Unpacking Conceptual and Procedural Meanings for Radian Angle Measure Through a Curricular Sequence: Henry's Case

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Researchers report that preservice mathematics teachers (PMTs) conceptualize radian angle measure in terms of π using memorization and calculational strategies. In this report, I unpack the mathematical meanings of one PMT, Henry (pseudonym), as he engaged in a series of three task-based interviews involving radian angle measure. Through engagement and reflection on these tasks, I describe shifts in Henry's understanding of radian by characterizing the amalgamation of procedural and conceptual meanings. Henry's initial meanings of radian involved procedural strategies, with an emphasis on memorized special angles expressed in terms of π. In response, the subsequent task-based interviews aimed to gain more insight into Henry's meanings and to challenge his dependence on procedural strategies. An analysis of Henry's strategy in the second and third interviews suggests shifts in his meanings of radian angle measure, including conceptual generalizations. I conclude by highlighting how tasks can be designed to build and support conceptual meanings.

Introduction

A coherent understanding of radian angle measure contributes to a coherent understanding of higher mathematics such as trigonometry, precalculus, and calculus (Moore, 2014; Paoletti, 2020; Thompson, 2008; Thompson et al., 2007). However, there are few opportunities for students to engage with radian angle measure in the curriculum (Mesa & Goldstein, 2017), and researchers have reported that learners struggle to conceptualize radian angle measure and how it relates to trigonometry (e.g., Moore, 2013, 2014; Moore et al., 2016; Tallman & Frank, 2020; Thompson, 2013; Topcu et al., 2006). Specifically, researchers demonstrated learners' dependence on procedures and calculational strategies to reason about the radian measure of an angle (Akkoc, 2008; Çekmez, 2020; Fi, 2003; Moore et al., 2016; Topcu et al., 2006). In this study, I aim to better understand individuals' meanings of the radian angle measure by characterizing one preservice mathematics teacher's (PMT's) thinking during engagement with a series of curricular tasks involving radians. I extend previous work in this area by describing the amalgamation of the PMT's procedural and conceptual meanings for the radian angle measure and by characterizing shifts in the PMT's meanings from procedural to conceptual generalizations that result from engaging with tasks that challenge initial procedural meanings. I first provide relevant background knowledge by expanding on Moore's (2013) conceptual analysis (Thompson, 2008) of the concept of angle measure and radian angle measure. I also describe the construct of mathematical meanings (Thompson et al., 2014) to clarify how I distinguish between procedural and conceptual meanings. Then I illustrate the PMT's meanings for the radian angle measure upon engaging with the first task, followed by a discussion of shifts in its meanings across subsequent tasks. I specifically argue that if learners (including teachers) are expected to have conceptual meanings for radian angle measure, then they must have experiences that support the development of those meanings.

Research on Angle Measure Meanings

Conventionally, angles are measured relative to a benchmark associated with a circle centered at the angle's vertex. Quantifying angle measure involves identifying a measurable attribute of angle where "the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, multi-linear) with its unit" (Thompson, 2011, p. 37). Following Thompson's (2008) conceptual analysis of angle measure, Moore (2013) described the arc approach to angle measure as involving the angle's subtended arc as a measurable attribute, measured relative to a specific, yet arbitrary, fractional amount of the circle's circumference. Most commonly, in degrees, the subtended arc is measured relative to $1/360^{\text{th}}$ of the circumference, while in radians, the subtended arc is measured relative to $1/(2\pi)^{\text{th}}$ of the circumference. Moore's (2013) teaching experiment provided precalculus students the opportunity to use ideas of proportionality to conceptualize angle measure with this arc approach as a fractional amount of the circumference. Additionally, researchers demonstrated that the arc approach to angle measure is beneficial for students' understanding of trigonometric (Moore, 2014) and inverse trigonometric functions (Paoletti, 2020). However, in-service teachers questioned the practicality of using ideas of proportionality and the arc approach to measure angles for understanding precalculus and calculus (Thompson et al., 2007).

Another approach to angle measure involves partitioning a familiar angle into equiangular parts (Lobato et al., 2010). Ninth-graders in Hardison's (2020) teaching experiment utilized partitioning to describe the measure of a 1° angle by equipartitioning a right angle into three 30° angles, then each 30° angle into three 10° angles, then one of those into ten 1° angles. The ninth-graders used ideas of proportionality (i.e., partitioning). They quantitatively attended to the space between the rays as a measurable attribute of an angle, measuring it by breaking an established benchmark into equiangular parts to describe a 1° angle.

These studies (i.e., Hardison, 2020; Moore, 2013) investigated meanings for angle measure as part of teaching experiments. In this study, I extend the previous work by examining and reporting on one PMT's meanings as he reflected on his own actions (Simon et al., 2004) during engagement with a series of curricular tasks involving radian angle measure. I also attend to the amalgamation of meanings, which I describe in the following section.

Mathematical Meanings

I take a constructivist perspective and build on the work of Thompson et al. (2014), who understand mathematical meaning as the assimilation of schemes associated with an understanding, including "the space of implications that the current understanding mobilizes" (Thompson et al., 2014, p. 13). This description of mathematical meanings builds on the duality between thinking and understanding, which are rooted in mental actions demonstrated through common cognitive characteristics in a given mathematical situation (Harel, 2008). Using Harel's (2008) and Thompson et al.'s (2014) descriptions, I interpret mathematical meaning as the understanding that enables a learner to reason about a particular concept (e.g., radian angle measure) in a given situation (e.g., a mathematical task) that evokes such reasoning. For example, PMTs described radian angle measure with an emphasis on special angles written in terms of π (Akkoc, 2008; Fi, 2003; Topcu et al., 2006). Additionally, Akkoc (2008), Fi (2003), and Moore et al. (2016) reported that PMTs incorporated procedural calculations with radian angle measure using the unit circle (Figure 1), which is a diagram of a circle centered at the origin of a Cartesian plane with a radius of 1, typically labeled with special angles in radians measured from the positive x-axis and written as integer multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ radian. Another version of the unit circle shows the special angles in radians along with their equivalent degree measures.

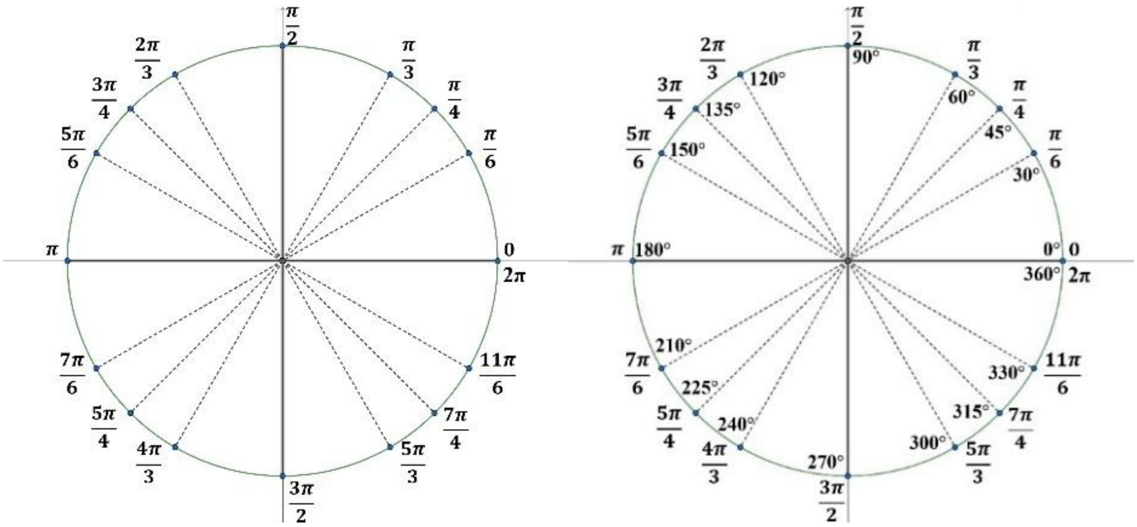


Figure 1. Typical Representations of the Unit Circle with Special Angles

Earlier findings (Akkoc, 2008; Çekmez, 2020; Fi, 2003; Moore et al., 2016; Topcu et al., 2006) suggest that PMTs' previous experiences could have resulted in mathematical meanings that incorporate procedures (e.g., conversion, calculations with the unit circle, memorizing special radian angles in terms of π) to reason about angle measure. However, the National Council of Teachers of Mathematics (NCTM) calls for learners to move beyond the procedural use of memorized facts toward flexibly applying mathematical problem-solving grounded in conceptual understanding (NCTM, 2014b). Additionally, NCTM identified building procedural fluency from conceptual understanding as an effective practice for learning and teaching mathematics (NCTM, 2014a). Since applying procedures is expected to build on a foundation of conceptual understanding, attention to the amalgamation of procedural and conceptual meanings is needed.

By procedural meanings, I mean the understanding that involves algorithms, memorized facts, and calculational strategies (Star, 2005, 2007) to enable a learner to reason about a particular concept. For example, PMTs used an algorithm to convert between degrees and radians, but struggled to conceptualize the results of their calculations (Akkoc, 2008; Çekmez, 2020; Fi, 2003; Topcu et al., 2006). Additionally, PMTs' meanings of angle measure incorporated a unit circle diagram for calculations (Akkoc, 2008; Fi, 2003; Moore et al., 2016). These findings suggest procedural meanings, in which the PMTs, through their previous coursework and experiences, have developed understandings of the radian angle measure that involve using algorithms, memorized facts, and calculational strategies.

By conceptual meanings, I mean the understanding that enables a learner to reason about a particular concept through explicit, generalizable connections (Star, 2005, 2007). For example, precalculus students in Moore's (2013) teaching experiment reasoned about angle measure using various multiplicative comparisons (i.e., between arc length and circumference and/or arc length and radii). Their description of angle measure as a "fractional amount of *any* circle's circumference that is subtended by the angle," was generalized by describing radian angle measure as "the subtended arc [that] is so many times as large as the corresponding circle's radius." (Moore, 2013, p. 242). These descriptions reflect various mental connections, suggesting conceptual meanings for angle measure.

The previous definitions of conceptual and procedural meanings are not meant to be understood as dichotomous. In fact, researchers (e.g., Maciejewski & Star, 2019; Nilsson, 2020; Nordlander, 2021; Rittle-Johnson et al., 2015; Star, 2005, 2007) have indicated the benefits of exploring how procedural and conceptual meanings relate to one another. Additionally, building procedural fluency from conceptual understanding is considered an effective practice for teaching and learning mathematics (NCTM, 2014a). The distinction in this study aims to extend existing research by reporting how PMTs' procedural meanings (Akkoc, 2008; Çekmez, 2020; Fi, 2003; Moore et al., 2016; Topcu et al., 2006) relate to conceptual meanings. Based on the literature discussed above, I anticipate that one PMT's meanings for radian will involve special angles written in terms of π (Akkoc, 2008; Alyami, 2024), with attention to proportionality (Alyami, 2022a; Moore, 2013), including partitioning (Hardison, 2020). Specifically, the research question guiding this study is "What is the relation between a PMT's procedural and conceptual knowledge for radian angle measure?" By answering this research question, I aim to extend existing research by reporting that a PMT's procedural meanings for radian relate to conceptual meanings.

Methods

Participant

The participant in this report was Henry (pseudonym), a PMT enrolled in a mathematics teacher preparation program at a large Midwestern U.S. university. Henry volunteered to participate in three separate task-based interviews and was compensated for his time. I did not offer any learning sessions on radian prior to this research; however, Henry likely learned about the radian measure during his K-12 and university education. Specifically, at the time of participation in the first task, Henry had completed two courses from the undergraduate calculus sequence. Additionally, by participating in a series of task-based interviews, Henry was expected to demonstrate initial meanings upon entering the study, as well as shifts in his meanings because of active reflection on his own actions (Simon et al., 2004).

Data Collection and Analysis

Henry participated in three think-aloud, task-based interviews involving radian angle measure. Think-aloud tasks are used to unpack participants' knowledge, in which participants use their own language to make sense of the task (van Someren et al., 1994). The semi-structured interview provided an opportunity to request elaboration (e.g., "How do you know that?" or "Would you draw a picture to represent what you are thinking?"). While the initial task aimed to model Henry's initial meanings for radian upon entering the study, subsequent tasks were adapted and/or selected to gain further insights into those meanings and to model shifts in them.

Each interview followed the guidelines of structured, task-based interviews (Clement, 2000; Goldin, 2000), lasted approximately one hour, and was audio- and video-recorded. Each interview was transcribed, and the transcripts comprise the data for this report. To unpack Henry's meanings of radian angle measure, I attend to the strategies he used when addressing each task. By '*strategies*,' I mean utterances and observable actions (writing, gestures, etc.) used to address various tasks, which were analyzed using conceptual analysis (Moore, 2013; Thompson, 2008). By characterizing Henry's strategies, I describe a collection of reasoning actions that were brought to bear through engagement with the tasks, representing the space of implication that resulted from assimilating to a scheme (i.e., Henry's meanings for angle measure). I used thematic analysis (Saldaña, 2013) to categorize Henry's mathematical meanings with attention to conceptual and procedural strategies.

To answer the research question, my analysis focuses on unpacking Henry's strategies throughout the three interviews with attention to conceptual and procedural meanings "so that more can be gleaned from the data than would be available from merely reading, viewing, or listening carefully to the data multiple times" (Simon, 2019, p. 112).

Findings

I organize this section chronologically to illustrate the development in Henry's understanding of the radian angle measure during the interviews. I then summarize the findings to explicitly characterize the shifts in his meanings for radians.

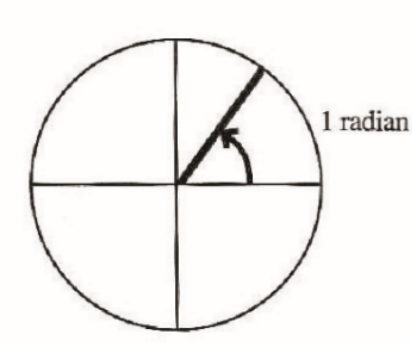
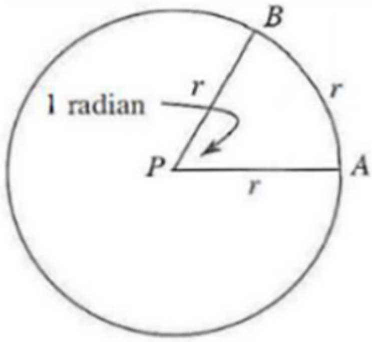
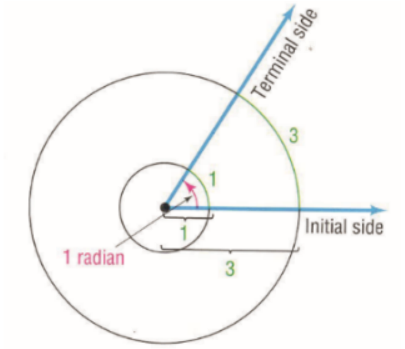
Interview 1: Describing Radian Given Various Diagrams

In the first task-based interview, I asked Henry to describe radian angle measure by examining different diagrams of radian (Alyami, 2022a, 2024). The diagrams were handed to Henry one at a time, in the order listed in Table 1, and the question was, "How would you describe radian angle measure using this diagram?"

The diagrams were evenly distributed based on the extent to which a quantitative operation was depicted. For example, Diagram 4 is categorized as quantitative because it depicts the radius's iteration along the circumference of the circle. Diagram 6 was considered non-quantitative because, although it presents multiple radian angle measurements along the circumference of the circle, it lacks a quantitative procedure for generating such measurements. Additionally, I placed this diagram last because it is common in U.S. high school trigonometry, which Henry had likely encountered before participating in this study.

While Diagram 1 does not depict radian angle measure in terms of π, Henry's description emphasized radian angle measure in terms of π. He stated, "The unit of 1 radian doesn't really mean a whole lot … since [radian] is always measured in π … radians as I know it is not something that's meant to be measured in integer units, it's meant to be measured in fractions of π." Henry continued referring to π when describing radian through Diagrams 1 to 5 (none of which depicted radian angle measure in terms of π). For example, when describing radian angle measure through examining Diagram 2, Henry indicated that "In the relationship between a circle's radius and its arc length, while it's described well by this … my brain keeps wanting this [points to the terminal side of the angle] to be just like a little bit larger into … $\frac{\pi}{3}$ radians instead of 1 radian." Additionally, Henry's emphasis on special angles in radians suggests that he developed thought patterns for reasoning about radian angle measure involving angles measured in terms of π.

Table 1
Radian Angle Measure Representations Used for the First Task-based Interview

<table>
<tr>
<td>Diagram 1 (Non-Quantitative)

(Kysh et al., 2009, p. 47)</td>
<td>Diagram 2 (Quantitative—Equivalence)

(Kysh et al., 2009, p. 46)</td>
<td>Diagram 3 (Quantitative—Equivalence)

(Sullivan & Sullivan, 2009, p. 355)</td>
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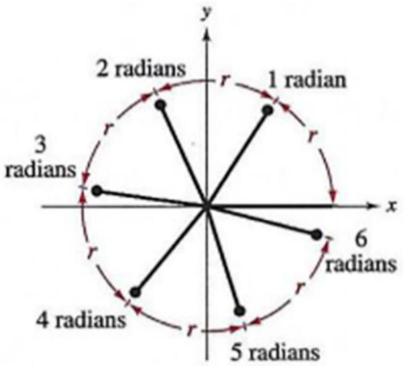
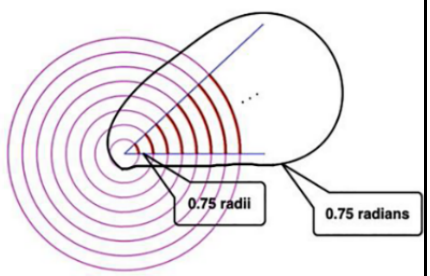
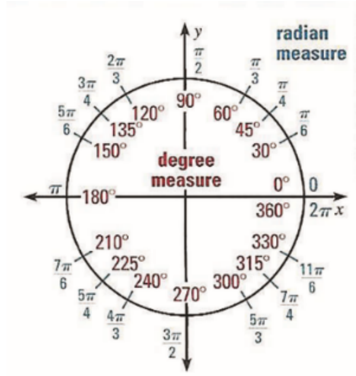
<tr>
<td>Diagram 4 (Quantitative—Iteration/Partitioning)

(Larson et al., 2008, p. 259)</td>
<td>Diagram 5 (Quantitative—Proportionality)

(P. W. Thompson, personal communication, 23 February 2018)</td>
<td>Diagram 6 (Non-Quantitative)

(Larson et al., 2010, p. 861)</td>
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</table>

While Henry emphasized radian angle measure in terms of π, he identified the arc approach to radian angle measure when given Diagrams 2-5. When looking at Diagram 2, Henry indicated that "there is the relationship between the lengths of the sides [points to the radius] and the arc length." When looking at Diagram 3, he elaborated that "the three-to-one ratio of the radii is the same as the three-to-one ratio [of] the arc lengths, and for me I'm just looking at this as a special case of that property in which the arc length just happens to be the same as the radius," which highlights his attention to the special case of equivalence. He used the same approach when describing radian through Diagram 4—"each of these arc lengths is the same length as the radius"—and Diagram 5—"each arc length is 0.75 of, the length of each arc is 0.75 times the radius." The previous statements suggest Henry's awareness of the multiplicative relationship between the arc length and the radius when measuring angles in radians.

Despite recognizing the proportionality involved when measuring angles in radians, Henry still preferred to describe radian angle measure in terms of π and questioned the practicality of the arc approach to angle measure. This is evident in his description of radian when looking at Diagram 6:

> This is the most satisfying representation of radian, when you multiply it by π, and you get easy fractions of a circle … you just have to immediately go to halfway around the circle is π … you kind

of just have to immediately identify π with the fraction of the circle and not spend time thinking about it's actually the physical length of the arc.

This excerpt demonstrates Henry's preference for the unit circle representation, along with special angles written in terms of π. While Henry's description of radian as "fractions of a circle" could be interpreted as partitioning, he emphasized the procedural multiplication of π with symbolic association to the circle. By the end of this interview, Henry's understanding of radian could be described as procedural, with conceptual meanings demonstrated by his awareness and acknowledgment of ideas of proportionality. Figure 2 represents Henry's emphasis on procedural meanings for radian angle measure without connecting to the conceptual meanings he demonstrated.

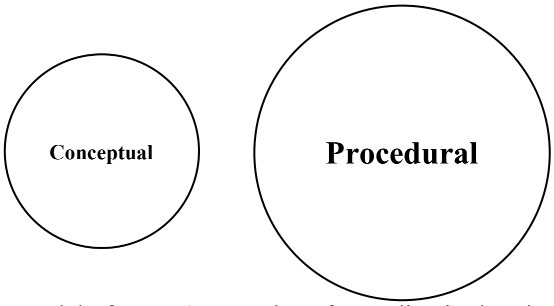

Figure 2. A Model of Henry's Meanings for Radian in the First Interview

Interview 2: Describing Radian Given Various Angle Measures

I adapted the second task from Thompson's *Teachers Promoting Change Collaboratively* (TPC^2) Functions Course, a project of the Center for Research on Education in Science, Mathematics, Engineering, and Technology.[1] The second task-based interview should have taken place a semester after the first; however, due to the COVID pandemic, it occurred a year later. Henry was asked to describe what it would mean for an angle to have the measure of 1 radian, and how he would determine the measure of angles in radians, given specific measures in degrees (i.e., 360°, 90°, 72°, 36°, 112°, x°). I included the measures of 72° and 36°, which can be written as whole fractions (i.e., $\frac{2\pi}{5}$ radians and $\frac{\pi}{5}$ radian respectively), in response to Henry's emphasis on describing radian angle measure as "easy fractions of π." I wanted to clarify whether Henry was referring to the few special angles memorized on a typical diagram of the unit circle or to conceptually partitioning the semicircle.

When Henry was asked to describe what it would mean for an angle to have the measure of one radian, he stated, "I remember this from the last study, but I'm going to take it a little bit to how I came to think of it after that study, how I came to think of it on my own." This suggests that Henry's description (below) resulted from reflecting after the interview with the first task:

> we learn the formula for circumference before we learn anything about angles … the circumference is 2π times the radius … this really didn't hit me until … we were asked … what's the length of this thing [points to arc in Figure 3] … it turns out that whatever that angle is, if you measure that angle in radians, then that's just the length of the arc. I mean, still times r … that was the part that kind of like, blew my mind … because … this circumference formula … [points to $C = 2\pi r$] is just a special case of the more general formula $S = \theta r$. This is what really made this significant for me … because

[1] http://tpc2.net/Courses/Func1F05/

all the way around is 2π, okay … what happens if you just make this 1 [points to θ in $S = \theta r$]? … that arc length equals the length of the radius (Figure 3) … it's just one radian.

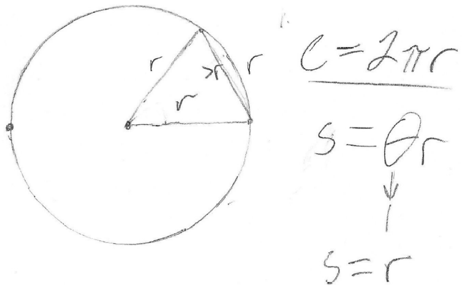

Figure 3. Henry Describing the Measure of 1 Radian Angle in Relation to the Arc Formula

While Henry referred to various formulas and calculations, his description involves connections between the radian angle measure and the arc length and circumference formulas. He recognized the circumference formula as a special case of the arc length formula, where the circumference is the arc length associated with a full-rotation angle. Henry also used the arc length formula to describe the measure of one radian, as the situation when the arc length subtending a one-radian angle equals the length of the radius.

Additionally, given the angle measure in degrees, Henry's strategy involved using the measure relative to special angles that he memorized. For example, to describe the measure of 72° in radians, Henry referred to the measure as $\frac{\pi}{3}$ + 12°, where 12° is $\frac{2}{5}$ of $\frac{\pi}{6}$ (Figure 4). As he implemented this strategy, Henry indicated that he had "these benchmarks for what are *easy fractions of* π that I can remember off the top of my head," referring to his use of the special angles $\frac{\pi}{6}$ and $\frac{\pi}{3}$ as memorized facts. However, instead of describing the measure of 72° as $\frac{2\pi}{5}$ in radians, he used $\frac{2}{5}\pi$ radians and indicated that $\frac{2}{5}\pi$ is more representative of the angle measure than $\frac{2\pi}{5}$ "because you have an inherent understanding that π is halfway around." Henry continued clarifying that π "would trace the full semicircle … I would argue that this [points to the $\frac{2}{5}\pi$] communicates more than this [points to the $\frac{2\pi}{5}$] does because this is … two-fifths of the way to a straight angle," suggesting that Henry was attending to the measure as a fractional amount of the semicircle.

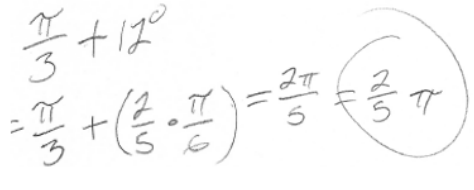

Figure 4. Henry's Description of 72° as $\frac{2}{5}\pi$ radians Relative to Special Angles

Henry's attention to angle measure in radians as a fractional amount of the semicircle was further evident in his description of 36° as $\frac{\pi}{5}$ radians, which he calculated in various ways, including using the special angles $\frac{\pi}{6}$ and $\frac{\pi}{3}$, but also as $\frac{1}{10}$ of 360° (Figure 5). Henry was surprised by $\frac{\pi}{5}$ radians, which satisfied his "easy fractions of π" description, yet he was not familiar with it. He stated that "36° being $\frac{\pi}{5}$, that's, that's such a great fraction, but wasn't on the unit circle. What was on the unit circle was $\frac{\pi}{4}, \frac{\pi}{6}$, and multiples thereof. Those are the ones that I remember first," acknowledging the influence of the unit circle diagram.

$$= \frac{\pi}{6} + \frac{1}{10}\left(\frac{\pi}{3}\right)$$
$$= \frac{\pi}{6} + \frac{\pi}{30} = \frac{6\pi}{30} = \frac{\pi}{5}$$

$$36^\circ = \frac{1}{10}(360^\circ)$$
$$= \frac{1}{10}(2\pi) = \frac{\pi}{5}$$

Figure 5. Henry's Description of 36° as $\frac{\pi}{5}$ radians Relative to Special Angles

Henry's description of 72° as "two fifths of the way to a straight angle" could reflect partitioning the semicircle into equiangular parts to represent the measure in radians. This conjecture was based on Henry's reflection on further measurements that are not depicted on a typical diagram of the unit circle. For example, he indicated that "a step further like 18, 18° is $\frac{\pi}{10}$ now," suggesting he might have been considering more ways to partition the semicircle to produce other radian angle measurements that he might not be familiar with.

By the end of the second interview, Henry's meanings for radian evidently involved both procedural and conceptual meanings, with a connection between the two (Figure 6). While using formulas and calculations, he described the circumference formula as a special case of the arc-length formula for a full rotation. Additionally, he used the arc length formula to define 1 radian as another special case, in which the arc length equals the radius. Henry did not emphasize or depend on procedural meanings; instead, he used procedural meanings as a vehicle to describe his conceptual ones through generalizable connections.

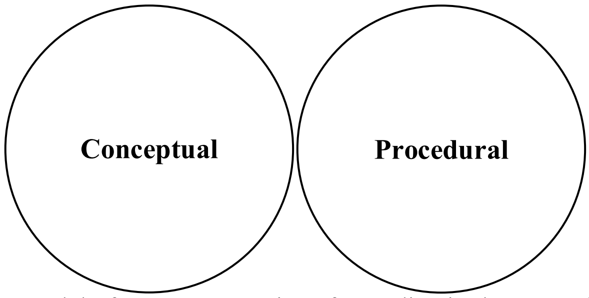

Figure 6. A Model of Henry's Meanings for Radian in the Second Interview

Interview 3: Describing Radian Given a Novel Context

The third and final task-based interview took place a semester after the second, and had two aims: (1) further to challenge Henry's dependence on the special angles, and (2) to confirm that Henry's meanings for radian angle measure involved partitioning the semicircle into equiangular parts. The task required Henry to engage with *Radian Lasers*, a digital (i.e., Desmos) activity that involves radian angle measure and light reflection (Alyami, 2022b), in which he had to input angles measured in radians to situate a laser and one or two mirrors so the laser beam would pass through three stationary targets. A benefit of the *Radian Lasers* task is that the angles needed to situate the mirror are not limited to common special angles (Figure 7), which would achieve the aims specified earlier.

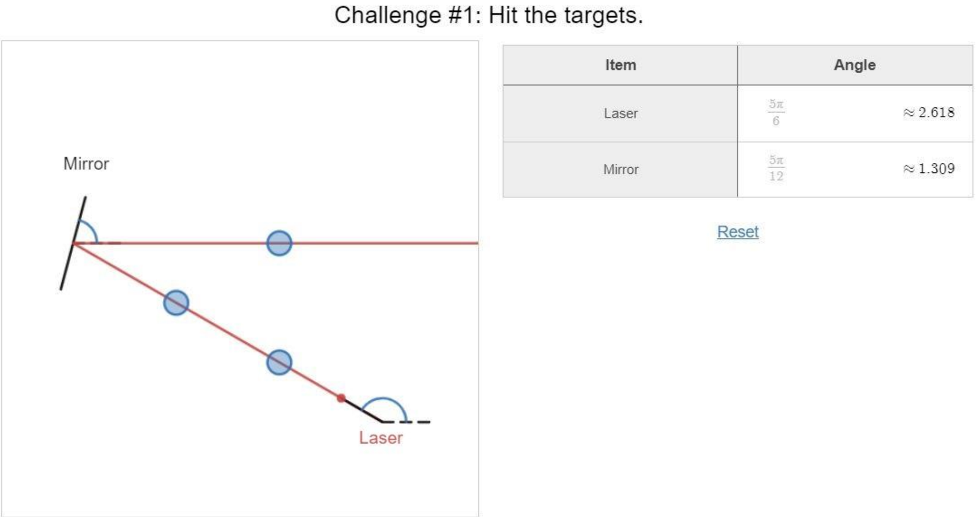

Figure 7. A Radian Lasers Challenge, With the Mirror Angle not a Common Special Angle

Radian Lasers challenged Henry's dependence on the special angles typically depicted on the unit circle diagram. For example, when working on Challenge 1 (Figure 7), Henry correctly positioned the laser first at $\frac{5\pi}{6}$ radians relative to the horizontal, which led the laser beam to pass through the first two targets and reach the mirror. However, for the laser beam to reflect off the mirror and pass through the third target, Henry initially positioned the mirror at $\frac{\pi}{3}$ radian, and indicated that "$\frac{\pi}{3}$ gives me too small of an angle." The next angle Henry tried for the mirror was $\frac{\pi}{2}$ (Figure 8), and he indicated that "$\frac{\pi}{2}$ gives me too big of an angle." However, Henry noticed that the two angles were "off by the same amount, just in opposite directions. Okay, so I must just need the value in between those two."

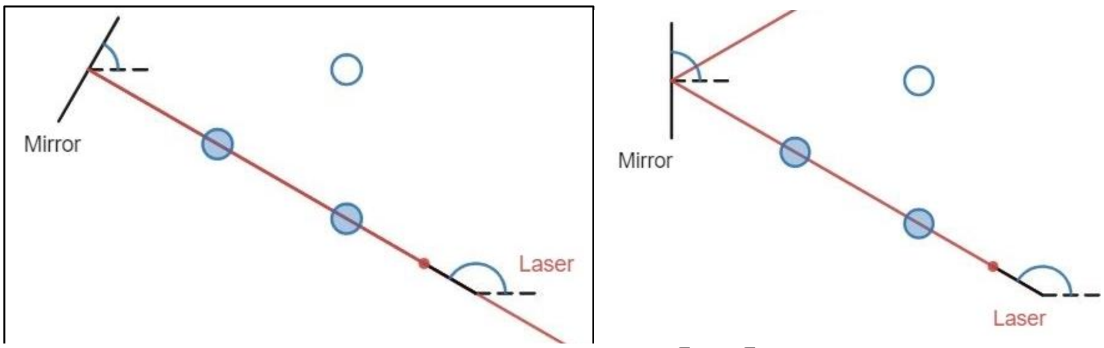

Figure 8. Situating the Mirror at $\frac{\pi}{3}$ and $\frac{\pi}{2}$ Radians

To find the value between $\frac{\pi}{3}$ and $\frac{\pi}{2}$ radians, Henry calculated the average to conclude that "the answer was $\frac{5\pi}{12}$." I use the following quote from Henry to discuss his own initial dependence on the special angles:

> Once I saw that this is a problem about radians, I had already decided that the answer had to be on the unit circle, and so I looked for every one of those [special angles] before I even thought about, well,

maybe I actually need to think about this problem, instead of just looking for the most applicable unit circle value.

Henry's statements illustrate both his reliance on unit-circle values and his acknowledgment of the limitations of such an approach. However, since the angles needed for this task were not limited to special angles, Henry reflected on the limitation of depending on unit circle values:

> The knowledge that we're trying to give to students when we introduce the unit circle … exercises like this will still get you to that same place … but they won't lock you into like, *the diagram.* Like everyone knows *the diagram*, it's got all the multiples of $\frac{\pi}{4}$ and all the multiples of $\frac{\pi}{6}$. That's *the diagram* … but this develops a better intuition of actually knowing that … $\frac{\pi}{n}$ radians is one n^{th} of the way around the semicircle.

Henry's statement suggests that even if students are introduced to the unit circle diagram, it can be done in a way that promotes conceptual understanding of radian measure, with students considering radian angles beyond the few special angles depicted on a typical unit circle diagram. While Henry initially used the special angles from the unit circle, he was able to flexibly think about radian angle measure beyond those special angles using conceptual meanings. His generalization of the radian angle measure as "easy fractions of π," to describe the fact that any fraction of π radians represents an angle that is "one nth of the way around the semicircle," confirms that his meanings for radian angle measure involved partitioning the semicircle into equiangular parts. Figure 9 represents Henry's meanings for radian angle measure after the third interview as involving overlapping procedural and conceptual meanings.

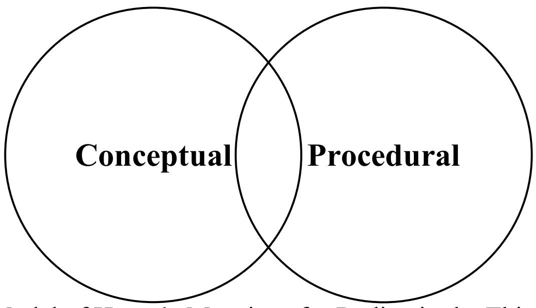

Figure 9. A Model of Henry's Meanings for Radian in the Third Interview

Summary of the Findings

As illustrated in the findings, Henry demonstrated various meanings for the radian angle measure. He initially emphasized procedural strategies, despite his awareness of the conceptual meaning of radian, as demonstrated by his attention to proportionality. However, while in the first interview he indicated that "the unit of 1 radian doesn't really mean a whole lot," two years later, he elaborated on the description of 1 radian using the arc approach to angle measure as a special case of the arc length formula, when the arc length and the radius are equal. This demonstrates a shift in Henry's meanings from minimizing the measure of 1 radian to generalizing and making various and explicit connections that involve proportionality in his description of radian.

While Henry continued to demonstrate procedural meanings through memorized facts and calculational strategies, he was able to incorporate connected conceptual meanings. In the second interview, when asked to describe 90° in radians, Henry stated, "I read this, the unit circle image comes up in my head, like boom, $\frac{\pi}{2}$, next question," suggesting a procedural meaning using memorized facts (i.e., unit circle and special angles). However, he initially indicated that "90° is a quarter of the way around the circle," and proceeded to describe the radian measure as "a quarter of 2π radians, same thing as $\frac{\pi}{2}$," suggesting a conceptual meaning of angle measure as part of a full-rotation angle.

In the last interview, Henry continued to reflect on his meanings for radian. He acknowledged the limitation of depending on the memorized facts as it became clear that his initial description of radian angle measure as "fractions of a circle" involved "all the multiples of $\frac{\pi}{4}$ and all the multiples of $\frac{\pi}{6}$" that are depicted on a typical diagram of a unit circle. He elaborated on how engagement with the task allowed him to generalize the description of a radian to apply to any angle as a fraction of the semicircle.

While Henry's procedural meanings for radian angle measure involved memorized facts and calculational strategies, his conceptual meanings involved the arc approach and equipartitioning the semicircle, depending on the task's context.

Discussion

Several researchers have documented that learners have an instrumental understanding of radian angle measure (Akkoc, 2008; Çekmez, 2020; Fi, 2003; Moore et al., 2016; Topcu et al., 2006), which could be a result of not having sufficient curricular opportunities to engage with radian angle measure (Mesa & Goldstein, 2017). This study extends existing research by demonstrating a PMT's mathematical meanings during engagement with three curricular tasks that involve radian angle measure. Henry's initial description of radian demonstrates the influence of common representations of radian in terms of π and in relation to the unit circle (Akkoc, 2008; Fi, 2003; Moore, 2013; Topcu et al., 2006). Additionally, during the first interview, Henry attended to the role of proportionality between the angle's subtended arc and its radius (Alyami, 2022a). However, as in-service teachers in Thompson et al. (2007), Henry questioned the practicality of using proportionality and the arc approach to measure angles.

Henry elaborated on the relationship between the arc length and radius at the beginning of the second interview by relating the formulas for measuring arc length and circumference, although this was not Henry's preferred description. However, using these relationships to describe one radian as the special case in which the angle's subtended arc and the radius are equal reflects Henry's multiplicative comparison between the two lengths (Thompson, 2011). While Henry demonstrated this comparison procedurally using the formulas, his description incorporated conceptual strategies that focused on *how* and *why* these formulas are related (Nordlander, 2021). Henry's description of the underlying concepts behind the procedures reflects effective mathematics teaching practice, namely building procedural fluency from conceptual understanding (NCTM, 2014a).

Henry's initial description of radians emphasized the special angles depicted on a typical unit-circle diagram. However, he acknowledged the limitation of this strategy upon encountering $\frac{\pi}{5}$, which he was not familiar with because "it was not on the unit circle." This illustrates how procedural strategies can be beneficial in limited situations (Thompson & Carlson, 2017). Eventually, Henry generalized his description of radian as "easy fractions of π" to include any fraction that is "one n^{th} of the way around the semicircle," suggesting a partitioning approach (Hardison, 2020). While Henry used conceptual strategies from the beginning, his procedural explanations outweighed his conceptual ones. However, since procedural strategies might not be devoid of conceptual understanding (Maciejewski & Star, 2019; Nilsson, 2020; Nordlander, 2021; Rittle-Johnson et al., 2015; Star, 2005, 2007), exposure to situations with radians in contexts that challenged procedural strategies, such as non-familiar diagrams and measurement (e.g., $\frac{\pi}{5}$, $\frac{5\pi}{12}$), allowed Henry to connect his procedural and conceptual meanings for radian angle measure.

Henry's two conceptual meanings of radian reflect the variations in identifying and quantifying a measurable attribute of an angle. Specifically, measuring the length of the subtended arc in units of radii of the circle containing the arc (Moore, 2013), and equipartitioning of the semicircle as an established benchmark (Hardison, 2020) that has the measure of π radians. The distinction between these two meanings lies in the measurable attribute Henry attended to, depending on the task's context. While Henry expressed a preference for the partitioning of the semicircle approach, his awareness of the arc approach to angle measure is notable, especially since multiple researchers have demonstrated the effectiveness of the arc approach to angle measure for students' understanding of trigonometric (Moore, 2014) and inverse trigonometric functions (Paoletti, 2020).

Curricular Implications

This report does not aim to villainize the use of the unit circle. In fact, students can successfully complete tasks using the unit circle (Weber, 2005; Moore et al., 2016). Henry himself was pleased by the unit circle representation despite acknowledging its limitations. Instead, I extend Moore et al.'s (2016) argument that "[i]f students are to understand the unit circle in a way that encompasses all circles, it is necessary that they have experiences that warrant the construction of meanings that entail such generality" (p. 239). Learners must have experiences that support the development of conceptual meanings for radian and the unit circle representation. However, learners are not given such experiences because there are few curricular opportunities for students to explore the concept of the radian (Mesa & Goldstein, 2017).

In this report, I identified that one PMT's meanings for radian angle measure initially entailed using special angles and the unit circle as benchmarks. However, given experiences (i.e., tasks) that problematized and limited reliance on special angles and the unit circle as fixed representations, Henry had the capacity to flexibly adjust how he used those representations once they no longer served him. The main implication of this report is the recommendation to capitalize on tasks designed to build and support conceptual understanding of the radian angle measure. Instead of tasks that emphasize procedures, using tasks that encourage learners to unpack the mathematical foundation behind the procedures potentially contribute to developing conceptual and productive meanings for radian angle measure.

Limitations

Before concluding, I mention noteworthy methodological and theoretical points. From a methodological perspective, this study explored one PMT's mathematical meanings through engagement in task-based interviews. The study aimed to describe the amalgamation of procedural and conceptual meanings of the radian angle measure and to characterize shifts in these meanings as a result of engaging in tasks that challenge reliance on procedural meanings. While Henry's reflection on his own actions (Simon et al., 2004) during engagement with the tasks was from a learner's perspective, he alluded to pedagogical approaches by reflecting on how tasks providing opportunities for using radian angle measure beyond the special angles (multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ radians) allowed him to flexibly reason with the unit circle. However, exploring how Henry's mathematical meanings may influence his teaching was beyond the scope of this study. Studies that investigate the influence of mathematical meanings of angle on teaching (e.g., Tallman & Frank, 2020) have the potential to deepen our understanding of the alignment between the development of mathematical meanings of angle and instructional decisions aimed to support mathematical learning. Additionally, this study focused on only one case of a PMT's meanings for radian angle measure using a series of curricular tasks that involve radian. Other learners may develop meanings different from Henry's. I have provided evidence that experiences (i.e., curricular tasks) that challenge reliance on procedural meanings for the radian angle measure can support the development of conceptual meanings. However, future studies extending engagement with the three tasks involving radian angle measurement, with larger samples from diverse backgrounds, are needed. Finally, the long stretch between the interviews was due to the COVID pandemic and was not intended, as Henry could have studied mathematical topics involving radians during those periods and/or forgotten what was explored during the interviews. However, Henry did mention the first task-based interview during the second, which shows his reflection on his own actions (Simon et al., 2004), even though the second interview was conducted two years after the first. Future studies using this task sequence over a shorter span would contribute to our understanding of learners' meanings of the radian angle measure as a result of engaging with the series of designed tasks.

Contributions and Concluding Remarks

This study contributes to existing research exploring conceptions of radian angle measure and how it relates to trigonometry (Akkoc, 2008; Çekmez, 2020; Fi, 2003; Moore, 2013, 2014; Moore et al., 2016; Tallman & Frank, 2020; Thompson, 2013; Topcu et al., 2006). While there are few opportunities for students to engage with radian angle measure in the curriculum (Mesa & Goldstein, 2017), this study demonstrated that engagement with curricular tasks that involve and focus on radian angle measure can potentially challenge procedural meanings for radian and contribute to the development of productive conceptual meanings. Teachers and curriculum developers are encouraged to capitalize on the reported findings to develop and use curricular approaches that allow learners to experience radian angle measure beyond the special angles depicted in a typical diagram of the unit circle.

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