Describing Lesson Study Designed for Improvement of Mathematics Teachers' Knowledge of Student Thinking

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The aim of this study is to explain how to shape the lesson study designed for improving mathematics teachers' knowledge of student thinking and to uncover the evidences of the mathematics teachers' knowledge of student thinking in their lessons. The lesson study composed of three cycles with five stages was carried out with three mathematics teachers. The data were collected from the interviews, the lesson observations, the video records of the lessons and their transcriptions, the researchers' field notes and the teachers' reflective diaries. The data were examined by the indicators regarding knowledge of student thinking in a detailed way. The designed lesson study increased the teachers' awareness of knowledge of student thinking. The teachers conducted the lessons not only depending on the content but also depending on the students' ideas.

Keywords: knowledge of students' thinking, lesson study, mathematics teachers, professional development, teacher knowledge.

Introduction

There is a common consensus about the fact that effective teachers have knowledge of students' mathematical thinking and ideas (Hill, Ball, & Schilling, 2008). The effective teachers could support students' conceptual learning and could realize their teaching by focusing on their students instead of the subject. In the report of The Common Core State Standards for Mathematics (CCSSM, 2010), the question of how teachers could be supported in focusing on students' mathematical thinking was pointed out (Wilson, Mojica, & Confrey, 2013). It is not meaningful to try to respond to the problems about learning without considering the learning environment. Research conducted with teachers provides meaningful results or implementations in many aspects like the learning environment, learning activities, students-teacher interactions and teachers' discourses. The researchers should enable teachers to attend professional development programs on how they could create learning environments to encourage their students to think. Nielsen, Steinthorsdottir and Kent (2016) emphasized that professional development programs enable teachers to have experiences of how to plan and realize instruction based on students' thinking. Kazemi and Hubbard (2008) stated the importance of focusing on student thinking for effective professional development and that teachers must learn how to elicit students' thinking and reasoning, design mathematical tasks for mathematical learning trajectories, and realize appropriate classroom discussions and group work so that they can trigger students' ideas.

Different professional development studies have been realized for mathematics teachers and pre-service mathematics teachers to improve their teaching (Baki, 2012; Cheng & Yee, 2012; Ilma, 2011; Lewis, Perry, & Hurd, 2009; Meyer & Wilkerson, 2011; Olson, White, & Sparrow, 2011; Sisofo, 2010; Tepylo & Moss, 2011). The main purposes of these studies were the improvement of mathematics teachers' knowledge of teaching mathematics. Besides, the studies on improvement of teachers' "Knowledge of Student

Thinking" (KoST) have recently gained momentum. KoST is defined as the basic component of the knowledge of teaching mathematics (Kung & Speer, 2009; An, Kulm, & Wu, 2004). Evidently, the practices for professional development such as video case studies and analysis and online discussions were utilized for improving KoST or increasing teachers' awareness of student thinking (Baş, 2013; Cengiz 2007; Fernandez, Llinares, & Vals, 2012; Hartman, 2012; Van Zoest, Stockero, & Kratky, 2010). Baş (2013) aimed to develop teachers' noticing of student thinking based on model and modeling perspectives by the seven-month professional development program which she conducted pre- and post-lesson meetings about classroom practices with four high school mathematics teachers. Cengiz (2007) tried to reveal how students' thinking was expanded and how teachers' beliefs and knowledge affected students' thinking by conducting group discussions on numbers and operations with six experienced elementary school teachers performing standards-based curriculum instruction. Fernandez, Llinares and Vals (2012) aimed to provide preservice teachers with professional awareness about students' mathematical thinking in online contexts. The preservice teachers worked individually or collaboratively on different activities and firstly discussed their work online. Then they talked about each other's work face to face and finally online. Hartman (2012) examined the effects of video-based intervention on pre-service teachers' understanding about early-age children's mathematical thinking. Van Zoest, Stockero and Kratky (2010) conducted focus group discussions on classroom video camera records with 14 beginning teachers to reveal and support the teachers' thoughts related to students' thinking. In these studies, the teachers or pre-service teachers participated in the environments which would support their professional developments about their noticing of students' thinking in the context of specific mathematical topics. Also, a knowledgable other generally interacted with them in these processes. In this direction, it is evident that mathematics teachers' KoST, the basis for knowledge of mathematics teaching, should be examined in the context of different mathematical concepts. Additionally, a professional development model supporting experienced mathematics teachers' KoST should be presented with the evidences from real classroom practices. It is important that mathematics teachers in a school can improve themselves by supporting each other without helping from a knowledgable other. The process of our study will be guiding teachers to understand how a path they can follow to develop themselves with their colleagues.

Teachers should attend professional development programs in which they interact with their colleagues to develop their mathematical content knowledge and to provide students' effective learning (National Council of Teachers of Mathematics, 2000). However, Simon (2013) emphasized that teacher education and professional development programs have largely been unsuccessful in the last 20 years in carrying out reforms because the teachers did not believe in the necessity of development and were resistant to change. Among the models that provide professional development, lesson study has many features of high-quality professional development (Borasi & Fonzi, 2002, Darling-Hammond & McLaughlin, 1996, Hawley & Valli, 1999, as cited in Perry & Lewis, 2008; Garet, Porter, Desimone, Birman, & Yoon, 2001). We also suggest that lesson study could be used effectively to improve mathematics teachers' KoST. We carried out a lesson study lasting nine-months with mathematics teachers for developing their KoST. As this process was a long period, it was effective in terms of mathematics teachers' progression. Also, it was conducted by focusing on the evidences of KoST in real classroom environments. When the importance of KoST is considered, our study could make a great contribution for mathematics education literature. In this context, the purpose of the study is to explain how to shape the lesson study designed for improving mathematics teachers' knowledge of student thinking and to uncover the evidences of the mathematics teachers' knowledge of student thinking in their lessons.

Conceptual Framework

Knowledge of student thinking. In the last 20 years, the studies of learning have focused on understanding how students think and how thoughts become more advanced over time (Mojica, 2010). Focusing on student thinking is a distinguishing characteristic of teachers' professional learning (Wilson, Mojica, & Confrey, 2013). Mathematics educators came to a consensus in creating new learning objectives that ensure focusing more on student thinking (Simon, 2006). Mathematics teachers who are aware of students' thinking make learning and teaching meaningful both for themselves and for their students. An, Kulm and Wu (2004) placed KoST at the core of pedagogical content knowledge. The teachers having KoST could reveal students' ideas and support their thinking and problem-solving skills (Wicks & Janes, 2006) and could consider students' needs and create opportunities to improve their understanding (Asquith, Stephens, Knuth, & Alibali, 2007). Hill and Ball (2004) stated that students' learning depended not only on teachers' content knowledge but also on the interaction between the knowledge of their students' learning and knowledge of teaching strategies. The experienced teachers could decide tasks appropriate to students' thinking and learning and properly conduct teaching (Mousley, Sullivan & Zevenbergen, 2007). KoST includes knowing students' understanding, conceptual difficulties and possible ways of learning. It also involves developing an awareness of what students think and do in mathematics lessons (Takker & Subramaniam, 2012). An and Wu (2012) have suggested KoST includes teachers' knowledge of how well students perceive mathematical concepts, understand possible misconceptions and develop appropriate strategies to overcome misconceptions. The teachers whose awareness of student thinking is improved can develop effective ways for conceptual understanding and prepare appropriate lesson plans. Considering the content of KoST, we can say that it includes all factors affecting student learning such as teacher-student interactions, in-class discourse, lesson plans, evaluating students' thinking and teaching approaches. When mathematics teachers discuss their KoST in detail, they can understand what approaches they should have for focusing students' thinking.

Lesson study. Hurd and Licciardo-Musso (2005) defined lesson study as a cycle focusing on teachers' planning, observing and revising a research lesson with cooperation. In the lesson study model, students and particularly students' thinking are considered significant during all the activities (Takahashi & Yoshida, 2004) and this makes teaching more practical and understandable by enabling teachers to develop a profound understanding about the content and students' thinking (Murata, 2011). During the lesson study, teachers infer students' thinking, solution ways and strategies, and search for the usage of tools to create classroom discussions (Yoshida & Jackson, 2011). Xu and Pedder (2014) explained that teachers conduct conceptual analysis in-depth and investigate students' prior knowledge and understandings related to the concept to prepare a plan in a lesson study process. Observing lessons as if the teachers were students may develop their KoST and may provide motivation to make their teaching more effective (Lewis, Perry, Friedkin, & Roth, 2012). Synchronously taking video records is of importance for the next stages of the lesson study process while observing the lessons. Alston, Pedrick, Morris and

Basu (2011) expressed that the use of video during the lesson observations in a lesson study model enhance teachers' natural ability to reflect on their practice.

White and Lim (2008) stated that lesson study assists teachers to design quality lessons and gain a better understanding of student learning. Lesson study is commonly used to support teachers' professional development (Stigler & Hiebert, 1999; Takahashi, 2017). Teachers improve their pedagogical content knowledge and support their students' learning by participating in lesson study (Department for Education (DfE), 2009). The lesson study requires active participation of teachers and enables them to make sense of their practices. Eskelson (2013) stated that the meetings of the lesson study supply teachers in active participation as opposed to listening to a lecture passively. By engaging in this process, teachers find opportunities to develop all the students' learning in the school.

We designed a lesson study lasting nine months and consisting of three cycles. Each cycle was composed of five stages: (a) research and planning, (b) implementing research lesson, (c) reflecting and improving research lesson, (d) implementing revision lesson, and (e) reflecting and improving revision lesson (See Fig. 1).

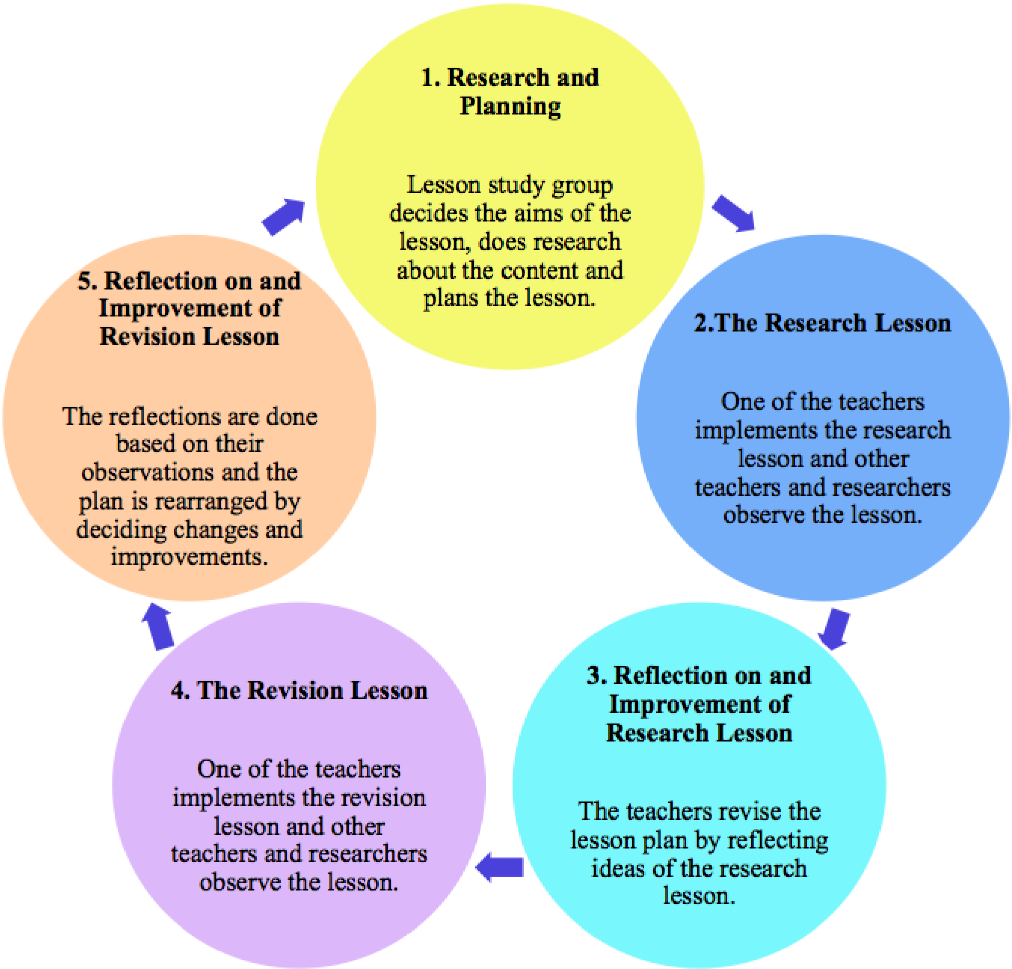


Figure 1. The Lesson Study Cycle

The teachers collaboratively worked and examined the concepts by considering possible students' thinking during the planning. In the planning and revision meetings, they discussed how the student would think about the concepts and in which ways they would learn them. Inherently, teachers would support one another in lesson study and their improvements would gain momentum. In this study, we tried to respond to the questions of "How to shape the lesson study designed for improving mathematics teachers' knowledge of student thinking? and "What are the evidences of the mathematics teachers' knowledge of student thinking in their lessons?". In this direction, we dealt with the three research questions as follows:

1. What knowledge of student thinking was in evidence during the first lesson study cycle?
2. What knowledge of student thinking was in evidence during the second lesson study cycle?
3. What knowledge of student thinking was in evidence during the third lesson study cycle?

Method

We conducted the study through a qualitative case study in which we examined the lesson study model designed for the development of the reflections of mathematics teachers' KoST on their teaching. Creswell (2013) defined case study as a research design in which a researcher deeply analyzes cases such as a program, an event, an action, a process or one or more individuals. In our study, we examined planning and revision meeting, lesson plans and the teachers' teaching in detail to reveal evidences of the teachers'development about KoST during these examinations, the cases which we focused on were the teachers' actions and discourses in each cycle. We aimed to reveal the teachers' development in the context of the lesson study cycles with these analyses. Therefore, we utilized case study design because we broadly dealt with each cycle of the lesson study designed for the improvement of KoST.

Participants

In this study, three mathematics teachers working at a high school in Turkey were chosen as participants by typical-case sampling of purposive sampling methods. The aim of the purposive sampling method is to obtain more information about the purposes and to choose the cases which will provide useful information. By means of typical situation sampling, a general view may be gained through working on the average situations (Patton, 2002). In our study, we studied with the teachers from only one school. As the lesson study requires that teachers collaboratively work, it is important that meeting time can be readily arranged. Realizing this process with the teachers in one school provided us with ideas about more comprehensive studies with teachers from different schools. Also, the participants of the study were the teachers who have been interacting with the researchers for a number of years. These interactions were derived from their school-based mentoring pre-service teachers of our university and their participation in some workshops and seminars held by us. Hence, one of the factors in selecting these teachers was our knowledge about their teaching approaches and routines. The second factor was their willingness to improve themselves in some respects such as mathematics knowledge, teaching approaches, motivating students, as they expressed in our workshops and seminars. The participants have also known each other since they studied in the same university, and worked in the same school for a long time but did not discuss with each other about their teaching. The only issue they shared was what would be dealt with in lessons and which subjects would be included in the exams, because mathematics exams were commonly held for all classes. They had never observed their lessons before. The idea of being observed by researchers and by their colleagues was distressing for them. After we told them that observing different teaching practices and discussing their efficiency as a part of lesson study were important, they became willing to join the study despite feeling anxious. The participants' real names were not used and pseudonyms were used. The gender, the educational status and the teaching experiences of the participants were given in Table 1.

Table 1
Information about the participants

Pseudonym	Descriptions
Ali (male)	He took a bachelor's degree in mathematics education and then a master degree in mathematics. He had thirteen-years of experience.
Ozden (female)	She took a bachelor's degree in mathematics education. She had thirteen-years of experience
Serin (female)	She took a bachelor's degree in mathematics education. She had thirteen-years of experience

We conducted semi-structured interviews with all the participants and observed their lessons. With these interviews, we tried to reveal teachers' opinions about lesson planning and what KoST was, and whether they had the approaches regarding KoST during their teaching, and how they considered the students' thinking in their lessons before lesson study. Also, we observed their lessons as we wanted to understand their actions and teaching approaches in the lessons which they prepared per se. Thus, we had more information about the teachers. Serin was teaching in a very traditional manner and failing to make her students active participants. Initially, she was lecturing, then she was giving definitions and properties regarding the subject. She was generally selecting one of the students who correctly responded to the question to present his/her solution on the board. She was not focusing on the students' incorrect solutions and she was not encouraging the students to think another way. She was going on her teaching without considering her students' thinking. Ozden, who was willing to improve herself, was trying to attend various workshops. However, she was conducting her lessons mainly depending on textbooks rather than students' thinking. Thus, we could not say that Ozden was flexible in planning and conducting her teaching. Moreover, she was asking only some certain pupils whom she considered to be more likely to correctly respond to the questions. Although Ali was trying to ask thought provoking questions to his students in comparison to other two teachers, he did not completely integrate these kinds of questions into his lessons. Sometimes, he was able to choose appropriate questions and examples. Yet, his pedagogical approaches were limited while he was practicing well-chosen questions. Also, he was not questioning the reasons for the incorrect solutions. In general, these three teachers' ways of teaching involved asking questions and considering the correct solutions and repeating these solutions one more time. Therefore, it would be important for the professional development of the participants to share their experiences. The teachers noticed each other's different actions and supporting aspects of their teaching, so their motivation in the process increased and their teaching became more effective. The teachers who had not share anything related to their teaching got used to working collaboratively and shared the points which they thought as important for students' learning. We thought that the interactions among the participants and researchers would promote the participants' improvements regarding mathematics knowledge, student knowledge and pedagogical content knowledge.

Data Collection

The data were gathered from interviews, observations of lessons, transcriptions of the lessons, reflective diaries and field notes. During three cycles, we conducted eight unstructured group interviews in total, three of them in the first cycle and three of them in the second cycle and two of them in the last cycle. We realized the unstructured interviews in the planning and revision meetings in the course of the lesson study. The purposes of the interviews conducted in the planning meetings were to understand the teachers' decisions about in-class activities, to reveal their thoughts leading these decisions, and to prompt them to interact with each other by encouraging them to think aloud. The purposes of the interviews in the revision meetings were to ask questions paving the way for criticizing the research lessons and to determine the actions/activities which they would change and their reasons. Additionally, we interviewed the teacher who implemented the lesson after each lesson about the extent to which she/he complied with the prepared the plan and to what extent her/his KoST was reflected in lessons. The purposes of individual interviews were to understand and extend the accuracy of our field notes. We recorded all the interviews with a video camera to do retrospective analysis and to prevent losing data.

We observed the lessons during the lesson study and in the course of these observations, we took detailed field notes by considering the KoST. We recorded the lessons to capture the teacher's and the students' discourse/actions/gestures and to follow the solutions on the board by two cameras. While one of them focused on the board and on the teacher, the other focused on the students. The purpose of the lesson observations and video camera recordings was to determine the cases which would be evidences for teachers' KoST. After the lessons, we transcribed the video camera records verbatim. We wrote the teacher's discourse, the students' discourses and their solutions verbatim. Additionally, we wrote the teacher's and the students' actions and gestures which we considered as necessary at the parts of video recordings in brackets in order to present the context better. The purposes of these transcriptions were also to examine all the data in detail and to make comparions between the data after the lessons. We asked the teachers to write individually reflective diaries after each cycle of the lesson study. In the reflective diaries, we asked them to explain their ideas about the interactions which they made with the researchers and their collegues, the actions which they and the students had in the classrooms, the changes which they made in the lesson plans and their reasons. Thus, we obtained the data which would support the observations and interviews.

Procedure

Initially, we informed the teachers about the content of the study. We aimed to support the teachers to get motivated, to explain our expectations and to increase their awareness regarding the importance of their professional development. At the beginning of the study, we firstly conducted the semi-structured interviews with the teachers. Then, we observed their lessons but the participant teachers did not observe these lessons. One of our purposes in this process was to determine the content of the seminar which we would give to the teachers and to understand their knowledge before the lesson study. We gave a seminar to the teachers introducing our purposes, the lesson study model, KoST, the lesson plan guideline. This lesson plan guideline was developed by Smith, Bill and Hughes (2008). This lesson plan guideline had the content which encouraged the teachers to think about the lessons in depth to consider the students' mathematical understanding. This guideline along with the lesson plan format was presented to the teachers. It prompted the teachers to focus on their students' thinking during the whole process.

This seminar process provided teachers with motivation to a study that they were not used to, realizing the importance of the study, adopting the study process beforehand, and discussing the reasons why working together would be necessary and useful. Then, the teachers decided the topics which they would teach. In this stage, the teachers, who concentrated on the implementation of the teaching at the two different grade levels, discussed the concepts which 9th and 10th grade students found difficult. After these discussions, they decided the subject ares of the radical expressions and trigonometry. The teachers taught "radical expressions" at 9th grade in the first cycle. For the second cycle, they implemented the lessons about "the trigonometric ratios in a right-angled triangle" at 10th grade and for the third cycle, they taught "coterminal angle and unit circle" at 10th grade. The study process was given in Table 2.

Table 2
The process of this study

Week	The content of the process
Week 1 October 4, 2013	We conducted semi-structured interviews with 3 teachers to learn their opinions about KoST and lesson study.
Week 2 October 11, 2013	We observed each teacher's two hour lesson before the lesson study process.
Week 3 October 18, 2013	We gave a seminar introducing the lesson study model and discussed the effectiveness of lesson study.
Week 4 October 25, 2013	We gave information about concept map, information map, concept cartoon, cooperative learning and brain storming, etc.
Week 5 November 1, 2013	We discussed the components of KoST.
Week 6 November 15, 2013	We introduced and discussed the lesson plan guideline. This format, different from a standard lesson format, included the questions those to be considered in the planning of a lesson during which students' thinking was especially taken into consideration, the tasks to be implemented in the classes and the reasons for these tasks to be chosen,

	the periods to be devoted to each task, the tools to be used, the roles of the teacher and the students in this process and their relations with the knowledge of student thinking. Additionally, the teachers decided to implement the first research lesson at 9th grade about radical expressions.
Week 7 November 22, 2013	We met to plan the research lesson but teachers could not complete the plan because they did not interact with each other effectively. The researchers noticed this situation during the meeting and encouraged them to make research about the concept together and to discuss their ideas up to the next meeting.
Week 8 November 29, 2013	The teachers continued to prepare the lesson plan and completed it (see Appendix A for a part of the lesson plan of the first cycle lessons).
Week 9 December 3, 2013	One of the teachers implemented the plan.
December 5, 2013	We met to discuss the research lesson, share the observations, to revise the plan.
December 6, 2013	The revision lesson was implemented and after the lesson, teachers met for evaluating it. They made revisions by considering the revision lesson. They decided to teach trigonometry at Grade 10 for next lesson study cycle.
Week 10 January 10, 2014	The teachers met to plan the second lesson study cycle but they decided to do more research about the subject because they thought the studies they did were insufficient. They also discussed the revision lesson.
Week 11 January 23, 2014	The teachers planned about teaching “trigonometric ratios in a right angled triangle”.
January, 27 –February 7 Mid-term Break	
Week 12 March 3, 2014	The research lesson of the second lesson study cycle was implemented.
March 4, 2014	The teachers talked and discussed the research lesson, they assessed it

	in the context of KoST, they revised the plan.
March 6, 2014	The revision lesson was implemented.
March 7, 2014	The teachers assessed the reflections of the revisions on teaching and students learning. They made refinements on the plan they considered as necessary. They decided that they would implement the subject of "coterminal angles and unit circle" for the third lesson study cycle and they did the lesson plan of the research lesson.
Week 13 March 10, 2014	The research lesson was implemented. Then, the teachers discussed the lesson. They revised the plan by changing what they considered necessary in the context of both KoST and the content.
March 11, 2014	The revision lesson was implemented. After the lesson, they talked about the reflections of the revisions on lesson.
March 14, 2014	We arranged a meeting to assess the study by considering which changes the lesson study provided with the teachers and what positive aspects of reflections of KoST on teaching were.
Week 14 May 9, 2014	We met for the teachers to understand their approaches in the context of KoST by presenting the analysis of all their lessons.
Week 15 May 15-16, 2014	We observed the 3 teachers' two-hour lessons.

Data Analysis

We participated in all the lessons and meetings, so we were familiar with the data. We realized the data analysis simultaneously to the data collection process. The main data for analysis were the transcriptions of the lessons and meetings. In this process, we considered the studies of An, Kulm and Wu (2004) and Lee (2006) to determine the evidences of KoST. During the analysis process, the KoST indicators which we considered were: building on students' mathematical ideas, promoting students thinking mathematics, triggering and considering divergent thoughts, engaging students in mathematical learning, evaluating students' understanding, motivating students learning, considering students' misconceptions and errors, considering students' difficulties, and estimating students' possible ideas and approaches.

While examining the transcriptions of the meetings, we focused on whether the teachers considered students' possible thinking, how they evaluated the students' possible thinking, how they integrated their thinking on the lesson plans, and the evidences of KoST. While examining the transcriptions of the lessons, we considered which approaches could be related to KoST, what approaches teachers used while considering students' thinking, and how they tried to reveal students' thinking. We examined all cases that could be related to KoST and these cases had important roles in the analysis. We reviewed again the transcriptions and video records together and deeply thought about the cases, and sought responses to the research questions. Then, we came together with teachers and discussed the sections related to the evidences of KoST we previously determined over the transcriptions and examined again these sections if needed. We tried to support their improvement by asking several questions to the teachers at this step. We guided teachers by emphasizing KoST with the questions such as“ Which section have been effective according to your plan?”, “What the students had difficulties in the learning process?”, “What would you change about this lesson if you wanted to do so?”, “What do you think that the best and the worst practice in terms of KoST was?”, “What did you do or what could you have done to determine whether the students learned or not?”, “Why was the time not enough for implementing all the activities?”, “When you considered the teacher’s and students’ actions after the teacher asked the question of ... , how could you have managed this process?”, “What would you have done to better understand student thinking as teachers?”, “When the student said that …, what would his/her thoughts be?”. Thus, we tried to reveal the cases about KoST more effectively and deeply.

The field notes and the reflective diaries were supportive to the main data for data analysis. We examined the field notes in the context of the parts in which the teachers considered students’ thinking, and determined their approaches and actions related to the KoST. In this direction, we employed these data to ask questions to the teachers in the meetings. We read the reflective diaries, determined the descriptor parts related to the KoST and presented them by the directly excerpts. At the end of the whole process, we completed the analysis stage by making retrospective analyses.

Findings

We handled each step of the three lesson study cycles while presenting the findings. The reason why we presented the steps separately was to reveal what the teachers realized and which evidence of KoST occurred in each step of different cycles to reveal how the lesson study affected teachers’ improvement. Also, we presented the excerpts related to KoST. While presenting these excerpts, we focused on in-class discourse including the teacher-students interactions. We exemplified the teachers’ activities in research lessons which were important in terms of KoST. Especially, we presented the activities, if they revised them after the research lessons.

1. Findings on the First Lesson Study Cycle

(a) Research and planning. In the first meeting of the planning, the teachers could not conduct an effective study and did not share their teaching experiences with each other. They considered the order of the content without regarding the students’ difficulties and understanding. The only case they considered was what the students learned about the square root expressions at the elementary school level. Especially, they focused their own content knowledge about the root expressions by depending on the curriculum and the textbook. They also discussed the negative root and the positive root while examining a concept cartoon, but they decided to ignore the difference between them in the context of the lesson. The teachers interacted more in the second meeting and completed the lesson plan. The content of the lesson plan was intensive and not focused on students’ thinking. Although we tried to draw their attention to students’ thinking and to support them to

discuss, they could not give up their routines. The teachers' expressions and the researcher's questions were as follows:

Ali:	We can ask a question related to multiplication of radical expressions and we can make them order these numbers [$x = \sqrt[4]{2}, y = \sqrt[6]{3}, z = \sqrt[12]{7}$]
Researcher:	What do you think about what the students will think?
Ali:	What will they think about the question, won't they?
Researcher:	Yes, what will students think?
Ali:	They think that 7 is bigger than the others.
Researcher:	What else?
Ali:	They will write these numbers in exponential, but, umm… what will they think really?
Ozden:	They will make equal the degrees of the roots, one by one.

(b) Research lesson. Ali could not completely implement the research lesson plan because of lack of time. As they did not address what difficulties students could have and did not discuss the students' possible solution approaches in the planning, he lingered over how to overcome students' difficulties. Additionally, because they did not discuss how the students related their prior knowledge to new concepts for better understanding, he realized rule-based teaching for the relation between the exponential expressions and the radical expressions. He expressed that $\sqrt[3]{8}$ was another form of $8^{\frac{1}{3}}$ and this teaching approach caused the students to have only procedural understanding without thinking of the reason why these expressions related with each other.

Ali:	There is a different representation of $\sqrt[3]{8}$. We can say 8^1 instead of 8 as an exponential number, so if you want to write $\sqrt[3]{8}$ as an exponential number, you can write $8^{\frac{1}{3}}$. That is, $\sqrt[3]{8}$ is an another representation of $8^{\frac{1}{3}}$. These expressions are the same, not different. Then you find the equality of $\sqrt[3]{8}$ by using that operation: $8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2^{\frac{3}{3}} = 2$

Ali gave the definition of the exponential numbers and then he asked the questions including variables such as $\sqrt{x^2} + \sqrt[3]{(x-y)^3} - \sqrt{y^2}$ although he had to ask the questions including numerical values such as $\sqrt{5} - 3$ according to the lesson plan. As Ali did not exemplify the definition by using the real numbers with the different roots after giving definition, the students had difficulties and continuously asked questions him. Ali noted the reasons of this situation in his first reflective diary as follows:

> As I did not instruct according to the lesson plan, some problems occurred. Also, I could not complete the lesson plan because its content was too intensive. Therefore, the plan was not completely implemented.

Another factor which interrupted the flow of the lesson was that Ali asked a question which was not included in the plan. In the following excerpt, Ali spontaneously asked how $\sqrt[12]{2^{13}}$ could be written in a different way. This question led the students to produce different ideas. Also, he revealed that students had incorrect ideas and tried to correct them.

Ali:	$\sqrt[12]{2^{13}}$ is a response to a question. But this response was not included in the choices. How can you find the response? What can you do?
Student 1:	$2^{\frac{13}{12}}$

Ali:	This is not such an answer in the choices, also.
Student 2:	$2^{\frac{26}{24}}$
Student 3:	I simplify.
Ali:	You cannot simplify $\frac{13}{12}$
Student 4:	$6\sqrt[12]{2}$
Student 5:	Is it $\sqrt{4}$?
Ali:	I am listening, what else?
Student 6:	$6\sqrt[12]{2}$
	ooo
Student 3:	$\sqrt{4}$, that is, 2.
Student 4:	$\sqrt{4}$, I say the same response.
Ali:	It is$\sqrt{4}$?
Student 5:	Yes, it is $\sqrt{4}$.
Student 7:	I think, $\sqrt{2}$.
Student 8:	$2\sqrt{2}$.
Ali:	What is $\sqrt{8}$?
Students:	It is equal to $2\sqrt{2}$.
Ali:	Because, you can write $\sqrt{8} = \sqrt{4}.\sqrt{2}$. Is it right?
Students:	Yes.
Ali:	Well then, what is $\sqrt[3]{16}$?
	ooo
Student 9:	If we write 16 equal to $2^3.2$, $\sqrt[3]{16}$ is equal to $2\sqrt[3]{2}$.
Ali:	Well, we can write $2^3.2$ instead of 2^4.
	$\sqrt[3]{16} = \sqrt[3]{2^3.2} = \sqrt[3]{2^3}.\sqrt[3]{2} = 2\sqrt[3]{2}$
	The cube root of 2^3 is 2. Now, think $\sqrt[12]{2^{13}}$ similar to this way.

(c) Reflection on and improvement of the research lesson. The teachers decided that the content of the plan would be reduced. They considered whether the negative and positive roots were necessary. When they examined the textbooks, they realized that these concepts were included in the radical expressions. They decided that the students would have incomplete mathematical knowledge and added these concepts to the revision plan. They stated that the students' motivation was lost due to the intensity of the content in the research lesson and decided to give historical information about the radical expressions.

They emphasized that the students had difficulties because Ali directly gave examples including variables as soon as he expressed the formal definition. They thought that the examples including numerical values such as $\sqrt{(\sqrt{5}-1)^2} - \sqrt{\left(2-\sqrt{5}\right)^2}$ needed to be given by expressing the fact that Ali's practice was problematic as seen in the following excerpt.

Ozden:	There were only examples including algebraic expressions. The examples related to the roots of real numbers were not used.
Ali:	Immm, this was the first problem of implementing the plan.
Ozden:	In the revision lesson, we should use these examples before asking algebraic expressions.

Also, because the students had difficulties in relating exponential expressions with radical expressions and the relation was lectured in a procedural way, they discussed that they had to teach in a different way and how this could be. To prevent students from memorizing this relation, they decided a different approach based on justification of this relation. Additionally, they added decimals examples to the lesson plan to provide students higher order thinking.

(d) Revision lesson. Ozden who implemented the revision lesson used mathematical language correctly while giving the definition of the radical expressions. This approach led the students to understand the concept and its properties more easily and supported the development of the students' existing knowledge. Additionally, by using the examples including relations of different concepts, she prompted the students to use their prior knowledge regarding the exponential expressions and encouraged higher order thinking.

In the revision lesson, Ozden enabled the students to understand effectively underlying ideas regarding the relation between the exponential and radical expressions. Ozden used the students' knowledge related to the exponential expressions and the solution of the equations including them. She explained how to solve the equation of $27^{x} = 3$ as follows:

Ozden:	$27^{x} = 3$ Which exponent of 27 is equal to 3? If you try to solve this by using knowledge of the exponential expressions, how can you interpret it? In such equality, if the bases are equal, the exponentials must be equal, too. So, If I write 3^3=27, this equality is so $(3^3)^{x} = 3$. $3^{3x} = 3^1$ As the bases are equal, the exponents will be equal too and we can write 3x= 1 and then, x $= \frac{1}{3}$. Right?
Students:	Yes.
Ozden:	That is, I can say this equality: $27^{1/3} = 3$ Right?
Students:	Yes.
Ozden:	Well, you know that what $\sqrt[3]{27}$ is.
Student 1:	The same thing.
Ozden:	So, what is this equal to?
Students:	It will be 3.
Ozden:	Yes, $\sqrt[3]{27}$ is equal to 3. Then, I can write $\sqrt[3]{27} = 3$. So, $27^{1/3} = \sqrt[3]{27}$.

Ozden handled the negative and the positive roots and she discussed what the real numbers were whose squares were equal to 64.

Ozden:	What are the real numbers whose squares are equal to 64?
Students:	8.
Ozden:	What else or only 8?
Students:	Also, the square of the -8 is equal to 64.
	ooo
Ozden:	Ok, if we use the knowledge regarding the exponential number, we can write: $x^2 = 64$ $x^2 = 8^2$ and $x = 8$ Also, we can think $x^2 = (-8)^2$ and $x = -8$. These numbers are square roots of 64. 8 is its positive root and -8 is negative root.

One of the information about the historical development of radical expressions which used to increase the students' motivations was effective. Ozden tried to give the idea of the importance of using notation.

Ozden:	We used the notations such as $\sqrt[3]{\ }$, $\sqrt{\ }$, $\sqrt[4]{\ }$, etc. while writing the radical expressions. Many years ago, mathematicians used a different notation. For example, to express the cube root, they wrote v$\sqrt{\ }$ or to write $\sqrt[4]{\ }$, they used

	vv$\sqrt{\ }$. That is, they increased the drawing number of the root symbol.
Students:	Wowww.
Ozden:	But they noticed the difficulty of this notation when the degree of the root was increased.

(e) Reflection on and improvement of the revision lesson. The teachers decided that the revision lesson was more realistic than the research lesson. They decided to use only one example which attracted the students' attention. They agreed on which teaching of the relation between the exponential and the radical expressions was reasonable. They discussed that the teacher clearly defined concepts by using the mathematical language effectively. Ali, in his reflective diary, stated the contributions and problematic parts of the first lesson study cycle as follows:

> As the examples were determined in the planning, examples, the examples were suitable for the subject during the lessons. The first plan was prepared well but it was not suitable in terms of the lesson period. In the revision lesson, the content was reduced, some examples which were spontaneously used in the research lesson were also handled in the revision lesson. Reducing the lesson content led revision lesson to be more understandable and more effective. The almost same questions were used to reflect the knowledge of student thinking in the research lesson and revision lesson and information about the historical development of root expressions were given in the revision lesson. One of these examples distracted the students' concentration and, on the other hand, another one attracted the students because they thought it interesting.

After the first cycle, the teachers realized that planning lesson by considering the students' thinking was important to overcome the students' difficulties. They noticed the importance of focusing on the students rather than only the content. However, they continued to their routines. With this cycle, the teachers gained awareness about which sharing experiences and thoughts would be effective. Observing the teacher and the students provided the teachers with opportunity to have critical perspectives to their own approaches. Ozden stated in her reflective diary that having the chance to observe the lessons through the eyes of teacher and student supported us to have different viewpoints.

> While observing the lessons, it was interesting to observe through both eyes of the teacher and the students and to try to understand in terms of both the teacher and the students. It enabled us to understand our own appropriate aspects as well as problematic aspects such as coping with students' difficulties, choosing examples, teaching methods.

As the plan was not completed in the revision lesson, the teachers decided to consider again the appropriateness of the content and the lesson period. Especially, they emphasized that the situations where the students could not understand or could have difficulties affected the flow of the lesson. They had the idea that considering student thinking was important by observing and discussing the lessons.

2. Findings on the Second Lesson Study Cycle

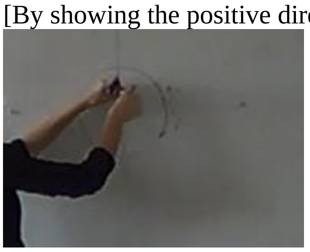
(a) Research and planning. The teachers examined the learning objectives about trigonometry in the elementary mathematics curriculum. The teachers examined the elementary mathematics curriculum to understand the students' prior knowledge and to support for them to relate their existing knowledge with the new knowledge because the students engaged in learning the trigonometry at Grade 8. Also, they made research about teaching trigonometry such as historical development, students' misconceptions, GeoGebra documents and videos in real life.

The teachers met two times to design the lesson plan. In the first meeting, they examined the curriculum and discussed the content and its order. They decided to question the students' prior knowledge such as trigonometric ratios in a right-angled triangle, trigonometric ratios in triangles with special angles during the research lesson. Then, they discussed whether the unit circle, angle and oriented angle would be included in the lesson plan and agreed on which beginning the lesson by teaching angle would be appropriate although the order of content was different in the curriculum. In the second meeting of the planning, they benefitted from textbooks as well as curriculum and often talked about students' prior knowledge and planning the activities.

(b) Research lesson. Ali began the lesson by asking questions for determining students' prior knowledge about ratio, trigonometric ratio, and trigonometric ratios in a right-angled triangle. As the content of the lesson was quite intensive, he did not implement the entire lesson plan. After teaching the unit circle and its equation, the lesson period was over. Thus, he did not teach to calculate the value of coterminal angle to a given trigonometric angle. The materials about the oriented angles discussed in the planning were not presented to students. Instead, Ali spontaneously related the plug of the board marker to oriented angles and enabled the students to understand the oriented angles concretely.

Ali: [A student came to the board. After she rotated the plug of board marker wrongly]
May you rotate the board marker? [the student is rotating.]
Which direction do you rotate?

Student: [By showing the positive direction.]



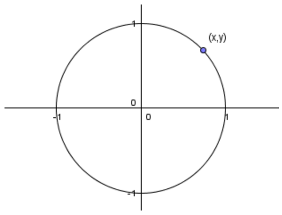
Ali: Ok, positive direction. To open the plug, you rotate it in a positive direction? Or negative direction?

Student: While closing, I rotate it in negative direction.

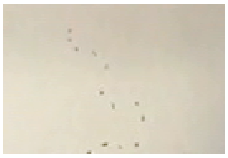
As some of the concepts in the lesson included what the students learned at 8th grade and 9th grade, he easily continued the lesson. He could generally predict the reasons of the students' difficulties, and enabled them to cope with these difficulties either by asking questions or by giving clues.

(c) Reflection on and improvement of the research lesson. After implementing the lesson, they reduced the content of the plan. They decided that the lesson should be finished after teaching the concept of the unit circle and its equation. They stated that the trigonometric identities given in the research lesson would be handled one after another instead of giving them in the separate parts of the lesson. As the research lesson had an intense content, the teachers noticed that the students could not especially understand the concept of oriented angle and unit circle. The teacher asked the students how they could construct the equation of a unit circle. Although he asked the students what the equation of a unit circle was, he explained it without waiting for them to state their ideas. They decided that they would make a change in the revision lesson and that it was necessary for students to construct of the equation of a unit circle by reasoning.

Ali: Who can find the equation of the unit circle?
Supposing that I determined a point of (x,y) on the graph of a unit circle. Remember that it is a unit circle and its radius

	is 1.  ooo
Student 1:	[Student interrupted him.] I know the equation, but I do not know how it will be found.
Student 2:	I know, too.
Ali:	What do you know?
Student 1:	$x^2 + y^2 = 1$
Ali:	$x^2 + y^2 = 1$ How did you acquire it? [The student came to the board.]
The student on the board:	I think that I cannot tell.
Ali:	I give you a clue. You can make use of the Pythagorean theorem.
The student on the board:	Will I draw a triangle on the graph?
Ali:	[The student drew the triangle.] No matter which point you select, you can find the equation of $x^2 + y^2 = 1$, because the radius is 1.

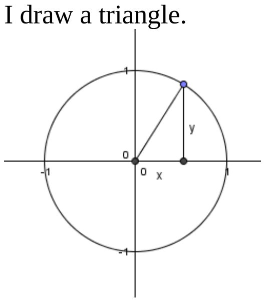
(d) Revision lesson. Ali taught the trigonometric identities after he introduced the trigonometric ratios in a right-angled triangle. He considered the students' ideas and responses and encouraged them to think more. He presented this approach while the students were expressing the definition of a circle.

Ali:	What is a circle? Can you tell me the definition of circle?
Student 1:	It has not an edge.
Ali:	Ok, who else wants to define?
Student 2:	It is a hollow closed circular region.
Ali:	Well, what else?
Student 3:	It is locus.
Ali:	That's great, it is absolutely locus. Ok, how is this locus?
Students:	Circular.
Student 1:	Infinite.
Ali:	The respond of circular is not sufficient.
Student 4:	Non-linear.
Student 2:	Yes, non-linear.
Ali:	Do I draw a non-linear locus for you? 
Student 3:	It is drawn around a certain diameter.
Ali:	I think, you can find the definition.
Student 5:	It is drawn by assembling the points whose distances from a point were equal.
Ali:	What are their characteristics?
Student 6:	Infinite.
Ali:	Which characteristic do they have?

Student 5: They have equal lengths.
Student 2: They are located from the center equidistantly.
Ali: Well then, we can define the circle that it is locus of points whose distance from the fixed point were equal.

The lesson continued in accordance with the revised plan. Ali prompted the students to write the equation of the unit circle algebraically. The excerpt indicating the discussion in this process was given below.

Ali: Who will find the equation of a unit circle? Its equation was expressed as $x^2 + y^2 = 1$.
[A student came to the board.]
The student on the board: I determined a point on the circle.
Ali: That's great.
The student on the board: The point is A(x,y)
I draw a triangle.



If I calculate it by using Pythagorean relation, it will be that equation is "$x^2 + y^2 = 1$".
Ali: Thanks. This means that the unit circle whose equation is $x^2 + y^2 = 1$ is the locus.

Ali encouraged the students to think by asking them to find the points whose coordinates were whole numbers on a unit circle, which was not in the plan.

Ali: Are there points whose coordinates are whole number on the unit circle?
Student 1: Why not, certainly there is.
Ali: Which ones?
Student 1: (1,0), (-1, 0)
Ali: Where is (1,0) located on?
Student 1: I show.
[She showed on the unit circle drawn]
Ali: There is not any point except (1,0), (0,1), (0,-1), (-1,0) Right? Because, the integer solution of the equation of $x^2 + y^2 = 1$ are only (0,1), (1,0), (0,-1), (-1,0).

In the revision lesson, different from the research lesson, the oriented angle and the unit circle were better understood by the students. Ozden expressed her thoughts in her reflective diary as can be seen in the following excerpts.

> I think, the revision lesson was more fluent and meaningful. The example about the integer solutions of a unit circle given spontaneously was important because the circle would be used in the context of the trigonometry. Also, the question emphasizing the geometrical definition of the circle encouraged the students to discuss. The content related to the trigonometric ratios in a right-angled triangle ratio was already good, so we did not do considerable change.

(e) Reflection on and improvement of the revision lesson. Teachers concluded that the revision lesson was suitable for the plan. In the second cycle, teachers reflected their KoST more in terms of the different components such as questioning the students' thinking, giving them a chance of thinking, presenting approaches for students to be motivated, and leading them to associate the mathematical concepts with real life. Although the teachers considered the students' thinking, they continued to make several directions to the responses which they expected. In other words, teachers frequently continued their routines of asking funneling questions as well as asking the questions focusing on student thinking.

3. Findings on the Third Lesson Study Cycle

(a) Research and planning. In this process, the teachers decided to ask the questions of finding the coordinates of the ending point of an angle on the unit circle. They discussed how the coterminal angles could be taught. They stated that the students would make only calculation errors in some questions, on the other hand, they expressed that the students could have difficulties especially in two questions. The teachers tried to respond to the questions in the lesson plan guideline which provided them with a focus on the students' thinking.

(b) Research lesson. Ali started his lesson with the repetition of the oriented angle and the unit circle. He asked questions about how to find the coterminal angles with degree. The students did not have any problem about degrees but they had difficulties about the coterminal angles with radian. One of the students thought that she would divide the angle by 2π to find the coterminal angle but later on she had difficulty because she could not make sense of it. Ali guided his student to make the operational steps and then he explained the way of finding the coterminal angle with radian.

Ali:	Find the coterminal angles with $\frac{21\pi}{2}$.
Student 1:	Will $\frac{21\pi}{2}$ be divided by 2π?
Ali:	Great! $\frac{21\pi}{2}$ will be divided by perimeter of the circle.
Student 1:	$\frac{21\pi}{2}.\frac{1}{2\pi}$
Student 2:	Is 2π equal to $360°$?
Student 1:	Yes. Then, it will be $\frac{21}{4}$. Now, what should I do?
Ali:	Divide 21 by 4.
Student 1:	Umm..
Student 3:	Now, we converted radian to degree, didn't we?
Ali:	What else?
Student 1:	I divided.
Ali:	I am explaining what your friend did. This angle is greater than 2π. We know that the diameter of a circle is 2π. In the way that we divide the angle by $360°$, we divide it by 2π.

(c) Reflection on and improvement of the research lesson. The teachers discussed an alternative way of teaching the coterminal angle with radian by considering the students' difficulties.

Ozden:	I think, the students could find the coterminal angles with degree but that they were not successful in finding those with radian.
Ali:	As the students had difficulties and I needed more time to overcome these difficulties, so, the lesson plan was not implemented completely.
Researcher:	What do you think to do about this issue?

Ali: We should handle the subject again and firstly we should think how we will do it.
Ozden: Yes, otherwise, they will not learn.

They determined that they would start the lesson with directly the coterminal angles different from the research lesson. Therefore, they did not reduce the content of the plan because they thought that there was enough time for all questions.

(d) Revision lesson. Ali continuously tried to understand the students' reasoning. He asked the students to explain how they solved the question and what the underlying reasons for their thoughts were. He frequently asked the students the questions such as "Why did you do that?", "Why did you think like that?" The reflection of this approach in the lesson was seen in the excerpt below:

Ali: Find the coterminal angle of -1210°.
The student on the board:

$$\begin{array}{r|l} 1210 & \underline{360} \\ -\underline{1080} & 3 \\ 130 & \end{array}$$

360-130=230

Ali: That's great. Explain to us why did you do such as that, please. Why did you subtract?
The student on the board: Because, the angle is negative.
Ali: How could you show that by making use of the unit circle?

In this lesson, teaching the coterminal angles with radians in a different way made it easy for students to understand. The reflection of this case was as follows:

Ali: What is the coterminal angle of 4π?
Students: Zero.
Ali: What is the coterminal angle of 7π?
Student 1: π
Student 2: If we divide it by 2π, it remains π.
Board: [Ali wrote the student's expression.]

$$\begin{array}{r|l} 7\pi & \underline{2\pi} \\ -\underline{6\pi} & 3 \\ \pi & \end{array}$$

ooo

Ali: [After he asked the coterminal angle with radian like $k\pi$, k an integer.] What is the coterminal angle of $\frac{25\pi}{3}$?
Student 3: We can find it by dividing 2π.
Student 4: It is multiplied with $\frac{12}{3}$ and there remains $\frac{1}{3}$.
Ali: Come to the board and explain it.
Student 4:

$$\begin{array}{r|l} \frac{25\pi}{3} & \underline{2\pi} \\ -\underline{\frac{24\pi}{3}} & \frac{12}{3} \\ \frac{\pi}{3} & \end{array}$$

Ali: That's great! Ok, what else can you find the multiples of 2π?
Student 5: We can write it as integer and improper fraction.
Ali: Ok, we will divide it by 2.
Student 5: $\frac{24\pi}{3} = 8\pi$ is integer part and $\frac{\pi}{3}$ is fraction part.

Ooo

Ali: We will subtract the multiples of 2π from it. That is, we will make out the acute angle if we substract the multiples of 2π from $\frac{25\pi}{3}$.

ooo

How many 2π can we write?

$\frac{25\pi}{3} = 4.2\pi + \frac{\pi}{3}$

(e) Reflection on and improvement of the revision lesson. They stated that teaching of coterminal angles with radians was more appropriate and that the students could more easily relate the operations in finding the coterminal angles with degree to those of radian. The teachers concluded that the students had more difficulties while solving the questions required the geometrical knowledge. They used their knowledge about trigonometry; however, they did not integrate their geometrical knowledge to the solution. In the third cycle, they performed practices to reveal students' thinking. They considered students' difficulties and errors and tried to overcome them. They started to change their teaching methods by sharing opinions and discussing the teaching episodes. They noticed the importance and influence of considering not only their own ideas but also their students' ideas.

Then, they talked about the all lessons in terms of KoST's indicators and generally discussed which actions relation to KoST were not performed in the lessons. When they came to the end of this process, they noticed the importance of students' thinking for learning conceptually. They started to consider students' thinking and realized their teaching process by focusing on students' thinking. They revised and enriched their own teaching approaches by interacting with each other. As they made the conceptual analysis, they took the opportunities to have deeper content knowledge. They began to think about the teaching episodes regarding the students' cognitive process which they did not think of before and decided on teaching approaches specific to the content.

Discussion and Conclusion

The teachers engaged in a quite productive process by the lesson study designed for improving mathematics teachers' knowledge of student thinking. The teachers got involved in a process which was different from their previous routines, and they gained experiences which would support their professional development by evaluating each step of the lesson study cycles.

During the planning stages of the lesson study cycles, the teachers conducted discussions about the concepts with their colleagues and the researchers. During these discussions, they initially made conceptual analysis to teach the concepts effectively and focused on the underpinnings of the concepts and ways to teach these underpinnings. Thus, teachers had the opportunities to improve their content knowledge both by questioning their own thoughts and by considering other teachers' ideas about the concepts in the planning stages. In the first cycle's planning stage, the teachers directly focused on the concepts. However, they did not consider how students thought about the concepts and how they would connect their teaching to take account of their students' conceptual learning. As the teachers had not previously shared their teaching practices with each other and they did not know what they would share, in the planning stage of the first cycle they focused on the concepts rather than students and their thinking. In the following cycles, they started to discuss what the students would think, how they would think, and how the concept could be taught better. They also started to relate mathematics to real life as well as relating mathematical concepts with each other. In this context, in the planning meetings, they discussed real life examples to be used in order to improve students' mathematical thinking and to motivate them. Although one of the teachers did not think the necessity of using real life examples before the lesson study, this teacher used real life examples for supporting students' learning during the lesson study. As the teachers prepared the lesson plans of the second and third cycles by considering the students' thinking, they made achievable plans for their teaching. Especially, they started to act more realistically while dealing with the difficulties and obstacles students could face. Although they had problems in terms of the lesson time and its content in the first cycle's lessons, they made more realistic plans in terms of the content, activities and time in the next cycles. When considering the three cycles, the planning stages were effective for both the improvement of the teachers' content knowledge and the development of their awareness on student thinking. Conducting the lessons on different topics at different grades supported teachers in terms of some positive aspects. Discussing and teaching different concepts instead of discussing one specific concept and making plans for teaching them supported the teachers' content knowledge. Also, they can decide what kind of strategies they should develop in different teaching processes after this study. The teachers made concept analysis for different concepts of different grades and shared their experiences by discussing students' possible difficulties, their prior knowledge and the relational concepts. In this process, their personal awareness of mathematics teaching and their content and pedagogical content knowledge were improved. Therefore, they can conduct effective mathematics teaching in their future courses. Meyer and Wilkerson (2011) said that teachers who discussed the content during the lesson study developed deeper understanding for mathematics teaching. Instead of only focusing on the concepts or the teaching of concepts, planning lessons by relating them is necessary to develop students' learning. Additionally, expecting teachers to discuss only the concepts and the content is not sufficient for their improvement. In this process, it may be important for the researchers to observe the teachers and to support them interactively. In our study, we encouraged the teachers to examine the concepts and their teaching by providing them with theoretical knowledge, and to consider students' thinking more. If we had not discussed the possible student thinking with them during the planning stages, we could not have expected them to change their routines and to realize the necessity of these changes. The lesson study gave a momentum for the teachers' improvement because of working collaboratively in the planning stage. Also, the lesson plan guideline supported the teachers to notice and to consider students' thinking by means of the questions included in the guideline. Teachers began to notice the importance of the questions and to consider them in the planning as the lesson study progressed. By handling these questions, they effectively planned the lessons in line with students' thinking. The prepared lesson plans facilitated the teachers' foci on the students and their thinking during the teaching. The implementation of the plans was shaped both through the teachers' experiences and their pedagogical content knowledge.

The content of the research lesson and the revision lesson in the first cycle were intense, so the teachers could not sufficiently examine students' thinking due to the anxiety of completing the whole lesson plans. Hence, we could say that the lesson plan directly affects teachers' approaches in the lessons. In the following lesson study cycles, the teachers who could prepare more appropriate lesson plans started to question students' thinking and reasoning in their lessons. They conducted lessons depending not only on the content but also on the students’ ideas. As the teachers habitually questioned the students, they had a chance to assess their own ideas and others’ solutions. The teachers tended to ask “Why did you think so…?”, “What would happen if it were….?”, “Is there anyone with different ideas?”, “How did you come to this result?” The discussion meetings held in the lesson study process were effective in asking these kinds of questions. As the teachers examined and discussed the concepts by considering students’ thinking, they were able to ask students to justify and explain the underlying reasoning of their ideas. They asked effective questions to students thanks to their content knowledge developed through the lesson study process. Additionally, they came to think of contingency moments because they discussed the possible situations in the context of students’ thinking. They also gave them enough time to encourage thinking mathematically and to enable the expression of different thoughts. The process supported teachers’ improvements in different ways by not only leading discussions during the meetings but also including lesson observations. Observing the students and the teacher at the same time and handling the interaction between them in the process of learning and teaching were a great opportunity for teachers. During these observations, the teachers realized that using mathematical language effectively and defining the concepts clearly were critical for the students to better understand. While it was hard for the teachers to observe the lessons from the students’ eyes before, getting involved in such a process enabled them to conduct these observations. It was important for teachers to actively engage in the process, especially by trying to understand student thinking while observing the lessons. They started to gain awareness about the ways in which they could exemplify abstract mathematical ideas for students and use different representations of the concepts that were important in the learning. Cheng and Yee (2012) emphasized when the teachers purposively listened to students’ ideas, they assessed student learning better and they could improve students’ mathematical knowledge. Similarly, Teyplo and Moss (2011) expressed that teachers’ content knowledge, student knowledge and pedagogical content knowledge developed when they participated in a lesson study. As these researchers stated, we found that the lesson study helped the mathematics teachers improve their mathematical content knowledge. Additionally, as students' thinking includes important components such as students’ mistakes, difficulties, obstacles, prior knowledge, and understanding of concepts, we propose that focusing on the students’ thinking can contribute to the participants’ professional development. Given the importance for teachers to have KoST and to perform their lessons in accordance with the content of KoST (An, Kulm &Wu, 2004; An & Wu, 2011; Carpenter, Fennema & Franke, 1996; Hill, Ball & Schilling, 2008; Norton, McCloskey & Hudson, 2011; Speer & Hald, 2008), it would be significant for teachers to be guided to examine their lessons in terms of KoST. The present study contributed to improvements of the teachers’ KoST by discussing and obtaining feedback from their colleagues on their teaching during the lesson study. In addition, the opportunities for teachers and researchers to meet, to plan, to interact and to reflect on events proved to be extremely supportive of improvements in teaching and learning. The teachers had a chance of assessing each other’s approaches in this whole process depending on cooperation. Perry and Lewis (2008) stated that teachers who participated in a lesson study focused on student thinking more because they used the reflection and feedback cycles many times. In our study, the reflection and improvement meetings held after the revision lessons and the last stage became quite important for the teachers to evaluate the changes and revisions they made after the research lessons. Along with the revisions related to the research lessons, they discussed the reasons for inappropriate approaches in the revision lessons and dealt with the new necessary revisions. At this later stage of the lesson study, teachers made more realistic evaluations. Therefore, reflections on the revision lessons during the lesson study supported the teachers' improvements in terms of preparing and implementing lesson plans.

The teachers who reflected their KoST on their teaching in a narrower context before the lesson study supported each other by working collaboratively, sharing their experiences and giving feedbacks about their teaching during the lesson study. Thus, they started to reflect their KoST on teaching more effectively and intensively. This finding is aligned with those of Fernandez, Llinares and Valls (2012) who noted that pre-service teachers having low levels of awareness about students' thinking changed their comments and developed new understandings when they interacted with those having higher levels of awareness.

We encountered some problems and difficulties throughout our study. It was not easy to include teachers in such a study at the beginning because we asked them to participate in a different process about which they had no information before. Yet, our positive dialogues with them increased their willingness to participate in the study. Especially, since the stages of planning and reflection are important in the lesson study model, we enabled the teachers to have an appropriate environment to explain their ideas. In future studies, it is important to consider this issue in terms of increasing the quality of the studies. Since it was a quite intense period, data collecting and analyzing were also demanding processes. We effectively worked in this process because of our previous experiences. Depending on this, we can say that working as a team instead of one researcher to conduct the long-term lesson study is a determinant factor for the efficiency of the study. This would be in keeping with the essence of the Japanese lesson study literature where there is reported direct involvement of university researchers and supervisors who join teachers in lesson study and contribute their knowledge and experience (Stephens & Isoda, 2007).

Future Directions for Research

The future studies involving lesson study can be carried out with mathematics teachers working at different schools. The different mathematical concepts at different grades may be also handled and the teaching episodes may be examined. Additionally, while teaching concepts, long-term lesson study processes can reveal some suggestions, instructional strategies and best practices. Pre-service teachers as well as in-service teachers should be involved in such studies, thus they will realize the importance of KoST and its content before becoming teachers. Pre-service teachers' KoST and their reflections in their school-based instructions can also be addressed.

Notice

This study is a part of Aytuğ ÖZALTUN ÇELİK's master thesis entitled "Professional development of mathematics teachers: Reflection of knowledge of student thinking on teaching". The thesis was supervised by Professor Dr. Esra BUKOVA GÜZEL depending on Dokuz Eylül University, Institute of Educational Sciences İZMİR.

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Appendix A. The lesson plan for teaching radical expressions developed in the first cycle of the lesson study

	Activity and Its Duration	The reason of selection the activity	Teacher's Action/Role	Student's Action/Role
1.Getting-started	Questioning (10') • The square of which positive number is the number in the root? • What is the value of $\sqrt{4}$ and $\sqrt{16}$? • What are the real numbers which make x^2 equal to -64? • What are the real numbers whose squares are equal to 64?	• To reveal the students' prior knowledge about square roots, to eliminate their incorrect knowledge and to complete their lacking knowledge, to provide for them to remind the concepts/topics/representations and to warm up for learning the radical expressions. • To discuss that the square root of negative numbers are defined in the different sets from the set of real numbers by considering mathematical language. • To provide for the students to comprehend negative root and positive root by usind the solution of the equation. • To prevend a possible misconception by providing for the students to notice the relation x values of $x^2 = 64$ with the square root.	He/she asks the determined questions, tries to reveal the students' thinking, expands their thinking by added questions if needed, tries to eliminate the students' incomplete knowledge and directs the lesson process by considering their prior knowledge.	He/she responds the questions, expalins the knowledge abput square root, listens his/her friends' explanations, shares his/her own thoughts and participates actively.
2.Working on it	Questioning (5') • What is the value of $\sqrt[3]{27}$? • What is the value of $\sqrt[3]{-125}$? • What is the value of $\sqrt[4]{256}$?	• To support for the students to think the roots with different degree by using and extending their existing knowledge related to the square root and to start thinking on radical expressions by means of these examples. • To present the examples, which provide to make the definition of the radical expressions concrete.	He/she tries to expand the students' existing understanding by supporting them to use their prior knowledge and has them think about the examples prepared for being able to understand the definition of the radical expression.	He/she reasons about the meaning of the radical expressions with different degrees based on his/her knowledge related to the square root, responds the questions actively and make connections among the concepts.

	• <u>Definition</u> (5') • When $n \in \mathbb{Z}^+$ and $n \geq 2$, • $\sqrt[n]{x^n} = \begin{cases} x \text{ if n is odd,} \\ \lvert x \rvert \text{ if n is even,} \end{cases}$	• To explain the definition which make generalize the radical expressions and to discuss the meaning of the concept by using mathematical language appropriately.	He/she supports the students to define the radical expression after examples about radical expressions with the different degree and leads for them to use mathematical language effectively.	He/she generalizes the radical expressions by relating the examples with each other, explains what expression the radical expression is equal to and uses the mathematical language appropriately.
	<u>Questioning</u> (20') What is the results of the following questions? • $\sqrt{(\sqrt{5}-1)^2} - \sqrt{\left(2-\sqrt{5}\right)^2}$ • $\sqrt[4]{(-16)^2} - \sqrt[5]{-3^5} + \sqrt[8]{(-16^4)^2}$ • If $x < y < 0$, what is the expression of $\sqrt{x^2} + \sqrt[3]{(x-y)^3} - \sqrt{y^2}$ equal to? • If $x < 0 < y$, what is the expression of $\sqrt[3]{(y-x)^3} - \sqrt{(x-y)^2} + \sqrt[4]{(-y)^4} - \sqrt[5]{(-x)^5}$ equal to? • If $a < 0 < b < c$, what is the expression of $\sqrt{(a-b)^2} + \sqrt{b^2 - 2bc + c^2} + \sqrt[3]{(a-c)^3}$ equal to?	• To do practices related to the definition of the radical expressions and to provide procedural fluency.	He/she gives enough time to the students for responding the questions, provides to eliminate possible difficulties by creating classroom discussions about the students' problematic solutions.	He/she tries to respond the questions by thinking the meaning of the definition and reminding the knowledge about absolute value, listens his/her friends' solutions and participates on the discussion actively.
2.Working on it	<u>Motivation</u> (3') To inform the historical development about the sign of square root. $vvv\sqrt{\ }$	• To increase the students' motivation to the lesson and the concept and to support their mathematical thinking by providing to enhance the meaning of the concept.	He/she gives historical information about the sign of the radical expression and so provides fort he students to make connections between the sign and the meaning of the radical expression.	He/she relates the meaning of radical expression with the information presented by the teacher.
	<u>Relation the radical expression with exponential expression</u> (7') $a^{n/m} = \sqrt[m]{a^n}$ Using the solution of the equation of $27^x = 3$	• To provide for the students to make sense of the relation between the exponential expressions and the radical expressions. • To provide fort he students to derive the equation of $a^{n/m} = \sqrt[m]{a^n}$ by comparing the solution of $27^x = 3$ with the expression of $\sqrt[3]{27}$.	He/she supports the students to conceive the relations between the exponential expressions and the radical expressions by using solution the equation including exponential expression.	He/she solves the equation including exponential expressions by using his/her prior knowledge and relates the exponential expressions with the radical expressions by means of this equation.

	Questioning (20') • If $\sqrt{3^{3x-3}} = \sqrt[3]{3^{4x+2}}$, what is x equal to? • If $\sqrt[5]{(0{,}00243)^x} = 0{,}09$, what is x equal to? • If $\sqrt[5]{32^{x-1}} = 8$, what is $\sqrt[x]{16}$ equal to?	• To support thier procedural understanding by doing practices about relation between radical expressions and exponential expressions.	He/she waits for the students to think and respond the questions, considers the erroneous solutions by listening their ideas and encourages all class to discuss the responses.	He/she uses his/her prior knowledge regarding solving equation with the exponential expressions to make the solution, explains the solution by justifications and compares the solution with his/her friends' solutions.
3.Closure	Questioning (10') How can you write the expression of $\sqrt[12]{2^{13}}$ differently?	• To reveal whether the students learned the meaning of the radical expressions, and to evaluate whether they will use the ideas discussed in the lesson.	He/she encourages the students to think by asking questions and tires to reveal different thinking, considers the students' expressions and evaluates their ideas.	He/she thinks how to write the expression differently by using the knowledge learned and explains his/her thoughts with their reasons.