A design study for an Italian fifth grade class following Davydov traces

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We present a design study developed in an Italian school. Taking inspiration from the work of the Russian psychologist V. V. Davydov, we have reformulated some activities of his curriculum for the first grade, in order to adapt them to a didactic project for a fifth grade class. In the paper we firstly expose our theoretical assumptions and the hypotheses that stand at the roots of the project. Then, after a brief description of the entire educational path, we present some excerpts of it in order to analyse the processes by which the pupils give sense to algebraic language. Finally we argue that Davydov's ideas anticipate the claims of Early Algebra, nowadays very popular within the research community.

Introduction

During the last years, the community of researchers in mathematics education has acknowledged the importance of the teaching of algebra since the early years of school education (see, e. g., the recent surveys Cai & Knuth, 2011, and Kieran, Pang, Schifter, Fong Ng, 2016;, and references therein). In Italy, algebra is officially introduced at the sixth grade, substantially as a set of rules of symbolic computation, but the last ministerial curricula urges to take care of relationships since kindergarten. Indeed one of the goals at the end of 5th grade is "Riconoscere e descrivere regolarità in una sequenza di numeri o di figure" (Recognise and describe regularities in a sequence of numbers or pictures) (MIUR, 2012, p. 63). Our research team has been working on early algebra for many years (see e.g. Iannece, Mellone & Tortora, 2010; Mellone, Spadea, Tortora, 2014). During this period it has been very illuminating for us to encounter the work of the Russian psychologist V. V. Davydov rooted in Vygotsky's sociocultural reflections, (in particular, with his paper Davydov, 1982), where we have found a strong consonance with our way of working, to the effect that now we can say that Davydov's ideas, constitute one of our main sources of inspiration. In his paper, Davydov proposes activities for the first grade, aimed at the recognition of arithmetical and ordering relationships between quantities, starting from the observation of different volumes of water contained in two identical cylindrical containers. From this, the understanding and the ability to manipulate algebraic expressions follows directly. The main shared idea between Davydov's proposals and our previous practices is that school activities about relationships between physical quantities can be powerful prompts for young pupils toward the acquirement of algebraic skills. But we also look at the domain of arithmetical manipulations as an invaluable context to recognise structural relationships between quantities: therefore we believe that every sort of activity with numbers should be mastered by pupils for this kind of exploration rather than as mere executions of algorithms.

Until now, we have implemented some didactic paths inspired by Davydov's work with several teachers of our country. In particular we have been studying and discussing with kindergarten and primary teachers of our research group the contents of (Davydov, 1982 and 1992) and, together with them, we designed educational paths for kindergarten (see e.g. Mellone et al., 2014) and for primary school inspired by them. During the implementation of these paths we had the opportunity to appreciate the ability of pupils to recognise additive and multiplicative relationships in several situations, by expressing them in the form of spoken and written language, but also through objects, manipulations and drawings.

In this paper we present one of these experiences developed in a fifth grade class in Naples. It is important to notice that our research work is not an attempt to transfer a part of Davydov's curriculum into an Italian class, but rather a design of a mathematics education path for an Italian fifth grade class in the light of Davydov's study. In particular, we were fascinated and provoked by his proposal to overtake the classical conception of number, by putting quantities at the very base of mathematical knowledge (Davydov, 1982). Indeed, we are aware of the differences between the cultural context and education needs of Davydov's research and the present Italian context (Mellone & Ramploud, 2015). In particular in Italy pupils have met numbers and numerals since kindergarten, while the path described by Davydov (1982) avoids the use of numbers, at least until the first grade. Altogether, experiencing the design and implementation phases of our work of adaptation of Davydov's proposals for the teaching of algebra to the Italian context, turned out for us as an opportunity to rethink some important aspects of the teaching of arithmetic and algebra in primary school.

In the sequel we will firstly clarify the theoretical assumptions that underlie our design study. Then we will describe the three main phases of the experimental path, emphasising the peculiarities of our didactical choices in comparison to Davydov's curriculum. For each phase we will present some salient excerpts of the course and some pupils' behaviours in which we can see different ways to focus/grasp and express algebraic relationships. In the last section, we will stress the importance (and suggest the possibility) to create a special educational environment in which the algebraic symbols can be introduced to express relationships between quantities, but afterwards may be developed to denote abstract relations.

Theoretical Framework

Basically, our design study followed a socio-cultural approach in which learning is conceived as a refinement of linguistic skills always deeply intertwined with the development of thinking and awareness (Vygotsky, 1978). Our approach can be placed within the mathematics education research stream that tries to catch the complexity of the individual thinking as materialised in the body (by means of gestures, facial expressions, eyes movements, etc.) and in the use of signs and artefacts (Radford, 2011). In particular, we rely on the following Radford's tenet

> Learning consists of positioning oneself reflectively and critically in historical forms of action and thinking. Functionally speaking, learning is conceptualised in terms of processes of objectification—i.e., activity bound social processes through which the students encounter and grasp the historically constituted forms of action and thinking (Radford, 2010, p. 73).

In this view, as opposed to mental cognitive approaches, thinking is not considered something that just happens 'in the head', but rather it is conceived as a social practice rooted in bodily movements and in the use of signs and artefacts (see e.g. ibid.)

According with the previous perspective, the analysis of pupils' behaviours that we will present in the sequel, makes use of the notion of semiotic nodes, "pieces of the students' semiotic activity where action, gesture, and word work together to achieve knowledge objectification" (Radford, Demers, Guzmán and Cerulli, 2003, p. 56).

We look at algebraic thinking as a kind of "discourse" (inter/intrapersonal) in which we can recognise attention to structure and relationships:

[…] learners exhibit this type of attention when they begin to focus on what stays and what changes – that is, "becom[e] accustomed to considering invariance in the midst of change" (Mason, Stephens and Watson, 2009, p. 13).

As is well known, Bourbaki (1974) distinguishes three basic types of mathematical structures: the algebraic, the order and the topological ones. In particular, he expresses his vision of Algebra in this way

> It is no doubt the possibility of these successive extensions, in which the form of the calculations remained the same, whereas the nature of the mathematical entities subjected to these calculations varied considerably, which was responsible for the gradual isolation of the guiding principle of the modern mathematics, namely that mathematical entities in themselves are of little importance; what matters are their relations. (Bourbaki, 1974, p. xxi).

According with this idea, the symbolic systems and the transformations rules of the algebraic language emerge from structural relationships among elements whose nature has little importance. When the relationship is seen as instantiation of a property, the relation becomes (part of) a structure seen in terms of an agreed list of properties taken as axioms, from which other properties can be deduced (Mason et al., 2009).

The process of recognising a particular relationship among elements of a set as instantiation of a general property, that means “seeing the general through the particular”, is a core process in the learning of mathematics (Mason, 1996). It is strictly linked to the experience of ‘example-hood’, which allows the learners to recognise how different cases are subsumed under a general law. The students recognition of a single fact as an example requires that they grasp the sense of what the example expresses, the enhancement of the features that make it ‘exemplary’ and the darkening of the features that make it particular. We agree with Davydov that observation and modelling of the relations between continuous quantities can be a rich and effective example-hood context for the algebraic relationships, provided it is inserted in an accurate educational path. Indeed, as we claimed in the Introduction, our work is prompted by Davydov studies, where features and potentialities of experiencing manipulation of quantities are deeply analysed (Davydov, 1975, 1982). In these studies the thesis is advanced that at the root of the algebraic structures, and of the concept itself of number, there is the notion of physical quantities. Nowadays this crucial idea is finding many followers all around the world (see for example Slovin & Dougherty, 2004; Iannece, Mellone & Tortora, 2010; Schmittau, 2011; Polotskaia, 2015). From this thesis, Davydov derives the necessity to let pupils work with quantities since the very beginning of the school path, by making manipulations on them and discovering relations and properties, rightly before working with numbers. Our proposal of combining Davydov’s curriculum with the Radford theory of objectification arose from the hypothesis that the activities proposed by Davydov can support pupils in the process of the objectification of the algebraic language. In other words, we believe that the exploration of suitable contexts with continuous quantities can support pupils making their own sense of the algebraic language that embodies historical forms of thinking (Radford, 2010).

According with this approach, pupils firstly discover the properties of quantities in concrete materials, then they learn how to record these properties by means of representations and algebraic symbols, and finally, via suitable teaching mediations, they accomplish a mathematical analysis of the relationships. In this perspective, the additive structure is rooted in the experience of comparing two different quantities and of trying to make them equal by means of one of two actions: either to decrease the bigger one (subtraction) or to increase the smaller one (addition). In this situation the crucial point is that in both cases the quantity to be added or subtracted is the difference between the two initial quantities. To support very young pupils in really understanding these relationships the role of bodily handling is gradually reduced:

> To prepare children for this shift an intermediate strategy of graphic representation is used. The children represent the physical quantities with two segments A and B […] Line segment A is

superimposed on line segment B. The difference, expressed in the form B – A is defined as being equal to x (Davydov, 1982, p. 234).

The author suggests to use a spatial metaphor (Figure 1), which allows you to visualise the relationships between the two quantities A and B and the role of the quantity x.

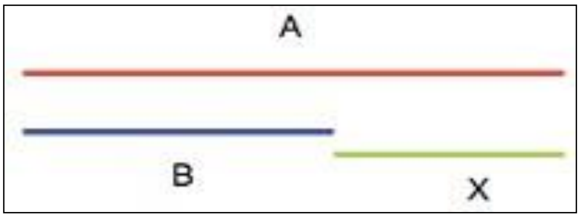


Figure 1.

This representation is employed in several school practices all around the world and is known under different names. For example it is known as pictorial equation in the tradition of many East Asia countries where it is used since the first years of primary school (Cai & Knuth, 2011). The name reveals a particular feature of this representation: namely, it can be used to interpret first grade equations in terms of relationships between quantities (for a different graphical model of equations, see Filloy and Rojano, 1989). Indeed, the solution of such an equation can be managed by suitably using the representation. Looking at the example of Figure 2: the quantities described by the two members of the equation are represented by two segments of the same length; then the representation is manipulated to match the two outer segments, whereby it is possible to read the value of the unknown as the length of the middle segment. In Figure 2 the manipulation of the segments is compared against the usual algebraic steps, to show the big difference between the two approaches.

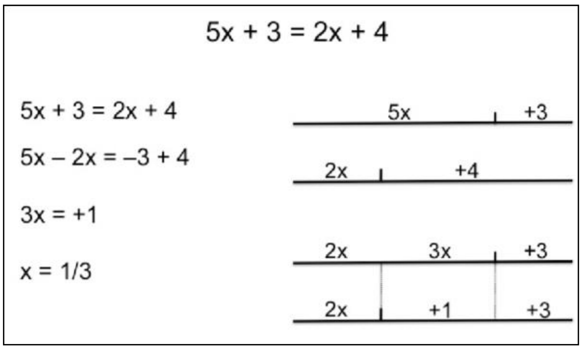


Figure 2

In Italian schools this representation is used from the 6th up to the 8th grade and completely disappears in the following grades. Indeed in Italy the use of this kind of representation is present in many textbooks for lower secondary schools, in particular it is proposed in the solving process of word problems, modelled by first grade equations. At this school level, it is used as a support for the introduction of the algebraic language. On the contrary, in higher secondary school, linear equations are presented as formal exercises to solve, without any reference to external contexts or graphic representations, like the one in Figure 2. We (Iannece et. al, 2010) have given this schema the name of Spatial Metaphor of Additive Structure (SMAS in the following) to underline the crucial role of the spatial category for the additive structure. The activities of exploration of relationships between quantities are supported by the use of such "intermediate strategy of graphic representation" (Davydov, 1982, p. 237), which have special perceptual and holistic features and become a strategical step toward the goal of catching structures and starting to use a formal/scientific language. Indeed, a graphical representation can play both as a cognitive support to generalise from numerical cases and as a perceptual metaphor of structural aspects of abstract entities, as suggested by Davydov. So, we agree with the researchers who acknowledge the importance of graphical models, either in general (for example Filloy and Rojano, 1989), or along Davydov's lines (like for example Schmittau, 2011; Slovin & Dougherty, 2004), claiming that experiencing this kind of representation is for pupils a good start and a powerful support for the learning of algebra. According with Davydov, we think that the SMAS, helps to reflect upon the additive relationships and to link the concrete experience with the algebraic representation. But in this study we want to show one more thing, namely the advantage of introducing and exploring this kind of representation tool in a class environment where the collective discussion is systematically employed, in order to obtain a refinement of students' reasoning ability, also due to their increasing awareness in understanding and using the algebraic language.

Indeed, the development of a proper use of algebraic language by students is a crucial issue in itself that is widely studied in literature. This acquisition is a very long and complex process, which can be connected with the analogous process by which the mastery of natural language develops, as many authors underline (see e.g., Cai & Knuth, 2011). As we will notice in the sequel, it is often possible to recognise in pupils' interventions different levels of linguistic competence, as well as their steps and results in managing symbols. In this perspective, we have found interesting in our study the notion of algebraic babbling (Malara & Navarra, 2003), a metaphor to describe the first naïve use of algebraic language by pupils, when improper uses or errors frequently occur.

Our research hypotheses and the design of our work

Our experimental work can be classified as a design study for its ecological and engineering features (Cobb, Confrey, diSessa, Lehrer and Schauble, 2003), in that the pupils have been engaged in an active process of sense-making of the algebraic language.

The work has been carried out in a fifth grade class that had completed a relatively traditional mathematics curriculum up to that time. The opportunity to propose a different approach to mathematics was provided when an expert teacher of our research group replaced the previous teacher. Many activities have been proposed and, in particular, those that Davydov devised for first elementary grades, that is, when children have not yet started to use numbers (Davydov, 1982). The reason why we proposed these activities to pupils that already know numbers, after spending four years in the practice of arithmetic operations, comes from the assumption that working on quantities contributes to the development of algebraic thought both before and after having explored numbers and arithmetic calculations. In our opinion, Davydov's theoretical reflections and their teaching consequences (where the classical conception of number is overtaken, by putting quantities at the very base of mathematical knowledge) may be intertwined with a more traditional path where also numbers are used since the beginning. In this way the mastery of the relationships between quantities does not necessarily precede the use of numbers at school, rather it supports the numerical knowledge and the comprehension of the arithmetical operations even if explored later, by promoting argumentations based on the properties of relations. Indeed, the simplistic view that identifies an "ontogenetic" one-way path from the arithmetic to the algebraic thought has been, by now, doubted (see e.g. Radford, 2010), while a more complex relationship has been suggested.

So, after a traditional course, in which the use of algebraic language was restricted to the formulas for the area and the perimeter of regular polygons, in the final year of primary school this group of pupils were engaged in the use of the algebraic language as a meaningful language rooted in the observation of relationships between quantities. The course shifted from working on concrete quantities to the use of algebraic language and the SMAS, up to the re-discovery of the underlying algebraic structures, supported by the use of those representations in problem solving activities.

The path was carried on during the whole school year at a rhythm of two hours per week and, time by time, it was revised during monthly meetings between the teacher and the whole research team. Moreover, for the entire duration of the time, a researcher of our team was in the classroom in order to collect data (audio and video records, pupils' representations) trying not to intervene in the educational activity. Of course in the year the class have also dealt with other mathematical activities, but the teacher has always cared that the contents of our didactic proposal and the rest of their daily mathematical work be closely intertwined and reciprocally supporting.

The path can be divided into three phases:

- Observation, description and graphical representation of simple equalities and inequalities between continuous physical quantities like water (Davydov, 1982). This phase was designed in order to favour a first use of algebraic language requested by the observation and manipulation of quantities, whose educational potential has been discussed in the previous section.
- Collective discussion oriented to building the SMAS as support to the use of algebraic language. This represents the point of innovation compared to the courses proposed by Davydov and from our point of view it is a crucial moment in order to work on pupils' reflections and on the growth of their awareness. The group discussion represents a necessary component of the shared construction and use of the SMAS, according to the social feature of our learning vision (see e.g. Radford, 2010).
- Identification and recognition of structures in several problematic situations, and the use of graphical and algebraic representations for solving them. This phase is meant both as an integral part of the development of algebraic thinking, and as a moment of verification allowing us to trace the effects of the course on pupils' reasoning. It also represents an innovation compared with Davydov's assumption that looks at 'the general' as prior to 'the particular' (rather than 'general-through-particular'), as the main vehicle of development of theoretical thinking in children. Moreover, the additional contexts offered by the word problems, are meant as opportunities for pupils to experience the effectiveness of the recognition of relationships among quantities in problem solving activities.
- In the following section we will describe the three phases of the experimental path, emphasising the peculiarities of our didactical choices in comparison to Davydov's curriculum. For each phase we will present some salient excerpts of the course. In particular we will show some pupils' behaviours and the consequent choices in didactic planning. Our main concern is to trace pupils' algebraic thinking during the path, in particular we are interested in observing and analysing the graphical representations used by pupils and their synergies with natural and algebraic language. Moreover we also want to recognise semiotic nodes (Radford et al., 2003) in order to reflect on their features and specificities.

Traces of pupils' algebraic thinking in the three phases of the experimental path

Phase 1

Following Davydov's theoretical layout, the children were given a task that involved the description and analysis of equalities and inequalities detectable from the observation of continuous quantities. To this end three identical containers, filled with different volumes of water were placed on the desk (fig. 3).

Figure 3.

The activity designed by Davydov for first-graders (Davydov, 1982) considered only two containers with different amounts of water. But for our group of fifth graders we preferred a richer situation. There was no scale on the containers, just as in the original Davydov's proposal, to avoid the use of numbers; moreover, in our case, since numbers were well familiar to the pupils, the teacher explicitly discouraged their use, saying to use the letters A, B, and C to express the relations of equality and inequality. In fig. 4, the page of a notebook of Giulia, one of the pupils, shows a drawing of the three containers. In the same page, just below the drawings, Giulia writes some inequalities and an equality where the letters A, B, C refer to the quantities of water in the containers. The drawing and writings of the pupil's notebook were reported on the blackboard and used as a prompt for a first discussion. The discussion was opened by teacher's request to observe the containers and the expressions on the board and to look for other relationships between the quantities of water.

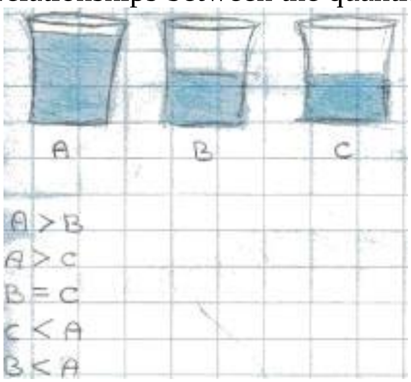


Figure 4.

Giulia: Miss, we only know that those two, B and C, are filled to the same level with the same amount of water, while the first has more water.

In Giulia's words ("B and C are filled…") the letters seem to be used as labels for the containers, while in the (in)equalities, as already noted, they play the role of variables. The use of the same letters for two different things constitute a symptom of a still immature use of the algebraic language. As we already discussed at the end of the section on theoretical framework, these still unsophisticated uses of the letters are predictable and have long been studied (see Malara and Navarra, 2003).

In the development of the discussion we could observe similar shifting of the meaning of the letters and even notice new subtle differences in the ways they are used

> Giuseppe: I know one more thing [referring to the relations written in fig. 4]: A plus B plus C minus B and C is equal to A [Giuseppe goes to the board and writes "A + B + C - B + C = A"]. If we sum A, B and C we get a certain quantity of water, then if we subtract B and C we get A again.
>
> Murmurs of disagreement
>
> Giuseppe: When I said "and" I meant that B and C should be subtracted together.
>
> Teacher: But did you write on the board exactly what you said in words: "A plus B plus C, minus B and C, is equal to C"? What should you write to show what you actually mean?
>
> Giuseppe: Ah, miss, I should put it in brackets [he corrects writing on the board "A + B + C - (B+C) = A"].
>
> Claudia: In fact, he added everything together and then subtracted two. But it is also possible to remove a different pair and get another result. For example if we had subtracted A and B instead of B and C, we would have been left with C [Claudia, invited by the teacher, writes her equality on the board "A + B + C - (A+B) = C"].

It is useful to stress that this path, being developed in the fifth grade and thus with pupils that already know the four operations and their properties, constitutes for them a new active process of sense-making of the additive structure in the context of quantities. Moreover in Claudia's words we can recognise really smart intuition about the possibility of generalising the specific relationship glimpsed by Giuseppe. Indeed her use of the word "everything" allows us to think that she is also shifting from looking at the particular context of that volume of water and starting to refer to any amount of water (maybe not only water). The part of the conversation reported here highlights how, in a quite natural way, the pupils discover some properties of mathematical relations and, so to speak, experience the fascinating sense of vertigo that arises from being in contact with the virtually unlimited possibilities of manipulation of these relations. Gradually, they become also aware of the necessity of expressing such relations in an appropriate language. The synergy between natural language and symbolic language to describe the various relations, stimulates a first refinement of the algebraic language and, in particular, the substitution of "and" with the "+" and a purposeful use of brackets.

The discussion develops for a while, up to a very interesting intervention by Flora.

> Flora: Miss, I have found another one. If we sum two B's, or two C's that is the same, and then we add once again C and then subtract A and C, the result is two B's or two C's. May I write it on the board? [Flora writes: "A + B + B + C - (A+C) = B×2 or C×2"]

Not all the children seem to understand the equality proposed by Flora, so the teacher invites her to explain her reasoning a little better and Flora provides a very effective explanation accompanying her words ("A plus B and B… and C… now I take off A and C and then two B are left here" by a drawing. Our video recording shows how the drawing is built according to her words (see figure 5).

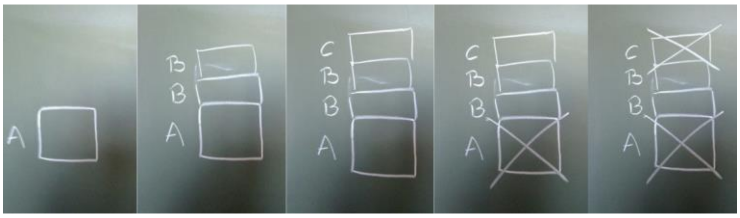


Figure 5.

Following Davydov (1982) we have claimed that the context of relations between quantities can act for its own nature as a rich source of examples for algebraic relations. In Flora's behaviour we can recognise the process of "seeing the general through the particular": indeed she initially describes a relationship observing the particular quantities of water in the containers, but at the same time she recognises that the relationship is valid for any quantity. In fact, even if the relationship "A + B + B + C - (A+C) = B×2 or C×2" has been conceived in the context of amounts of water, the drawing shows that her relationship also applies to areas of rectangles, in particular those drawn by her, but also potentially others. Consequently the letters, used initially as names for amounts of water in the containers, develop, at this point, toward their use as algebraic variables. Furthermore, a semiotic node (Radford et al., 2003) is visible in Flora's behaviour in which many registers (natural language, algebraic language, the gestures in building the drawing and the drawing itself) are coordinated in the attempt to grasp the relationships between the quantities.

Following this first phase, to induce the recognition that the equality results from the "transformation" of an inequality, the teacher asked pupils "What can we do to make the quantity C equal to the quantity A?"

> Flora: If A was ten glasses and C was three glasses, to make them the same I would need seven glasses!
>
> [a minute of silence]
>
> Teacher: I have these two amounts, A and C, I want to make them equal, how do I do it? I can either make C equal to A, or A equal to C; how do I do this?
>
> Giuseppe: I need to subtract!
>
> Martina: To make C equal to A, I have to add water.
>
> Teacher: We subtract water from A to make it equal to C [she writes on the board "A - ? = C"], we add water to C to make it equal to A [she writes "C + ? = A"]. What should there be instead of a question mark?
>
> Claudia: May I write? [Claudia writes on the blackboard "C + (A – C) = A, or A – (A – C) = C"]
> Teacher: And how would you call this quantity?
>
> Claudia: We have only these letters, that's why I wrote it this way … but we can call it D.

In this excerpt of conversation we can notice that Flora finds the answer using numbers, but without writing out numbers or calculations. Moreover Flora chooses small numbers to explain her reasoning and this is for sure a choice that allows to focus on the structure rather than on the procedure. What is more, the use of conditional shows that those numbers are used "generically", in other words they can be replaced by others; in this sense Flora seems to understand the process of generalisation towards which the course is going. However Flora's contribution arrives too early compared to the level of understanding achieved by the rest of the class, so the teacher repeats the question. Immediately the two courses of action are stated: removing or adding water. At this stage the teacher decides to reactivate the algebraic language writing two algebraic expressions on the blackboard: A – ? = C and C + ? = A. In this short extract of the discussion the influence of the teacher appears very strong and reveals her need to converge towards a formal expression, even at the cost of giving away too many answers of her students.

Claudia's comment, proposing the two "advanced" algebraic expressions, may seem surprising, but it is likely that previous experience in writing out equalities enabled her to execute this kind of algebraic representation with ease. In turn, the comments of the teacher lead to the introduction of a name, D, to indicate an unknown number, and thus begins the path towards the procedures typical of equations.

The activities carried out in this first phase enabled the students, at least at a level of collective discussion, to identify two possible actions/transformations that create equality between the amount of water in the containers: adding water to the smaller quantity (C) or removing water from the larger quantity (A); and to find a new quantity of water, called D, an element present in both transformations, and finally to identify this quantity D, which can be removed or added, as D = A - C. This last important passage is particularly delicate because it implies the ability to express an unknown quantity as a function of preceding unknown quantities, treating all of them as if they were known, an ability that is indispensable for solving equations.

Phase 2

In this phase the pupils were required to express the relationships and the transformations observed in the previous lessons by means of a suitable graphical representation. The type of representation was not proposed by the teacher (as occurs in Davydov), but was constructed by the students through a long analysis, comparing and sharing views. In this way it was possible to observe whether the construction of representations by the pupils in the context of a collective discussion guided by the teacher, might converge towards the SMAS; and, more in general, whether this type of mediation would help children to use autonomously the SMAS, or any other form of graphical representations, even later on.

Referring to the situation of Phase 1 the teacher asked pupils to draw a representation of the relationship involved, using segments. Some of these drawings were reproduced on the blackboard. Then the teacher chose one of the representations (fig. 6) to start a discussion with the aim of finding the most effective way to represent the passage from inequality to equality. During the discussion new drawings would be added on the blackboard.

Figure 6.

Irene: In my opinion this picture is wrong because C and D are within A, they are the same as A.

Alessandro: As it is, it indicates only one quantity… that is A with quantities C and D… so it would be an even larger container.

Flora: A plus C plus D. Yes this segment shows a sum.

Teacher: And is there any way to say that D is inside A?

Giuseppe: I think that we should remove those bars and make it clear that that segment is the container with inside A, D and C. [Giuseppe proposes the representation in fig. 7]

Giuseppe: Miss, the picture I drew is to say that in container A there are A, D and C.

Luca: Like this [fig. 8] in A there is D and C.

> Claudia: Miss, Irene's picture [fig. 10] is the same as mine [fig. 9] except that I didn't draw two segments but only one.

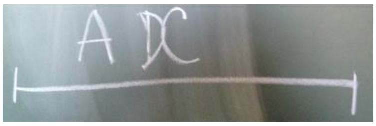


Figure 7

Figure 8

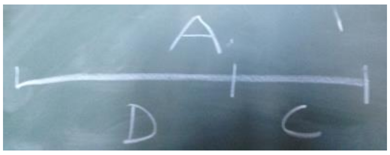


Figure 9

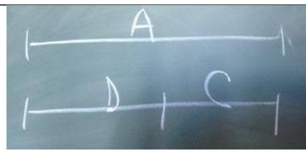


Figure 10

To save space we have proposed only a short excerpt from a long discussion on the choice of the most effective representations, capable of synthesising and expressing the relationships observed. But even in these few sentences it is possible to note the need for precision that leads the children to search for clearer and clearer representations, a need that conflicts, however, with the presence of still some uncertainty in the use of letters. For example in Giuseppe's second intervention we note that the letter A refers both to the container and to the amount of water. One of our hypotheses is that the use of graphical representations favours a faster and more decisive evolution of the use of letters towards a formally correct language.

The last representations drawn by the children on the board bear witness to how their thought had evolved substantially during the discussion and thanks to sharing ideas. In particular, the last three representations (fig. 8-9-10) are emblematic because when looked at in succession, they trace the evolution of the collective thought that converges, under the teacher's guidance, towards the SMAS.

Phase 3

This last phase constitutes a moment of testing: the pupils are shown a series of word problems that can be solved using equations, or rather comparing quantities and expressing one of them in terms of other possibly unknown quantities. One of the objectives was to test if the pupils recognised in the problems the structures they had studied during the course. In particular, we wanted to observe whether they used qualitative graphical representations to reach the solution and if the representation built up collectively in the previous phase figures in any recognisable way. The first problem brought to the class is called Carletto's candies:

> Carletto is a very greedy little boy. For his birthday he receives a gift box with 28 candies. Each day he eats twice as many candies as the day before. In three days Carletto finishes them all. How many candies did Carletto eat each day? Explain how you found it out and try to draw a graphical representation.

The pupils are asked to solve the problem individually using drawings as much as possible. After this phase of individual work, began a joint discussion about the different solutions.

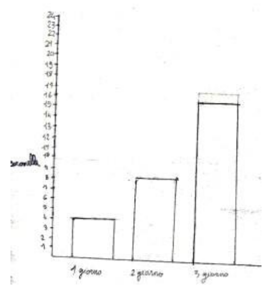
> Alessandro: Miss, I drew a segment and divided it into 28 parts representing the candies that Carletto eats in the three days... then I tried it once with a single candy on the first day, then with two, then with three... and I gathered with a bow the candies per day... I got the answer with four candies, so I stopped.

> Claudia: I did the calculation in my head... I had already understood that it wasn't possible with one, because it becomes two and then four... how could it reach twenty eight? Then I went directly to using three, but it doesn't work ... so I started by four and I found the answer.

What Alessandro and Claudia said constitutes a remarkable example of the application of the "trial and error" method, sustained by excellent control and feedback procedures. Other children comment referring less accurately and explicitly to methods of problem solving that are, again, traceable to the trial-and-error method.

Some graphical representations are also shown to the pupils (Claudia's drawing in fig. 11 and Giulia's in fig. 12), but for the moment these do not help to explain the structure of the problem with greater awareness, as this will occur later.

Figure 11



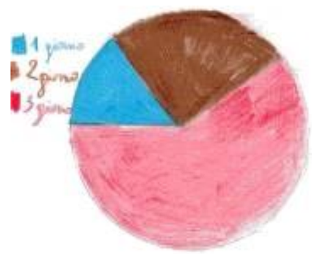


Figure 12

After this first trial-and-error phase, the teacher synthesises the various observations and solution procedures, in order to direct attention to the algebraic structure of the problem. She writes on the blackboard

> Since we know that Carletto finishes all 28 candies in 3 days, and that each day he eats twice the number of candies as the previous day, he must have eaten 4 candies on the FIRST DAY, 8 candies on THE SECOND, and 16 ON THE THIRD.

> Teacher: At this stage our question is where the number four comes from.

Contrary to what occurs usually, the new question posed by the teacher gets the students very involved, as they were by no means gratified by having already got the correct answer. Continuing to explore the "already solved" problem to discover its structure presupposes an attitude to research that the teacher and the students are learning to grow together.

Luca, in order to justify the number 4, performs "28 divided by 7 equals 4", but he can't explain where the number 7 comes from. It is interesting to observe how clearly he describes in his exercise book (fig. 13) the number 28 as the number of "the candies in the box" and 4 as the number of "the candies he eats in one day" (to be precise, this is strictly true just for the first day), while the scribble near the arrow from the number 7 makes clear the difficulty of identifying its role in the structure of the problem.

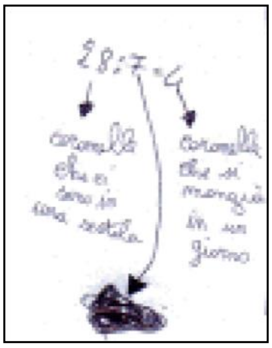


Figure 13. “Sweets in the box”, “sweets he eats in one day”

The pupils continue to reason at the blackboard building groups of four and showing how on the first day there is one of these groups, the second day there are two, and the third day there are four, for a total of seven times 4. A picture made by Flora (fig. 14) in his exercise book, where the SMAS can also be recognised, expresses well this achievement.

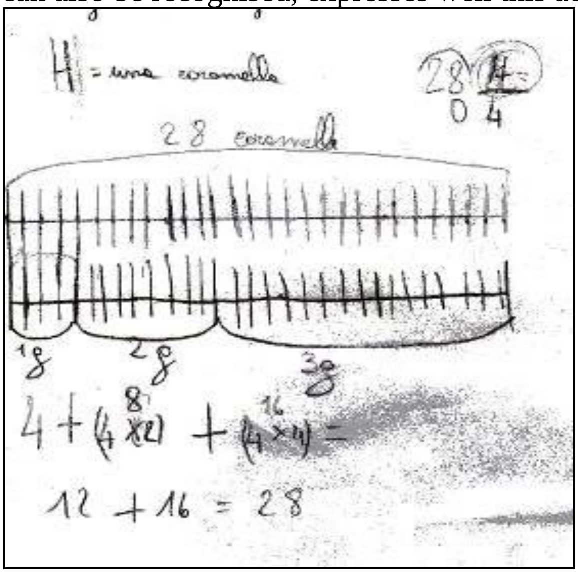


Figure 14.

So the teacher decides to let Flora speak,

Flora: The first day Carletto eats one portion of candies, and we don’t know how many... [Flora draws a square, see fig. 15] the second day he eats twice as many as the first one, so two portions [she draws two squares]… the third day he eats twice as many as on the second day, so four portions [she draws four squares]. Now twenty eight candies overall can be divided by the seven portions I have identified, and I know the value of each portion… that is 4, the number of candies in each portion. So seven is born before four.

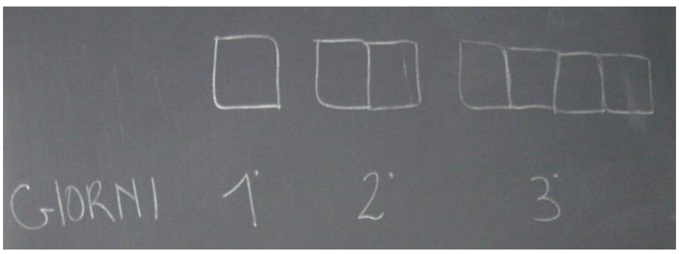


Figure 15.

Another semiotic node (Radford et al., 2003) is clearly recognisable in Flora's behaviour: indeed her words and her gestures to build the drawing in fig. 15 are coordinated in the attempt to grasp the relationships among the portions of candies eaten in the three days. Moreover Flora's words "the first day Carletto eats one portion of candies and we don't know how many" paraphrases perfectly the concept of unknown, and the use of the square in her representation is a nice example of proto-algebraic notation (Radford, 2011, p. 23) (Analogous examples of the use of the square as a proto-algebraic notation have been noticed by us in another design study, see Mellone, 2011). Furthermore her representation, together with her last comment: "Seven is born before four", seem to perfectly describe the equation $x + 2x + 4x = 28$, or $7x = 28$.

Some conclusive reflections

Davydov's curriculum was born in a particular cultural and historical context characterised by the spread of Vygotsky's sociocultural reflections (Davydov, 1995). These factors contributed to the design of the farsighted Davydov curriculum centred on the early use of algebraic language for detecting relationships between quantities. Even if nowadays we have a different context in comparison to the Russian milieu of those years, we think that the ideas at the root of that curriculum are precious insights for the present issue of early algebra in school (see, e.g., Cai & Knuth, 2011).

We have described a possible way to use his intuitions in our particular education context. In particular we used the activities that Davydov imagined for first elementary classes, that is before children start to use numbers, in the fifth grade: the idea of proposing these activities to a group of pupils that not only know numbers, but have also spent four years carefully studying their operational functioning, comes from the hypothesis that working on quantities contributes to the development of algebraic thought both before and after having explored numbers and arithmetic calculations. This fact has its importance both at a cognitive and at a methodological level in as much as the sum of activities carried out in classrooms has been a sort of backtracking, a meta-cognitive path for the pupils and the teacher in search of the "meaning" of mathematical structures, that was lost in the intricacy of syntax and procedures.

We have observed interesting behaviours in which the context of quantities has acted as a rich example-hood domain for the algebraic relationships. In particular, in Claudia's words "In fact, he added everything together and then subtracted two. But it is also possible to remove a different pair and get another result" and in the representation made by Flora in fig. 5, we observed two different ways of shifting from the use of letters as marks for particular amounts of water, to the use of letters as algebraic variables. Furthermore, during the use of graphical representations at the blackboard (fig. 5 and fig. 15) we could identify in Flora's behaviours two semiotic nodes that are important indicators of a learning process in place (Radford et al., 2003). The problem solving situation built on the context of quantities and the large use of collective discussions about the graphical representations were the ingredients which allowed the semiotic nodes to occur.

Already in Davydov's approach graphical representations are considered crucial tools to be used by the teacher: "In a school subject, intermediate means of description have crucial significance because they mediate between a property of an object and a concept" (Davydov, 1982, p. 237). In our design study we decided to make pupils build our intermediate graphical representation mean SMAS during a collective discussion lead by the teacher, in order to give space to social interaction and to support pupils to become able to express their ideas and to be aware of the use of suitable representations. In this way we could observe the efforts made by pupils to master, so to say, some syntactical rules underlying the use of the graphical representations. This is recognisable, for example, in Giuseppe's words "I think that we should remove those bars and make it clear that that segment is the container with inside A, D and C".

For the last phase of our activity, in which the pupils were involved in problem solving activities, we have shown a particular interaction, developed for Carletto's problem. Here the guidance of the teacher was crucial to turn a routine mathematical activity in an Italy primary school, such as solving word problems, into an activity oriented to the discovery of a mathematical structure. In order to face Carletto's problem the pupils used the SMAS and other spatial representations as well: the previous phase in which the SMAS arose as an object of discussion, not imposed by the teacher, may have helped pupils to feel free to find other kinds of spatial representations. Of course, the variety of spatial representations is a clear sign of pupils' efforts along their processes of sense-making. Finally, in the last representation of Carletto's problem by Flora (fig. 17), her use of a square as a symbol for the unknown quantity, reveals an approach to a problem situation directly pointing towards a mature algebraic language.

Finally, we are aware that our proposal of using the activities of Davydov's curriculum in the fifth grade can be seen in conflict with his assumption that 'the general' prior to 'the particular' is the main vehicle of development of theoretical thinking in children. Nevertheless, our "non-orthodox" use of his activities comes from two main exigencies. Firstly, we wanted to meet the Italian tradition of kindergarten and primary curricula, according to which pupils work with numbers and numerals since the beginning. Secondly, our choice comes from our belief that a more complex relationship occurs in ontogeny between 'the particular' and 'the general', and in particular between arithmetic and algebra (see e.g., Mellone, 2011).

According to Radford, we believe that "algebraic thinking does not appear in ontogeny by chance, nor does it appear as the necessary consequence of cognitive maturation. To make algebraic thinking appear some pedagogical conditions need to be created" (Radford, 2010, p. 79). In this perspective our design study was an attempt to create these pedagogical conditions. In our opinion, the exploration of suitable contexts with continuous quantities (Davydov, 1982) can support pupils in grasping historic forms of thinking and in finding their personal meanings of the algebraic language. The results, as seen in the observation of many interesting behaviours of the pupils, seem to be positive. In particular, the representation proposed by Flora (fig. 5), together with her arguments, reveals, in our opinion, the true potentialities and the need of the use of spatial representations (not only the SMAS), in connection with the algebraic symbols, also for use in the upper secondary school. We believe that Flora's performance occurred thanks to her experience with Davydov's activities, even if, of course, we don't have scientific evidence of this claim that might deserve further investigation.

Further research could also concern the design of a curriculum for the first class, in which Davydov's proposals are intertwined with traditional curricula. In this sense, it would be interesting to compare several research experiences on school traditions of different countries, variously interwoven with Davydov's curriculum.

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