additive reasoning instead of multiplicative reasoning, a pattern that continued throughout the instructional task sequence. After the initial assessment, the instructional task sequence with multiple steps was presented. Figure 5 includes the sequence with exemplar tasks for each step, as well as for each type of task. Note that the actual interview task sequence had additional tasks for each type of task.

Step 1: Recursively partitioning the unit and determining the measure of subpart.
Task 1. The bar on the screen is 1-unit long. Partition the unit bar into halves. Next, partition one of the parts into thirds. Pull out one subpart. What fraction of the unit is the subpart?



Step 2: Taking a part of a part and determining its measure.
Task 2. The bar on the screen is $\frac{1}{4}$ of a unit long. Pull out $\frac{1}{3}$ of the bar. What fraction of the unit is this new bar?



Task 3. [Use bar representation only if needed.] Imagine you have a bar that is $\frac{1}{3}$ of a unit long and you pulled out a subpart that was $\frac{1}{5}$ of the bar. What fraction of the unit did you pull out?

Task 4. What is $\frac{1}{4}$ of $\frac{1}{6}$? Why?
Step 3: Obtaining $\frac{1}{mn}$ from $\frac{1}{n}$
Task 5. If you had a bar that was $\frac{1}{5}$ of a unit long and wanted to make a bar that was $\frac{1}{10}$ of a unit long, what would you do?

Task 6. $\frac{1}{40}$ is what fraction of $\frac{1}{8}$?

Step 4: What was the part that was partitioned?
Task 7: If you had a bar, and you made a new bar by taking 1/6 of it, and the new bar was 1/30 of a unit long, how long was the original bar?

Figure 5. Task Sequence 3 – Recursive partitioning

For Task 1, Nina partitioned the unit bar in half, partitioned one of the parts into 3, pulled out a subpart, and asserted that the part she pulled out is either 1/3 or 1/5 of the unit. When suggested to verify whether 1/3 was correct, she said she did not know how to check her answer. Checking for correctness, or, in other words, checking whether the piece she pulled out is 1/3 of the unit, requires reverse reasoning of partitioning and determining the size. It did not occur to Nina that she could simply iterate three times the bar that she believed was 1/3 of a unit in order to confirm whether its size matched the unit. Because Nina successfully completed the task sequence where she iterated the unit fraction to make the whole unit, she was suggested to iterate the subpart to obtain the unit and make a comparison with the actual unit bar. Nina iterated the subpart three times (see Figure 6) and eventually changed her answer. She correctly stated that the subpart would be 1/6 of a unit. Her later explanation was,

> This is just half of the whole unit [pointing to the partitioned half of the one-unit bar at the top]. There is like three imaginary parts here [pointing to the unmarked half of the one-unit bar]… So, this would be one-sixth of the whole unit… if you were to divide this [unmarked half] into the same number that we divided this [marked half], you would see that there are six parts in the whole unit. If you were to pull out one, that would mean this [subpart] is one-sixth.

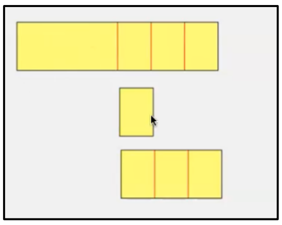


Figure 6. Nina's drawing on Fraction Bars application for Task 1 of Recursive Partitioning sequence

To help Nina notice the relationship of recursive partitioning with multiplication, she was asked, “How is ‘six’ related to your partitions?” Nina responded, “Six is… like three times two… and if you first divided the whole unit in half and then divided one of those halves into three different parts, that leads you to six parts, making up the whole [unit].” For a later Task 1-type task, when finding 1/4 of 1/5, Nina stated that the size of each subpart is 1/20 of a unit and explained, “If there is five parts and then four units within those parts… that would mean that there is twenty parts or twenty small units in total.” At the end of the Task 1-type tasks, Nina was able to determine the number of subparts in a unit by multiplying the parts and the number of subparts in each part when she was able to see the entire unit on the screen, even though not all the parts were physically partitioned into subparts.

In Task 2, unlike Task 1, the entire unit was not physically available. When given a bar that was 1/4 of a unit long and asked to pull out 1/3 of the bar, building on her current reasoning, Nina used multiplication and stated, “If this is one fourth, and then that one fourth was divided three times… then one of those smaller [parts] would be… one-twelfth… Four times three is twelve.” During another Task 2-type task, when given a bar that was 1/5 of a unit long and asked to pull out 1/6 of the bar and determine the size of the bar, Nina said, “One-thirtieth… Six times five is thirty, so if you have... If this is one-fifth and then you divided that one-fifth into six, then if you have five of those divided into six pieces, you would have thirty smaller [parts].” While determining the size of the subparts, Nina consistently used multiplication to determine the number of subparts in a unit when the entire unit was not physically available on the screen. During the interview, Nina completed Task 1- and Task 2-type tasks, but we did not initiate any Task 3-type tasks due to time constraints.

During the next interview, which occurred seven days later, we began with an assessment question (i.e., “What is 1/4 of 1/5? Justify your answer.”) presented in the abstract form. Nina answered 1/9 and explained that 4 plus 5 is 9. She continued, “If it’s one-fifth of a unit, then that fifth would be divided up into four parts which would then get you one-ninth of all of those sections of the one-fifth.” Nina imagined partitioning only one of the 1/5 pieces instead of partitioning all of the fifths in the unit. She again exhibited incorrect additive thinking during this decontextualized task. Considering that we had not completed the sequence, it was predictable that she would revert to her incorrect additive thinking. Next, we started the task sequence over. Nina correctly solved Task 1 and explained her reasoning, which was similar to her reasoning at the end of the previous interview. However, for Task 2, she incorrectly answered 1/7 or 1/3. When asked 1/3 of what, she said “One-third of one-fourth” correctly. During her justification, Nina iterated the bar that was 1/4 of the unit and explained,

> If this is one-fourth, then if you wanted to create the whole unit, you’d iterate it four times [...] And then, if you were to divide each of these into three parts, it would be three, six, nine, twelve…So, it’d be one-twelfth actually.

Nina later said that instead of adding 4 and 3, she should have multiplied. When asked why multiplication works, she said, “Because you have three [subparts], four times.” Once she iterated the bar into the whole unit, using a similar reasoning that she exhibited in Task 1, she was able to apply a multiplicative reasoning. Again, using bars helped her correctly solve the task as well as explain her reasoning clearly. When asked Task 3-type tasks without the bars and later Task 4-type tasks, Nina provided accurate answers and explained her reasoning using multiplication correctly. Such evidence indicated that Nina developed an understanding of Steps 1 and 2 in recursive partitioning tasks.

Next in the sequence was Step 3, obtaining 1/*mn* from 1/*n,* which included reverse reasoning of partitioning 1/*n m* times and determining the size of the subpart. In the original sequence from Simon, Placa, Kara, & Avitzur (2018), the reverse reasoning was supported with the tasks that included forward reasoning using a trial-and-error method. For Task 5, students would be asked trial-and-error tasks such as,

> If you had a bar that was 1/5 of a unit long and wanted to make a bar that was 1/10 of a unit long: Would taking 1/3 of the bar work?; Would taking 1/5 of the bar work?; What fraction of the bar would give you a bar that is 1/10 of a unit? (p. 196)

Such tasks were modified into "If you had a bar that was 1/5 of a unit long and wanted to make a bar that was 1/10 of a unit long, what would you do?" due to the consideration about PSTs' prior exposure to fraction multiplication tasks as well as their potential awareness of the multiplicative relationship between the denominators.

For Task 5, Nina initially stated that she would iterate the bar twice to get 1/10 from 1/5. When asked to demonstrate using the bars, she iterated the bar twice. Her use of bars again prompted her to revise her response. She partitioned the original bar in half, pulled out one, and explained, "This is one-fifth. So, if I wanted to get the whole unit, I would need four more, so it'd be two of these in each one-fifth, and five times two is ten. So, this should be one-tenth." In another Task 5-type task, Nina was asked, "If you had a bar that was 1/6 of a unit long and wanted to make a bar that was 1/24 of a unit long, what would you do?" Without using the bars, Nina provided a complete explanation that also involved reverse reasoning with multiplication. She mentioned she would need to partition it into 4 parts and explained that she arrived at 4 because 6 times 4 is 24. When asked to explain in the measurement context, she said, "If you had one-sixth of the unit and you divided it into four parts in that... It needs to be repeated six times to get one-twenty-fourth for the whole unit." Nina later completed Task 6, which was in the abstract form (i.e., "1/40 is what fraction of 1/8?"). She explained with multiplication and division, "Eight times five is forty, and forty divided by eight is five," and later provided a comprehensive explanation in the measurement context: "If you had one-eighth of a unit, and then divided that one-eighth into five parts, then that subpart of the one-eighth would be one-fortieth."

Next, we continued with reverse tasks involving the initial bar with an unknown size. When presented with Task 7, "If you had a bar and made a new bar by taking 1/6 of it, and the new bar was 1/30 of a unit long, how long was the original bar?", Nina provided a complete explanation, evidencing her reverse reasoning, "So, the original bar is one-fifth, and if you're dividing that one-fifth into six parts… In order to get the whole unit, you would repeat the one-fifth five times to get one-thirtieth." When asked how she knew it was 1/5, she said,

> It's coming from the number of bars it takes until you get thirty of the subparts… You are taking the unknown size bar… dividing it into six parts, and then repeating that for each one until you get a total of thirty subparts… and that would happen five times.

Overall, the task sequence for recursive partitioning enhanced Nina's comprehension of the recursive partitioning concepts and facilitated her development of understanding for tasks involving both forward and reverse reasoning. The utilization of the Fraction Bars application within a measurement context proved pivotal in the initial tasks of the sequence as well as in subsequent steps. While articulating her solutions, the use of bars not only assisted Nina in promoting her understanding, but also in improving her reasoning.

Conclusions

This study aimed to investigate the conceptual development of a PST regarding fraction concepts, focusing on the fraction-as-a-measure meaning. The research examined the design and effectiveness of task sequences tailored to enhance PSTs' understanding of fraction concepts within a measurement context. Analyzing Nina's responses to these sequences provided valuable insight into the conceptual growth in fraction understanding and highlighted the challenges and difficulties faced by PSTs in this regard. Notable patterns across three task sequences revealed Nina's evolving knowledge of fractions, her approach to tasks involving reverse reasoning, the influence of task modifications, and the significance of measurement context and the Fraction Bars application as scaffolding tools.

In general, the task sequences initially designed for elementary-level students required further modifications when adapted for the PSTs. Tailoring instructional sequences to the specific needs and prior knowledge of the PST was still crucial for promoting conceptual learning of fraction concepts as well as improving explanations and justifications on the underlying logical necessities for the task solutions.

PSTs demonstrate a fragmented understanding of fractions, possibly incorporating incorrect additive reasoning. Despite progress, Nina exhibited fragmented understanding in tasks involving reverse reasoning. Instances of incorrect additive reasoning, like determining 1/4 of 1/5, underscore the requirement for additional instructional support. The results indicate the necessity for further instruction to achieve a more comprehensive grasp of fraction concepts. Moreover, PSTs would benefit from improvement in their ability to articulate and justify their explanations.

PSTs encounter challenges in tasks that require reverse reasoning. For example, in Task Sequence 1, Nina demonstrated proficiency in forward reasoning by deriving fractional quantities from given bars. However, difficulties arose in reverse tasks, particularly in reconstructing the unit from a non-unit fraction. Similarly, issues with answer verification revealed incomplete reverse reasoning. Task modifications designed to address this misalignment emphasized the importance of carefully designed sequences that promote both forward and reverse reasoning. Tasks involving reverse reasoning require detailed analysis, and the sequence should be thoughtfully engineered to ensure their effectiveness. Engaging students with the forward process while solving a reverse task could help them anticipate the reverse process (Simon, Kara, Placa, & Sandir, 2016). The results of this study indicated that further unpacking of the task sequence is necessary to support students' reasoning in such reverse tasks.

The findings emphasize the importance of explicit support in developing multiplicative reasoning. Task sequences that gradually introduce and reinforce multiplicative thinking can contribute to overcoming struggles observed in PSTs' experiences when learning fraction concepts.

Integrating a measurement context proved beneficial in promoting the fraction-as-a-measure meaning. It also enhanced reasoning and justifications, evident in Nina's experiences. In several tasks, Nina initially employed incorrect additive reasoning, but rectified and clarified her justifications while contextualizing them in the measurement setting. For instance, Task Sequence 2 focused on comparing fractions, showcasing Nina's evolving ability to deliver clear explanations. The measurement context played a pivotal role in promoting Nina's understanding; Nina seamlessly integrated the fraction-as-a-measure meaning within the measurement context without additional prompts. Future instructional designs should explore ways to embed measurement contexts effectively in fraction tasks to enhance PSTs' conceptual understanding.

The Fraction Bars application proved valuable for the PST, aiding in clear explanations and justifications. In the third sequence, recursive partitioning, Nina initially struggled with multiplicative thinking. The Fraction Bars application emerged as a crucial scaffolding tool, aiding Nina in connecting abstract concepts to physical actions. Development of her reverse reasoning and the generalization of multiplicative reasoning highlighted the positive impact of this scaffolding process. Mathematics teacher educators should consider integrating such tools to enhance PSTs' understanding and reasoning skills.

These conclusions address how to enhance PSTs' conceptual understanding of fraction topics and offer valuable insights into designing effective task sequences within a measurement context. To develop and promote PSTs' understanding of fractions using the fraction-as-a-measure meaning, it is essential to tailor instructional sequences to their needs, focus on reverse reasoning, provide explicit support for multiplicative reasoning, integrate measurement contexts, and utilize scaffolding tools. Task sequences should be designed to gradually introduce more sophisticated concepts and include carefully integrated manipulatives that promote understanding, ensuring PSTs can articulate and justify their reasoning effectively. Future studies should explore further the sustained impact of such HLT-based task sequences on PSTs' long-term conceptual development and explore variations in task sequences across diverse and large student populations.

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