

Preservice Elementary Teachers' Understanding of Operations for Fraction Multiplication and Division

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This study examined preservice elementary teachers' change in their understanding of fraction operations while taking a mathematics methods course focused on grades 3-5. The preservice teachers ($n = 48$) completed an assessment before and after the course's unit on the teaching and learning of fractions. Specifically, the preservice teachers were asked to explain why the traditional algorithms for multiplication and division of fractions work. Additionally, they were asked to identify errors in students' work and provide rationales for why the strategies were faulty. All responses on the pre- and post-assessments were coded using an existing framework for the assessment of understanding. Paired t-tests indicated a statistically significant improvement on most items on the assessment and on the total test score. However, an in-depth analysis of how the scores changed gave insights into the proportion of preservice teachers who demonstrated no change, improvement, or regression on each question. The percent of teachers improving on each question ranged from 33% to 46%. Implications for mathematics teacher education programs and future research are discussed.

Keywords: elementary education; preservice teachers; teacher knowledge; fractions; reasoning and proof

The need to develop students' conceptual understanding in mathematics, as outlined in policy and recommendation documents (e.g., Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics [NCTM], 2014; Australian Association of Mathematics Teachers, 2006), means that teachers need to possess a deep understanding of mathematics (Ball & Bass, 2002; Ball, Thames, & Phelps, 2008). Consequently, part of teacher preparation is to provide activities and assignments that deepen conceptual understanding among preservice teachers (PSTs). Yet, over the past two decades, research has indicated gaps in PSTs' conceptual understanding of mathematical ideas, particularly for PSTs in elementary education, the focus of this study (e.g., Simon, 1993; Baturo & Nason, 1996; Ma, 1999; Newton, 2008; Chinnappan & Forrester, 2014). Learning about fractions and the operations is typically a challenging area of study for both elementary students and their teachers (Ma, 1999; Mack, 2001; Lamon, 2007; Harvey, 2012) as knowledge of whole numbers is transferred to this "new" set of numbers. There has been a sizable amount of research to suggest that elementary PSTs lack conceptual understanding of fractions, but there is little research on *how* to improve PSTs' understanding of fractions (Olanoff, Lo, & Tobias, 2014). The current study aims to begin to fill that gap. The purpose of the study was to examine how elementary PSTs provide explanation and justification for algorithms for multiplication and division of fractions before and after an instructional unit focused on the teaching and learning of fractions in a mathematics methods course.

Theoretical Framework

This study draws on two theoretical perspectives: mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008) and understanding (Skemp, 1976). To quote Shulman (1987), "to teach is first to understand" (p. 14). In other words, teachers need to develop a deep

understanding of their content material, know how to present the material in a variety of ways, and make connections within and across disciplines (Ball, 1990; Shulman, 1987). Building on the work of Shulman (1986) who had recognized the special type of knowledge that teachers need, Ball, Thames, and Phelps (2008) presented a re-conceptualized framework for this deep and specialized understanding of mathematics, called mathematical knowledge for teaching (MKT). Organized into two domains, subject matter knowledge and pedagogical content knowledge, the current study focuses on one category of knowledge within the subject-matter-knowledge domain, *specialized content knowledge* (SCK). SCK is the knowledge of mathematics that is specific to the work of teaching (i.e., interpreting solution strategies, evaluating alternative algorithms).

Teachers' SCK in mathematics is dependent upon their own *understanding*, which brings us to the second theoretical perspective that informed the work of this study. Specifically, we used the work of Skemp (1976) who defined the ideas of relational versus instrumental understanding. According to Skemp, relational understanding is knowing both what to do and *why* (e.g., why we can multiply by the reciprocal when dividing fractions), whereas instrumental understanding is engaging in mathematical procedures accurately and efficiently but not knowing why (what he called 'rules without reasons'). Researchers have argued that Skemp at one point viewed understanding as somewhat dichotomous (Sfard, 1991) – one has relational OR instrumental understanding – but we utilize his ideas from a continuum perspective.

This study examined PSTs' levels of *understanding* of multiplication and division of fractions through open-ended assessment items. The assessment items represented the aforementioned category of SCK. However, in this study, our focus is on measuring levels of *understanding* with the premise that a teacher's SCK is dependent upon where his/her understanding falls on a continuum further detailed below (Biggs, 1990).

Relevant Literature

Elementary Teachers' SCK in Mathematics

Research has emphasized that in order for teachers to be able to answer students' questions of "why," they must have a more robust relational understanding than that of their students (Ball, Thames, & Phelps, 2008). As Hill, Rowan, and Ball (2005) found in their study of first and third grade students, teachers with higher amounts of SCK in mathematics supported their students' achievement in mathematics in a positive direction. Similarly, scholars have found a positive and strong association between teachers' SCK and the quality of mathematics instruction they offer to their students (Hill et. al., 2008; Walkowiak, 2015).

Many PSTs come into their teacher preparation program lacking understanding of common algorithms, especially related to fractions, used in elementary school mathematics (Young & Zientek, 2011; Newton, 2008; Caglayan & Olive, 2011; Kajander & Holm, 2011). It proves challenging for these PSTs when they are expected to teach conceptually, based on reform movements, but have never experienced this type of learning themselves (Lubinski & Thomason, 1998). However, if we provide these teachers opportunities to make sense of problems, provide reasoning for their justifications, and grapple with solving problems, (i.e., develop their SCK) "they will not necessarily teach as they were taught but will teach for understanding and reasoning" (Lubinski, Fox, & Thomason, 1998, p. 252).

Specialized knowledge for fraction multiplication and division. Researchers have examined specialized mathematical knowledge in fraction multiplication and division. In their review of the literature on mathematical content knowledge for teaching fractions, Olanoff, Lo, and Tobias (2014) noted three primary findings. First, elementary PSTs typically know how to multiply and divide fractions using procedures (i.e., an instrumental understanding), but they often lack an understanding what the procedures mean (i.e., relational understanding). Second, research over time has moved to almost exclusively focusing on fraction multiplication and division to other operations and general fraction concepts. Finally, and speaking to the significance of the current study, the researchers found a lack of studies focused on how to improve PSTs' understanding of fraction ideas.

Regarding multiplication, there have been similar findings over the years. PSTs struggle with solving fraction multiplication problems and apply erroneous or unnecessary procedures such as using cross-multiplication (Newton, 2008) or finding common denominators (Young & Zientek, 2011). Furthermore, PSTs have struggled to represent fraction multiplication with a story problem (Luo, 2009) perhaps because PSTs often conceive that multiplication always makes larger (Tirosh & Graeber, 1991).

There is a larger amount of research on fraction division, and much of the findings are similar. PSTs are typically not able to explain why the "invert and multiply" rule works (e.g., Li & Kulm, 2008, Ball, 1990) and struggle to write story problems that represent a division expression with fractions (e.g., Simon, 1993; Rizvi & Lawson, 2007). Eisenhart et al. (1993) followed Ms. Daniels, a student teacher who tried explaining the "invert and multiply" rule to a student in her class and abandoned the explanation midway through when she realized the example she was using pertained to multiplication, not division. Ball (1990) gave an example of Allen, an elementary education major who also struggled with the "invert and multiply" rule for division of fractions and could not generate an example that did not reference multiplication. Much of the challenge for our work as mathematics teacher educators is to prepare PSTs to answer these types of questions effectively and not give the reasoning "because it's the rule". To do so requires the development of SCK through a deep conceptual understanding of the mathematics throughout the PSTs' teacher preparation program.

Understanding

To measure understanding in mathematics, researchers have typically used open-ended assessments and interviews that then must be analyzed (e.g., Tanisli, 2011). One analysis tool, and the tool used in this study, is the Structure of the Observed Learning Outcome Taxonomy (SOLO; Biggs, 1999) as displayed in Figure 1. The nature of the SOLO Taxonomy is one such that student learning is examined as they move from a lower level of understanding (Preoperational) to a higher, more abstract, level of understanding (Extended Abstract). As students progress through the levels, they retain the traits from the previous level; in other words, each level builds on the previous. In the Preoperational level, students incorrectly interpret the problem or provide an incorrect solution. During the Unistructural level, students work on procedures; however, once they reach the Multistructural level they are beginning to describe what is happening within a problem. Once students reach the Relational level, they begin to explain and analyze the situation, with the final level, Extended Abstract, showing that students can generalize a problem in a much broader sense. According to Newton (2008), PSTs' mathematical understandings often rest on rule-based reasoning and lack flexibility, characteristic of understanding at the Unistructural level of the SOLO taxonomy.

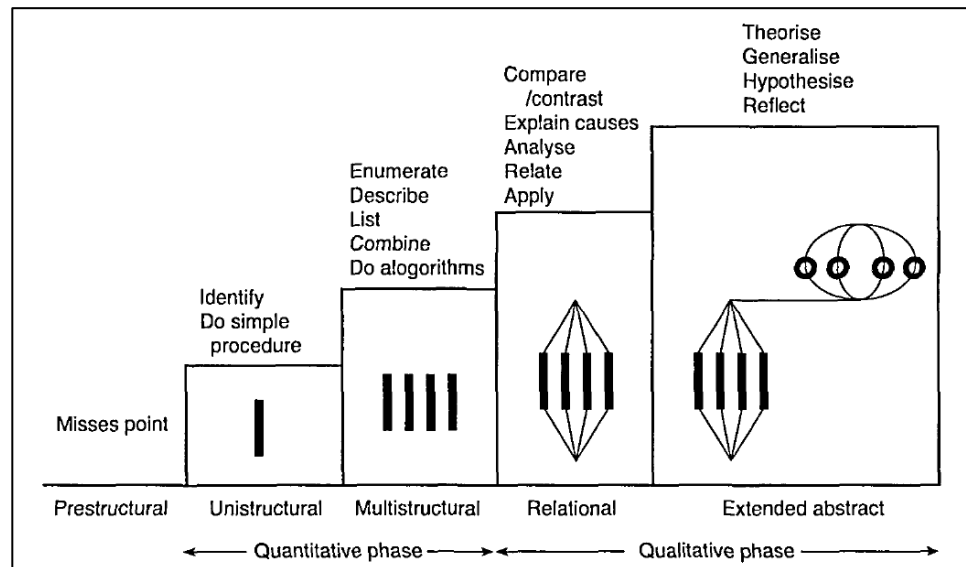


Figure 1. SOLO Taxonomy (Biggs, 1999, p. 67)

The SOLO taxonomy can be related to Skemp's descriptions of instrumental and relational understanding (1976). Figure 2 displays the relationship, as conceptualized in this study. The preoperational level indicates no understanding so it does not match either of Skemp's constructs. However, the Unistructural level can be likened to Skemp's idea of instrumental understanding. In the context of algorithms, we utilize Skemp's language of "rules without reasons." The three upper levels of the SOLO taxonomy represent Skemp's idea of "relational understanding" but along a continuum.

SOLO Taxonomy (Biggs, 1999)	Preoperational	Unistructural	Multistructural	Relational	Extended Abstract
Skemp (1976)		Instrumental	Relational		

Figure 2. Relationship between SOLO Taxonomy (Biggs, 1999) and Instrumental/Relational Understanding (Skemp, 1976)

Research Questions

With the continued emphasis on conceptual understanding in documents such as the recently released *Principles to Actions* by the National Council of Teachers of Mathematics (2014), there is a need to better understand PSTs' levels of *understanding* and the impact of instruction in elementary teacher preparation programs on those levels. Furthermore, since fractions have been difficult historically for both children and adults (Lamon, 2005), the field of mathematics teacher education needs research to bring increased clarity to how we can best prepare teachers to teach fractions to their students. The current study seeks to address these needs. Specifically, the two research questions were: (1) Do elementary preservice teachers *change in their understanding* of fraction multiplication and division during an instructional unit

on the teaching and learning of fractions in a mathematics methods course?; and (2) If changes occur, *how* do elementary preservice teachers change in their level of understanding of fraction multiplication and division?

Methods

Participants

The participants in this study were 48 PSTs or undergraduate students (juniors) in an elementary education program who were enrolled in a mathematics methods course, the second in a two-course sequence and focused on multiplicative reasoning in grades 3-5 (The first methods course was focused on additive reasoning in grades K-2). All participants were “traditional” college students, meaning they entered college immediately upon graduation from high school. Data was collected from students in two sections of the course.

The teacher preparation program was STEM-focused; this focus meant all participants had taken 12 credit hours (4 courses) of mathematics prior to beginning their full-time education program as juniors (as well as 4 science courses and 1 engineering/design course). Forty-six (95.8%) of the participants had taken one or two Calculus courses, either in high school or at the college level. Nineteen (39.6%) participants had taken *Calculus for Elementary Teachers* as part of their undergraduate course requirements; this course focuses on developing a conceptual understanding of Calculus topics and making connections to the elementary mathematics curriculum. The mathematics background of these students is important to note because the focus of this study was on the pre-service teachers’ conceptual understanding and level of justification they can provide to students. One can see from the level of mathematics achieved by these students that they have experienced the content they will be expected to teach; however, they may or may not be able to explain the ideas conceptually.

Intervention

Data was collected from students in two sections of a mathematics methods course focused on mathematics teaching and learning in the upper elementary grades (grades 3-5). The course focuses on multiplicative thinking and ideas (e.g., multiplication, division, fractions); it is designed to develop PSTs’ understandings and skills in: mathematical content; children’s thinking about mathematics; and pedagogies to develop conceptual understanding and problem solving among elementary children. The course was taught by a total of four instructors, two teaching one section and the other two teaching the other section. Two of the instructors were also the researchers for this study. All lessons were created as collaboration between the four instructors; therefore, all participants experienced the same tasks and activities during the class sessions.

After the pre-service teachers completed the pre-assessment, they participated in a five-week unit of instruction during which they learned about foundational ideas of fractions (e.g., equal sharing, equivalency, comparing, area/set/linear representations) as well as operations with fractions. Class sessions occurred weekly for 2.5 hours per week for the five weeks. Table 1 displays the foci of each of the five class sessions. We find it important to share an overview of the entire instructional unit because the building of fraction ideas in the first two class sessions serve as a foundation for subsequent concept development; however, it is important for us to note that multiplication and division of fractions, the focus of this study, were addressed in sessions 3, 4, and 5 of the unit. Examples of tasks completed during the instruction period are provided in

Appendix A. The activities were selected to build PSTs' conceptual knowledge of fraction-based algorithms as well as their ability to recognize common student errors relating to fraction multiplication and division. Class readings were also given to help further PSTs' understanding related to the given topics.

Table 1
Foci of Class Sessions

Weekly Session	Topics/Foci
1	<i>Pre-assessment</i> Equal Sharing (in context of Sub Sandwich Problem*) Defining Fractions Common Misconceptions
2	Representing Fractions (set, area, linear) Iterating/Partitioning Equivalency & Comparing
3	Adding/Subtracting Fractions Finding Fraction of a Fraction (in context of Playground Problem*)
4	Representing Fraction Multiplication Contextual Problems (Fraction Division)
5	Representing Fraction Division Student Errors in Fraction Division <i>Post-assessment</i>

*from Fosnot & Dolk, 2002

Measure

Before and after the instructional sequence, participants were asked to complete an assessment (see Appendix B), created by the first author, regarding questions related to algorithms for fraction multiplication and division. Questions 1 and 3 on the assessment focused on their ability to unpack why the traditional algorithms in the United States work. For multiplication, the traditional algorithm is multiplying across numerators and denominators. For division, the traditional algorithm involves keeping the first fraction (dividend) the same, changing to multiplication, and making the second fraction (divisor) its reciprocal (i.e., “invert and multiply,” “keep, change, flip”). The second and fourth questions on the assessment involved recognizing if an error in student work exists and explaining why the error occurs.

Prior to instruction, the participants had not worked with multiplication and division of fractions in either of their methods courses.

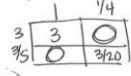
Procedure

Participants completed the pre-assessment on the first day of the unit of instruction on fractions. Answers were coded according to the levels of the SOLO taxonomy with numbers from 0 to 4 assigned to each of the levels, with Preoperational being a 0 and Extended Abstract being a 4. Two independent raters coded 25% ($n = 12$) of the pre- and post-assessments in order to check for inter-rater consistency. 98% of the two raters' codes were either exact matches or within one scale point of each other (with 76% exact match agreement). For any discrepancies in codes, the two raters met and discussed the codes to reach consensus. The remaining 75% of the assessments were coded by one rater. Table 2 provides the coding rubric for the items on the assessment. To provide further information about the application of the SOLO taxonomy in coding, we have included example responses for Assessment Question 2 in Table 3.

Table 2
Coding Rubric using SOLO Levels

SOLO Level (Numerical Code)	Assessment Questions 1 and 3	Assessment Questions 2 and 4
Preoperational (0)	The PST misinterprets the question or does not provide a response.	The PST misinterprets the question or does not provide a response.
Unistructural (1)	The PST recites a rule as a means of justification; he/she does not include any conceptual justification.	The PST can identify something is wrong with the student's work, but he/she cannot pinpoint exactly where the mistake is occurring.
Multistructural (2)	The PST recites a rule, but he/she begins to give a conceptual justification for a specific example; however, the justification is not complete or is incorrect.	The PST can describe the student's approach and can identify the student's mistake, but he/she does not know how to correct the student's mistake.
Relational (3)	The PST provides an explanation in which the idea is fully explained conceptually in terms of a specific example.	The PST correctly identifies the mistake, and he/she explains and corrects the mistake in terms of the specific example.
Extended Abstract (4)	The PST gives a generalized proof for the mathematical idea using variables and/or words.	The PST can explain the student's method in generalized terms.

Table 3
Examples of Student Responses for Each Coding Level for Assessment Questions

SOLO Taxonomy Level	Example Response for Assessment Question 2	Justification for SOLO Level
Preoperational (0)	2. Henry is trying to solve $1\frac{1}{4} \times 3\frac{3}{5}$; his work and thinking is shown below. Is Henry's method valid? Why or why not? $1\frac{1}{4} \times 3\frac{3}{5} = (1 \times 3) + (\frac{1}{4} \times \frac{3}{5})$ yes, it works because you are still multiplying the same numbers $3 + \frac{3}{20} = 3\frac{3}{20}$	Incorrectly answers the problem.
Unistructural (1)	2. Henry is trying to solve $1\frac{1}{4} \times 3\frac{3}{5}$; his work and thinking is shown below. Is Henry's method valid? Why or why not? $\frac{5}{4} \times \frac{18}{5} = \frac{90}{20} = 4\frac{1}{2}$ $1\frac{1}{4} \times 3\frac{3}{5} = (1 \times 3) + (\frac{1}{4} \times \frac{3}{5})$ $3 + \frac{3}{20} = 3\frac{3}{20}$ Henry's method is not valid because he is multiplying with mixed numbers and he needs to use improper fractions. He got a completely different answer with no way to make it correct using his method.	Recites the rule of turning the mixed numbers into improper fractions.
Multistructural (2)	2. Henry is trying to solve $1\frac{1}{4} \times 3\frac{3}{5}$; his work and thinking is shown below. Is Henry's method valid? Why or why not? $1\frac{1}{4} \times 3\frac{3}{5} = (1 \times 3) + (\frac{1}{4} \times \frac{3}{5})$ $3 + \frac{3}{20} = 3\frac{3}{20}$ No, because Henry is missing 2 parts of the problem. He still needs to multiply every part of the problem $\rightarrow (1 \times 3) + (1 \times \frac{3}{5}) + (3 \times \frac{1}{4}) + (\frac{1}{4} \times \frac{3}{5})$, therefore his problem is incorrect	Understands that the shown method is incorrect. However, the student doesn't explain why Henry must multiply all parts in the problem and instead recites a rule.
Relational (3)	2. Henry is trying to solve $1\frac{1}{4} \times 3\frac{3}{5}$; his work and thinking is shown below. Is Henry's method valid? Why or why not? $1\frac{1}{4} \times 3\frac{3}{5} = (1 \times 3) + (\frac{1}{4} \times \frac{3}{5})$ $3 + \frac{3}{20} = 3\frac{3}{20}$ No he's missing the rest of the distributive property. He needs to find the values for $1 \cdot \frac{3}{5}$ as well as $3 \cdot \frac{1}{4}$ to complete this method. 	Understands the numbers are not fully decomposed and wants to teach Henry how to view the problem in terms of an area (array) model.
Extended	*This is a hypothetical example response, not	Makes mathematical

Abstract

from our data.

(4)

$A^{b/c} \cdot D^{e/f}$

	A	b/c
D	$A \cdot D$	$\frac{Db}{c}$
e/f	$\frac{Ae}{f}$	$\frac{be}{cf}$

$(A + b/c) \cdot (D + e/f)$

Since $A^{b/c}$ can be written as $(A + b/c)$ and $D^{e/f}$ can be written as $(D + e/f)$ they can be shown in an array model just as with whole numbers.

Ex) $14 \cdot 15 = (10+4) \cdot (10+5)$

	10	4
10	100	40
5	50	20

So you must multiply all parts in the array.

generalizations beyond the situation in the problem using words, symbols, and/or pictures.

Analysis

After all pre- and post-assessments were coded, individual item codes (or scores) were entered into a database. Total scores for each assessment were calculated for each participant by finding the mean of the four individual item scores. For the first research question, paired t-tests were conducted for individual item scores and for total assessment scores to determine if there was a statistically significant *change* from pre- to post-assessment. For the second research question, pre- and post- assessment scores were examined in more detail to gain an overall picture of the nature of the changes. Specifically, we examined the number of participants whose scores increased, stayed the same, or decreased for each question. Additionally, we calculated frequencies for each type of change (e.g., 0 to 1, 1 to 3) for each question. These frequencies were utilized to determine patterns or unique changes in the participants' understanding on a given question.

Results

Research Question 1

Paired t-tests were used to determine if there was a statistically significant difference in individual item and total assessment scores from pre- to post-assessment. A Bonferroni Correction was applied; we used a p-value of .01 to check for significance since five t-tests were conducted. There was a statistically significant improvement on questions 1, 2, 4, and on the total assessment score. Table 4 displays the means, standard deviations, and test statistics.

Table 4
Results of t-tests

	Pre: Mean (SD)	Post: Mean (SD)	Difference	99% CI of Difference	t (df)	p*	Cohen's D
Q1	.54 (.92)	.98 (.96)	.44	[.00, .87]	2.69 (47)	.0099	.47
Q2	.65 (.53)	1.17 (.95)	.52	[.16, .88]	3.91 (47)	.0003	.68
Q3	.37 (.53)	.75 (1.04)	.38	[-.05, .80]	2.35 (47)	.0228	.46
Q4	.73 (.94)	1.44 (1.35)	.71	[.18, 1.24]	3.59 (47)	.0008	.61
Total	.57 (.46)	1.08 (.68)	.51	[.26, .76]	5.54 (47)	<.0001	.88

*p-values are rounded to the ten-thousandths to display the p-value for Q1 as below .01.

Research Question 2

After determining there was a statistically significant change on most of the items and on the total assessment score, we moved onto the second research question to dig deeper into *how levels of understanding changed* between the pre- and post-assessments. Table 5 displays the frequencies (i.e., number of participants) for change in understanding from the pre- to post-assessment as measured by the SOLO taxonomy. It is important to note that the percentage of PSTs remaining the same from pre-to post-assessment ranged from 35.4% to 47.9%. As we present findings for each assessment question in the sections below, we first describe an overview of how the data was spread across the levels of understanding. Then, due to the focus of our research on examining the *change* in the PSTs' levels of understanding, we focus on the PSTs who either increased or decreased in their level of understanding.

Table 5
Frequencies: Change on SOLO Taxonomy of Understanding from Pre to Post

	<u>Increase</u> Frequency (%)	<u>Remained the Same</u> Frequency (%)	<u>Decrease</u> Frequency (%)
Q1	22 (45.8)	17 (35.4)	9 (18.8)
Q2	22 (45.8)	21 (43.8)	5 (10.4)
Q3	16 (33.3)	23 (47.9%)	9 (18.8)
Q4	21 (43.8)	22 (45.8)	5 (10.4)

Assessment question #1. Figure 3 shows the number of participants at each level of understanding for both the pre- and post-assessment for question #1. On the pre-assessment, the majority (33 of 48) of the participants scored at the Preoperational level. For the post-assessment, most participants were somewhere from the Preoperational to the Multistructural level of understanding.

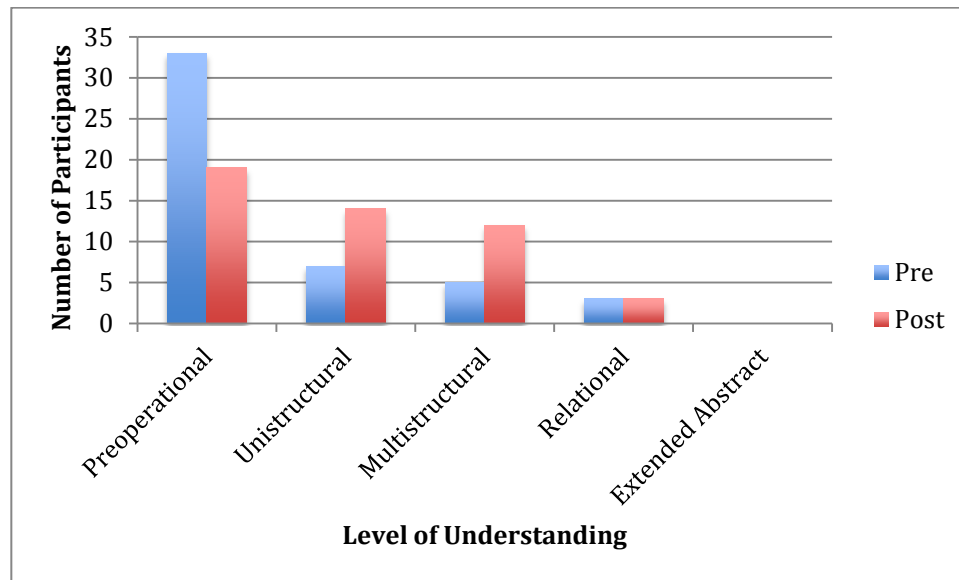
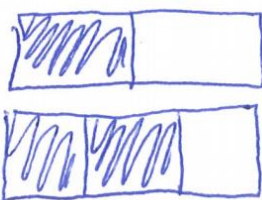


Figure 3. Assessment Question #1 Results.

Of the 22 participants who increased in levels of understanding (Table 5), 21 increased one or two levels of understanding; therefore, we highlight the work of one of those participants. The selected example in Figure 4 illustrates a participant who had a Preoperational level of understanding on the pre-assessment, and upon completion of the post-assessment, the participant had moved to a Multistructural level of understanding (Figure 5). During the pre-assessment, the participant responded with “I don’t know” for why the method works. However, upon completion of the post-assessment, the participant used a conceptual example of why the method is valid, but she neither clarified how her rectangular representation was created in relation to the problem, nor explained the relationship between the 6 parts and 2 parts that she mentioned. Therefore, her conceptual explanation is not complete. This example shows one type of increase we noticed in the data among the 22 participants who increased in their scores on Question #1.

1. Sarah is unsure why when you multiply two fractions such as $\frac{2}{3} \times \frac{1}{2}$ that you are allowed to multiply straight across. Give an explanation to Sarah why this method works.



I don't know

Figure 4. Example of Preoperational level (0), Question 1 (pre-assessment).

1. Sarah is unsure why when you multiply two fractions such as $\frac{2}{3} \times \frac{1}{2}$ that you are allowed to multiply straight across. Give an explanation to Sarah why this method works.

The denominators multiply to create the total number of parts. The numerators determine how many of those parts you are dealing with.

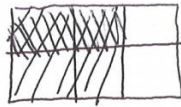
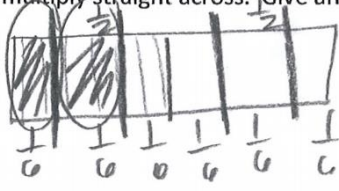
ex:  6 parts total, ~~1~~ 2 parts looked at

Figure 5. Example of Multistructural level (2), Question 1 (post-assessment).

Regarding the nine participants who decreased from pre- to post-assessment, most (or 7 of 9) participants decreased by one level. The selected example in Figure 6 illustrates a participant who moved from Relational to Multistructural levels of understanding from pre- to post-assessment (Figure 7). During the pre-assessment, the participant gave a full conceptual explanation to the student about why the method works. However, while the post-assessment shows a visual representation of the mathematics, the participant did not explain the representation and how it connects to the traditional algorithm.

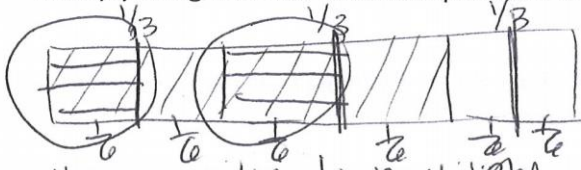
1. Sarah is unsure why when you multiply two fractions such as $\frac{2}{3} \times \frac{1}{2}$ that you are allowed to multiply straight across. Give an explanation to Sarah why this method works.

 $\frac{2}{3} \times \frac{1}{2} = \frac{2}{6}$

*this method works after you explain the area model. you can see that when you put up halves into thirds you get 6 pieces.

Figure 6. Example of Relational level of understanding (pre-assessment).

1. Sarah is unsure why when you multiply two fractions such as $\frac{2}{3} \times \frac{1}{2}$ that you are allowed to multiply straight across. Give an explanation to Sarah why this method works.

 $\frac{2}{3} \times \frac{1}{2} = \frac{2}{6}$

It is possible to multiply straight across, because you are able to demonstrate it conceptually using a fraction bar model.

Figure 7. Example of Multistructural level of understanding (post-assessment).

Assessment question #2. Figure 8 displays the number of participants at each level of understanding for both the pre- and post-assessment for question #2. The pre-assessment had 60.4% (29 of 48) of the participants scoring at the Unistructural level of understanding. For the post-assessment, 39.6% of the participants were at the Unistructural level of understanding (19 of 48); however, 15 of the 48 participants had moved into either the Multistructural or Relational level of understanding. Of the participants who changed in their score on Question 2 ($n = 27$), most of them increased (22 of 27); therefore, we give attention to that particular group of students as we further examine how they changed.

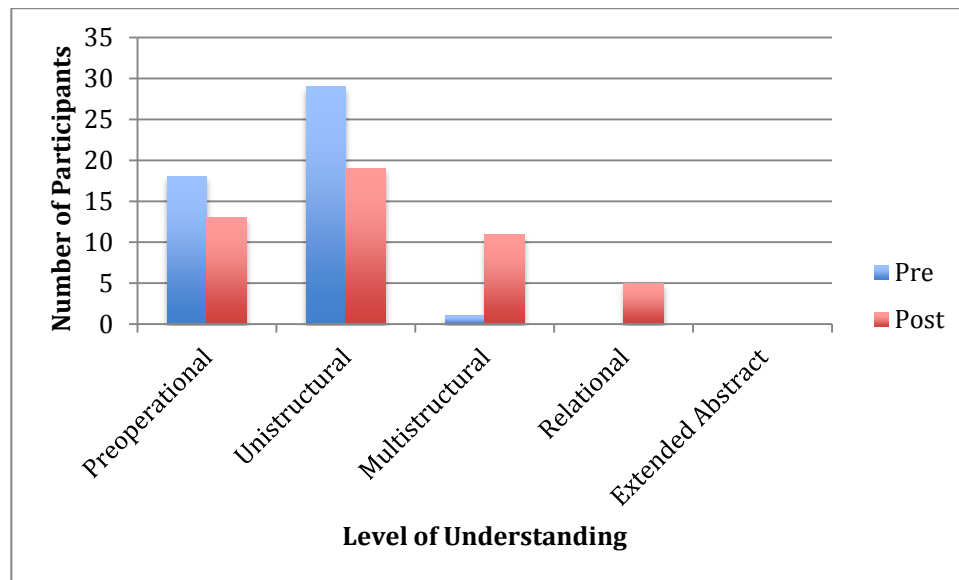


Figure 8. Assessment question #2 results.

An increase in understanding on Question 2 occurred for 45.8% ($n = 22$) of the participants; most of them ($n = 21$) increased one or two levels of understanding. The selected example in Figure 9 illustrates a participant who had a Unistructural level of understanding during the pre-assessment. At the post-assessment, the participant demonstrated a Multistructural level of understanding (Figure 10). On the pre-assessment, the participant responded with not being sure if the method Henry used was valid or not. On the post-assessment, the participant recognizes that Henry's method is not accurate; however, the participant does not give a complete explanation of why the missing partial products in Henry's response must be found.

2. Henry is trying to solve $1\frac{1}{4} \times 3\frac{3}{5}$; his work and thinking is shown below. Is Henry's method valid? Why or why not?

$$1\frac{1}{4} \times 3\frac{3}{5} = (1 \times 3) + \left(\frac{1}{4} \times \frac{3}{5}\right)$$

$$3 + \frac{3}{20} = 3\frac{3}{20}$$

$$\frac{5}{4} \times \frac{18}{5} = \frac{90}{20} = 4\frac{10}{20}$$

I'm not sure if Henry's method is valid, since he is switching from multiplication to addition.

Figure 9. Example of Unistructural level of understanding (pre-assessment).

2. Henry is trying to solve $1\frac{1}{4} \times 3\frac{3}{5}$; his work and thinking is shown below. Is Henry's method valid? Why or why not?

$$\frac{5}{4} \times \frac{18}{5} = \frac{90}{20} = 4\frac{1}{2}$$

$$1\frac{1}{4} \times 3\frac{3}{5} = (1 \times 3) + \left(\frac{1}{4} \times \frac{3}{5}\right)$$

$$3 + \frac{3}{20} = 3\frac{3}{20}$$

$$\frac{3}{4} + \frac{3}{5} + \frac{3}{20} = \frac{15}{20} + \frac{12}{20} + \frac{3}{20} = \frac{30}{20} = 1\frac{1}{2}$$

No, Henry also should have multiplied $1 \times \frac{3}{5}$ and $3 \times \frac{1}{4}$ and added those products as well as $3 + \frac{3}{20}$.

3. John is working on $\frac{5}{8} \div \frac{1}{2}$. He is unsure why you are allowed to invert and multiply when you divide these two fractions. Give an explanation to John about why this method works.

$$\left(\frac{4}{2}\right)$$

Figure 10. Example of Multistructural level of understanding (post-assessment).

Assessment question #3. On the pre-assessment, a majority ($n = 31$) of the participants scored at the Preoperational level of understanding as shown in Figure 11. For the post-assessment, most participants were still at the Preoperational level of understanding ($n = 28$) indicating very little change on this question overall. However, there was some progress among a subsample of the participants. Specifically, 22.9% of the participants demonstrated a Multistructural or Relational level of understanding on the post-assessment (versus only 1 participant on the pre-assessment). Since the nine participants who decreased on this question all moved down one level, just like on assessment question 1, we give our attention to the 16 participants who increased in their levels of understanding.

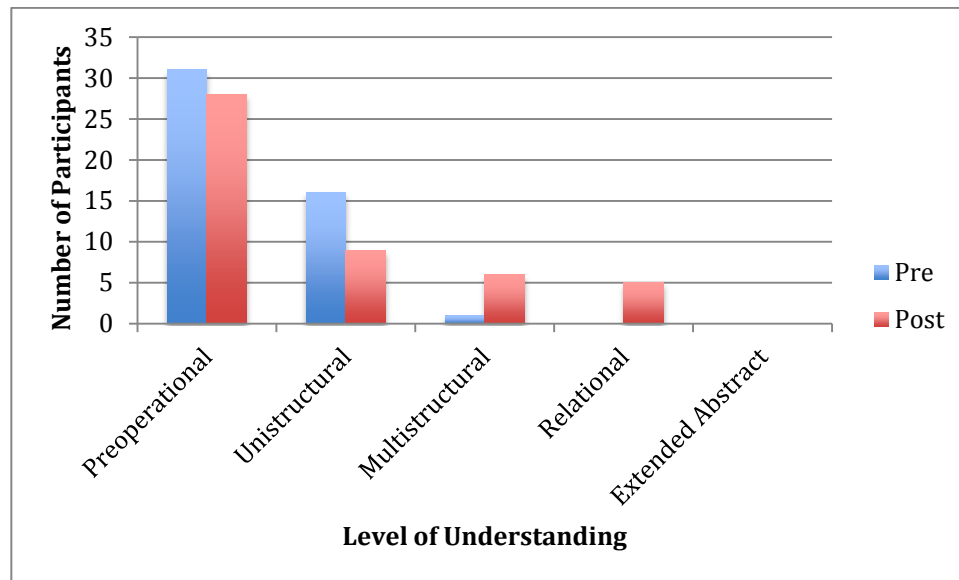


Figure 11. Assessment question #3 results.

Sixteen of 48 participants showed an increase in their levels of understanding on Question 3. Digging deeper, we determined that 8 of those 16 participants increased by one level of understanding, and 8 of the 16 increased by more than one level of understanding. Furthermore, of those that increased more than one level, 5 of the 8 participants ended at the Relational level of understanding. While this was only 10.4% of our sample, we provide an example of a participant who increased two levels of understanding and demonstrated a Relational level at the post-assessment. We have chosen to include the example in our findings to provide an understanding of the type of thinking displayed by this sub-group; we address the implications of this example in our discussion below. Figures 12 and 13 illustrate a participant who moved from Unistructural to Relational. On the pre-assessment, the participant provides an answer for the problem, but it mostly is a worded version of the algorithm. On the post-assessment, the participant uses the popcorn example from the methods class to explain the numbers in the problem conceptually.

3. John is working on $\frac{5}{8} \div \frac{1}{2}$. He is unsure why you are allowed to invert and multiply when you divide these two fractions. Give an explanation to John about why this method works.

Inverting allows you to change the problem to a multiplication problem. Multiply is the inverse of division so with fractions you can invert one number to make this easier to solve.

Figure 12. Example of Unistructural level of understanding (pre-assessment).

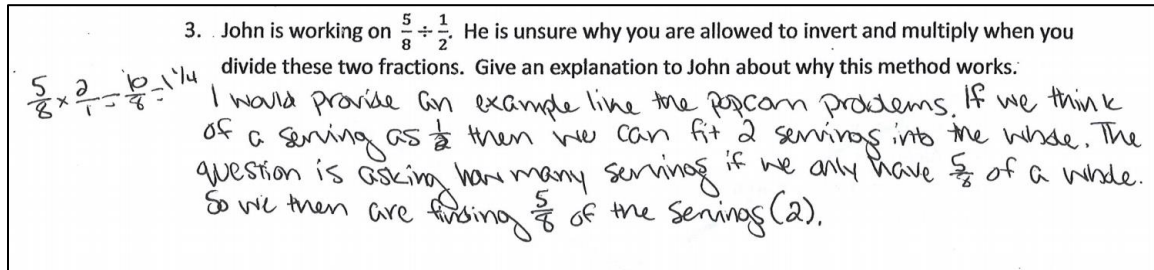


Figure 13. Example of Relational level of understanding (post-assessment).

Assessment question #4. The pre-assessment had exactly one-half ($n = 24$) of the participants scoring at the Preoperational level of understanding (Figure 14). For the post-assessment, most participants were either at the Preoperational level ($n = 19$) or Relational level of understanding ($n = 18$). This finding is quite interesting since most participants are split between two very different levels of understanding at opposite ends of the scale (We refer to Relational as the “end” here since we did not use Extended Abstract in our coding).

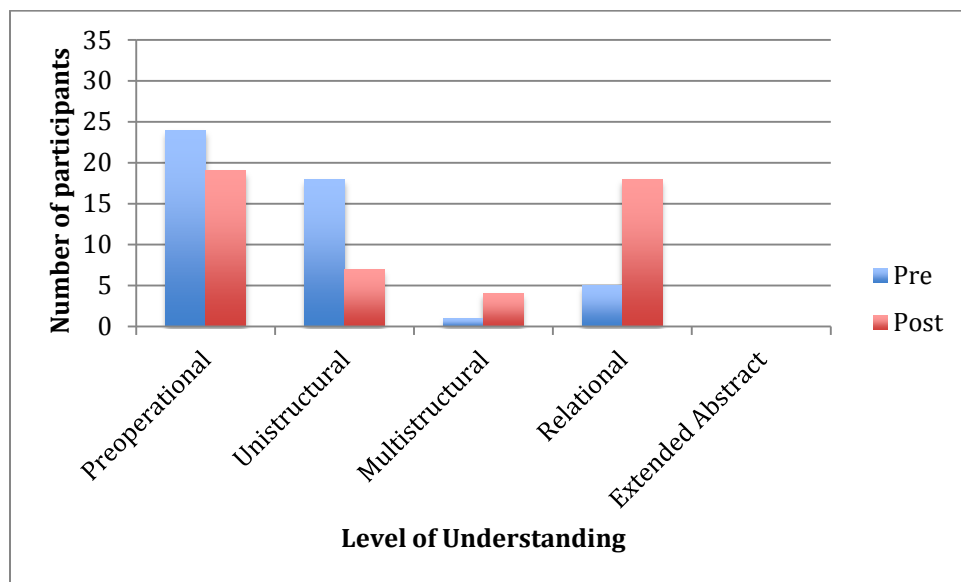


Figure 14. Assessment question #4 results.

Of the 21 participants (43.8%) who increased in their levels of understanding, 15 of them increased by two or three levels. In fact, nearly all of them (14 of 15) demonstrated a Relational level of understanding on the post-assessment. In Figure 15, a participant's work shows a Preoperational level of understanding on the pre-assessment when the participant incorrectly says that Abby is correct in her solution. However, on the post-assessment (Figure 16), the participant correctly identifies that the $\frac{1}{4}$ leftover should be considered in relation to the $\frac{1}{2}$ piece because $\frac{1}{2}$ is the unit in this case, displaying a Relational level of understanding.

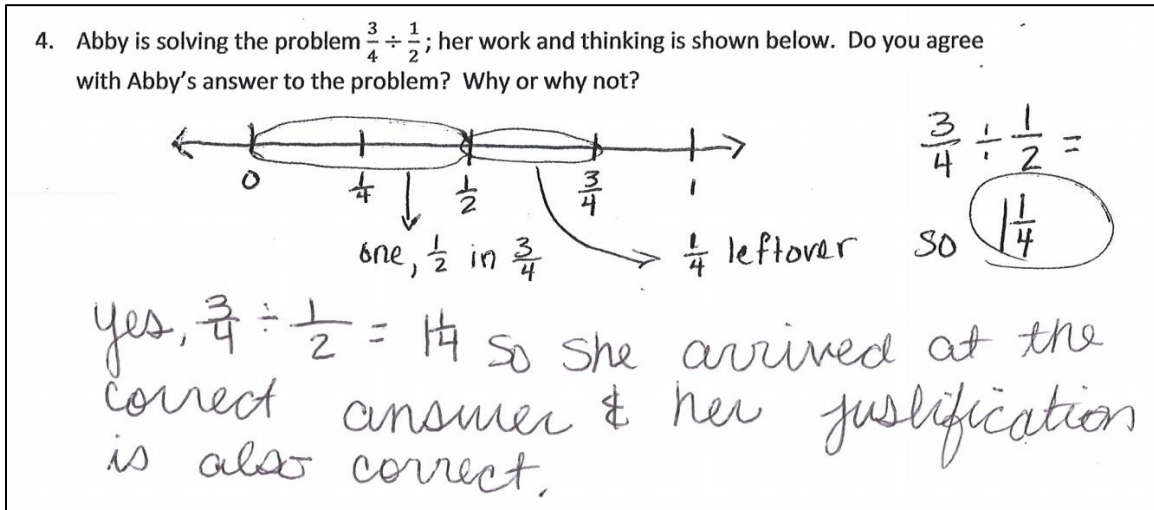


Figure 15. Example of Preoperational level of understanding (pre-assessment).

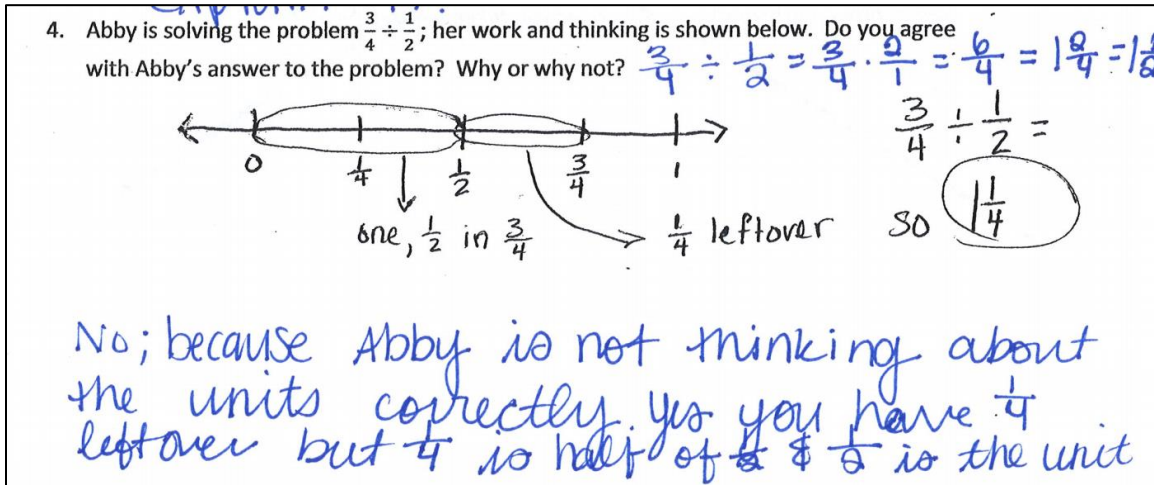


Figure 16. Example of Relational level of understanding (post-assessment).

There was a relatively small group of participants who decreased in their understanding (5 of 48), and three of them decreased by one level. However, we draw attention to the other two participants because they moved from either a Multistructural or Relational level of understanding on the pre-assessment, to a Preoperational level on the post-assessment. We highlight this type of change (i.e., more than one level) because it is important for us to understand what happens when someone decreases more than one level in understanding. This only occurred a total of 4 times in our data (2 participants on Q1 and 2 participants on Q4), but in a small sample size of 48 PSTs, it is important that we are attentive to these outliers. We focus on the participant who moved from a Relational to Preoperational level, dropping three levels in understanding according to the coding rubric (Figures 17 and 18). On the pre-assessment, the participant understood that they were trying to find how many $\frac{1}{2}$'s were in $\frac{3}{4}$. They also realized that the $\frac{1}{4}$ leftover is half of the $\frac{1}{2}$. In contrast, on the post-assessment, the participant incorrectly identified the leftover piece to be $\frac{1}{3}$. While the participant seems to recognize the notion of a "new whole" in regards to the answer, it appears the participant is using $\frac{3}{4}$ as the unit, rather than thinking about how many halves fit in the $\frac{3}{4}$, since $\frac{1}{4}$ is $\frac{1}{3}$ of $\frac{3}{4}$.

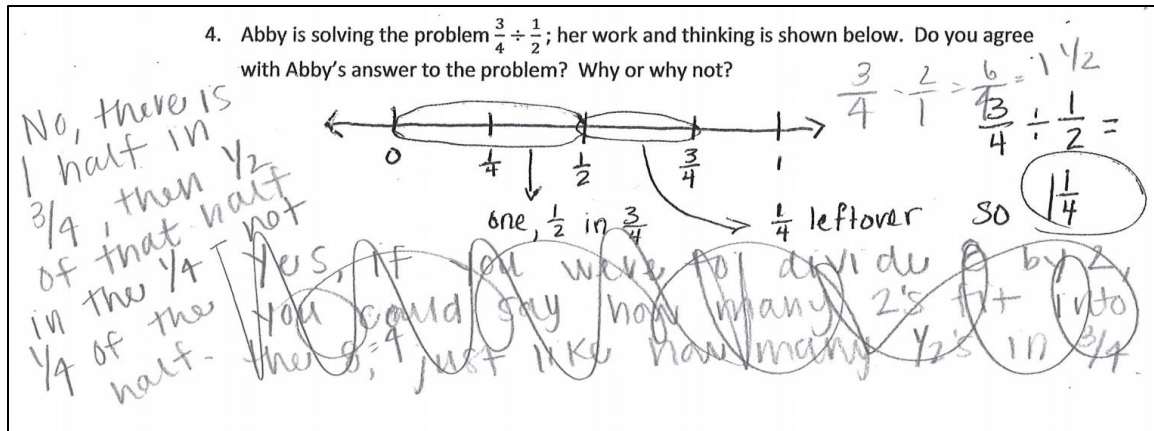


Figure 17. Example of Relational level of understanding (pre-assessment).

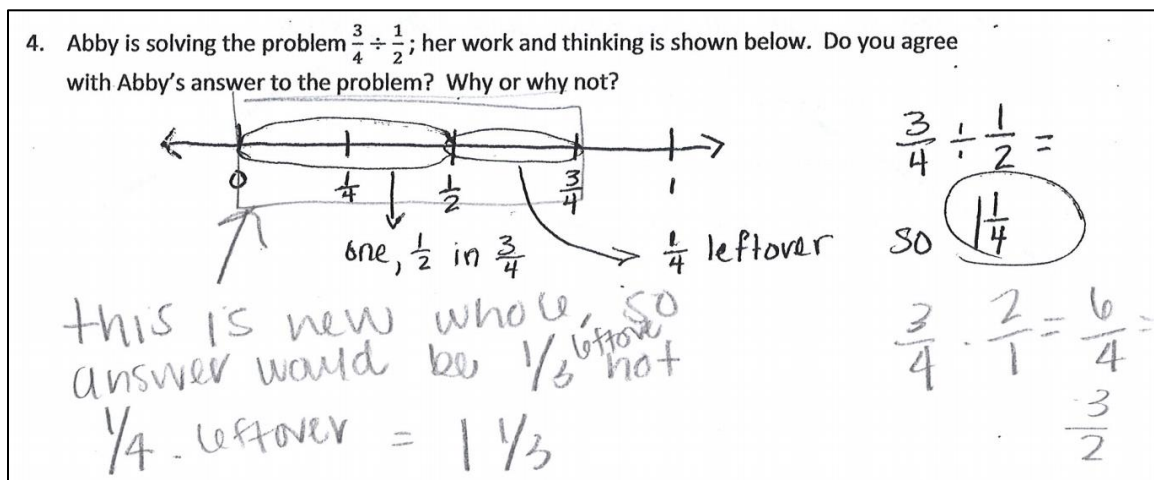


Figure 18. Example of Preoperational level of understanding (post-assessment).

Discussion

This study aimed to fill a void in the literature in examining how PSTs change in their understanding of fraction multiplication and division while enrolled in a mathematics methods course. Below we summarize the findings for each of the two research questions and simultaneously highlight the potential implications for practice and research in teacher preparation in elementary mathematics. We also outline study limitations.

The first research question examined whether the PSTs changed in their levels of understanding of fraction multiplication and division. The overall scores and the individual item scores from pre- to post-assessment indicated a statistically significant change on all questions except question 3. This indicates the five-week instructional unit had an impact on overall understanding of fraction multiplication and division, which in turn means a deeper specialized mathematical knowledge (Ball, Thames, & Phelps, 2008). However, the mean on the post-assessment for all questions and the total assessment was somewhere in the range of Unistructural to Multistructural (ranging from 0.75 to 1.44). Clearly, the goal would be to move all PSTs to at least a Relational level of understanding on unpacking traditional algorithms and

diagnosing/explaining student errors, that is, to be further along on the continuum of relational understanding as outlined in our theoretical framework (Skemp, 1976). This finding augments past research that speaks to the need for more time in teacher preparation programs on developing understanding of fractions (e.g., Newton, 2008). In this study's context of a five-week instructional unit, it is plausible that the amount of change that could take place in such a short duration of time is limited.

For our second research question, we conducted a deeper examination of our data to understand how the PSTs had changed in their understanding as demonstrated on each assessment question. On the first assessment question focused on the traditional multiplication algorithm and why it works, nearly half of the PSTs demonstrated an increase in understanding. Like the entire sample of PSTs, most of the ones who improved started at the Preoperational level of understanding so it was not surprising to find changes in understanding among this group who started with the lowest level of understanding. Unfortunately, changes in improvement were limited to one or two levels of understanding. Furthermore, it is important to recall from Table 5 that just over one-third of the participants did not improve on this question. When we debriefed as an instructor group the findings on this assessment question in relation to our course session activities, we noted our lack of sufficient attention to why the algorithm worked. While we spent considerable time exploring why, for example, $\frac{1}{2}$ of $\frac{3}{4}$ is $\frac{3}{8}$, we did not explicitly tie their new understanding back to "why we multiply straight across." Going forward, we have made revisions to our instructional unit to allow time for these connections to be explicitly made, not assuming students are going to make those connections on their own. This type of attention to teacher preparation curriculum is significant. Researchers have recently been involved in this type of work (Callis & Chapin, 2015), and our findings speak to the importance of the continued examination of teacher preparation curriculum and implementation as it relates to the development of PSTs' understanding and knowledge (Ball, Thames, & Phelps, 2008) of mathematical ideas.

Nearly half of the participants improved in their understanding on assessment question 2 focused on a student error when multiplying mixed numbers. For the ones who improved, most of them demonstrated a Multistructural or Relational level of understanding on the post assessment. While this finding is promising, we again engaged in discussion as an instructor team about why more than half of the PSTs did not demonstrate an improved understanding. We had given explicit and intense attention to the partial products approach for multiplying whole numbers in a rectangular area model (e.g., $24 \times 36 = (20 \times 30) + (4 \times 30) + (20 \times 6) + (4 \times 6)$), an example of specialized content knowledge for teachers (Ball, Thames, & Phelps, 2008). Subsequently, we gave it very quick attention during the sessions on multiplying fractions, but much of our focus was on multiplying fractions less than 1. Increasing the amount of time for the development of mathematical ideas seems urgent (Chinnappan & Forrester, 2014) to allow for connections to be made across mathematical strands (e.g., whole numbers and fractions).

On assessment question 3, focused on making sense of the "invert and multiply" algorithm for division, one-third of the PSTs demonstrated an improved understanding; however, this was not a statistically significant change. Half of those demonstrating improvement (or 8 PSTs) were at a Multistructural or Relational understanding on the post-assessment; they typically leveraged the contextual example used in class (See Appendix for Popcorn Problem) to make sense of the traditional algorithm just as the participant did in the sample work displayed earlier in the paper (Figures 12 and 13). Our findings are both expected and promising. Past research has shown that the division algorithm is challenging for elementary teachers to explain

(Li & Kulm, 2008, Ball, 1990), making our findings expected. However, since the division algorithm has been such a source of challenge in past studies, we found it promising that one-third of our participants made progress. Our findings and the difficulty of the topic warrants more time needed in the course in order to fully develop the ideas conceptually.

Similar to the first two assessment questions, nearly half of the PSTs demonstrated improvement on the assessment question 4 focused on diagnosing and making sense of a student's erroneous approach to dividing fractions. However, in this case, a larger number of participants ended at the Relational level of understanding. As explicated in the results section, there was a split in understanding on the post-assessment (19 at Preoperational, 18 at Relational). It seems important to investigate why some students made such progress on this assessment item, even jumping 2 or 3 levels in understanding, and why other students made no progress. Perhaps, students' past experiences and prior knowledge can explain this almost dichotomous look of the results. There was explicit attention to this type of error in the methods course on the same day of the post-assessment, a limitation of this study; however, it would still be important to examine PST characteristics that might explain differences in performance. It is plausible that students just need more time to grapple with the ideas and to make sense of the mathematics conceptually; learning is a process that takes time (Brown, Collins, & Duguid, 1989).

Nearly all of the PSTs in this course (98%) had taken some form of calculus before enrolling in their methods coursework. Although their level of mathematics taken was beyond what they would need to teach elementary school mathematics, this did not improve their explanation of common fraction algorithms. We know from previous studies that more mathematics does not improve conceptual understanding (Ball, 1990); however, a further examination of whether courses such as Calculus for Elementary Teachers actually improves algorithm justification would be beneficial.

Overall, the course itself was not designed for the students to do formal proofs, but rather to explain algorithms to elementary students in such a way that made sense conceptually. Therefore, it was not expected that PSTs would fall into the Extended Abstract category and thus was confirmed by none of the answers to the assessment being coded as such. Questions on the assessment also did not lend themselves to having PSTs provide explanations in such an abstract way; however, if the questions were reworded more generally or if teachers were probed in interviews, some may have moved from Relational to Extended Abstract in terms of their understanding.

It is important to acknowledge two limitations of this study. First, since the study was exploratory in nature, we utilized researcher-created assessment items. Perhaps, different wording of questions would have resulted in more detailed answers from students (e.g., "explain your answer using pictures and words" or "use more than one strategy to show why this algorithm works"). Second, the timing of the post-assessment is not ideal for a couple of reasons. The assessment taking place during a class session could have influenced the results on the last question of the assessment, in particular, since there was explicit discussion about units in the context of fraction division during that class session. Also, the post-assessment day coincided with the last day of class for the semester, but in future work, giving a delayed post-assessment would give the PSTs more time to make sense of the mathematical ideas and to show what might be a truer measure of their understanding. Additionally, it would be beneficial to administer the post-test again after PSTs gain more experience during their student teaching to understand continued changes, if any, in fraction understanding. Finally, it is important to also understand which components of the intervention contributed, or did not contribute, to deepening

understanding. Future studies should focus in on an in-depth analysis of the teaching unit itself, as well as focus on how teacher educators select and implement tasks in mathematics methods courses with PSTs.

The results from this study hold true with the results of Ball's (1990) study, such that elementary PSTs' "notions of mathematical explanation seemed to mean restating rules" (p. 138). By strengthening PSTs' conceptual knowledge of mathematics and helping them form solid justifications as to why procedures work, the difficulties secondary students face when they are abruptly introduced to proof in the upper grades may be diminished (Stylianides et al., 2007). This study only scratched the surface of trying to understand the impact of instruction on PSTs' understanding of mathematical ideas. More research is needed to understand the features and structure of elementary teacher preparation programs that afford or constrain PSTs' opportunities to deepen their understanding of mathematics. Overall, the findings in this study showed evidence of the instructional unit having impact on PSTs' understanding of multiplication and division of fractions. Therefore, instruction on content and pedagogy relative to fractions seems to be an important first step in improving PSTs' understandings, hopefully with a trickle-down effect to instructional practices in elementary classrooms.

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Appendix A – Sample Course Tasks

+ Playground Task

■ Carol Gardens

■ Lot is 50 yds x 100 yds

■ $\frac{2}{5}$ will be playground

■ $\frac{3}{4}$ of the playground will be blacktop

■ Flatbush

■ Lot is 50 yds x 100 yds

■ $\frac{3}{4}$ will be playground

■ $\frac{2}{5}$ of the playground will be blacktop

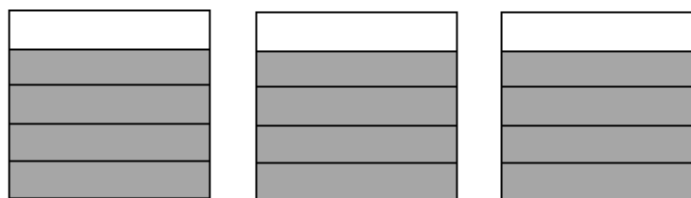
Where are students likely to find more blacktop space? Use models and pictures to represent your solution.

From *Young Mathematicians at Work* (Fosnot & Dolk, 2002)

1. Maisey draws a picture like the one shown to depict $3 \cdot \frac{4}{5}$. Maisey concludes from her picture that

$$3 \cdot \frac{4}{5} = \frac{12}{15}$$

Because 12 pieces out of 15 are shaded. Is Maisey right? If not, where is her reasoning flawed?



Adapted from *Mathematics for Elementary Teachers* (Beckmann, 2012)

Popcorn Problem #1**Name** _____

At the Carolina Popcorn Shoppe on Western Boulevard in Raleigh, the managers direct their employees as to how much popcorn constitutes a serving when a customer orders one. There are currently 6 cups of cheese popcorn at the serving counter. How many servings can be made if the customer is given:

- a. $\frac{1}{2}$ cup of cheese popcorn?
- b. $\frac{1}{6}$ cup of cheese popcorn?
- c. $\frac{1}{3}$ cup of cheese popcorn?
- d. $\frac{2}{3}$ cup of cheese popcorn?
- e. $\frac{5}{6}$ cup of cheese popcorn?

Solve this problem using pictures or another tool.

Can you write a mathematical equation to represent your model?

Describe the pattern that you notice, if any, across the sub-questions a-e.

Appendix B – Pre/Post Assessment

ELM 410

Name: _____

1. Sarah is unsure why when you multiply two fractions such as $\frac{2}{3} \times \frac{1}{2}$ that you are allowed to multiply straight across. Give an explanation to Sarah why this method works.

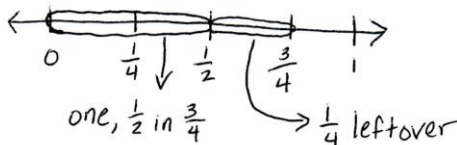
2. Henry is trying to solve $1\frac{1}{4} \times 3\frac{3}{5}$; his work and thinking is shown below. Is Henry's method valid? Why or why not?

$$1\frac{1}{4} \times 3\frac{3}{5} = (1 \times 3) + (1\frac{1}{4} \times \frac{3}{5})$$

$$3 + \frac{3}{20} = 3\frac{3}{20}$$

3. John is working on $\frac{5}{8} \div \frac{1}{2}$. He is unsure why you are allowed to invert and multiply when you divide these two fractions. Give an explanation to John about why this method works.

4. Abby is solving the problem $\frac{3}{4} \div \frac{1}{2}$; her work and thinking is shown below. Do you agree with Abby's answer to the problem? Why or why not?



so $\frac{3}{4} \div \frac{1}{2} = 1\frac{1}{4}$