Early Curricular Experiences with Nonnumeric Quantities, Evidence of an Enduring Perspective

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In this study we explore possible long-term effects of an adaptation of the El'konin–Davydov elementary grades curriculum, Measure Up or MU. The objectives for the study are to assess how students relate an equation of nonnumeric quantities to a length representation, and if former MU students develop and retain a perspective characteristic of the curriculum. Data were collected from thirteen former MU students and fourteen students who did not receive MU instruction were instructed together with the MU students in middle and high school programs. Findings show that former MU students reasoned about lengths as generalised quantities, applied a measurement-based method for marking and labelling quantities, and justified a representation of relationships given by an equation. Implications are discussed for how a measurement context in elementary mathematics supports meaning making in the later study of algebra, particularly with regard to variables and multiple representations.

Introduction

Early mathematics instruction in the U.S., as in many other countries, typically begins with counting and working with numbers. Counting discrete items and assigning numbers to quantities has been widely assumed as basic foundation for mathematics instruction at the elementary level. With an awareness of numbers, children then learn arithmetic and associate symbols (i.e., +, −, ×, ÷, =) with specific operations to find a numerical answer. This approach to early mathematics becomes problematic when students advance beyond arithmetic and are introduced to mathematical symbols in nonnumeric contexts where letters replace numbers in mathematical expressions. In the transition from arithmetic to algebra, students expect algebraic procedures to lead to a result (i.e., a number) as they have come to expect from their work with these same symbols in arithmetic procedures (Linchevski & Herscovics, 1996). Young students commonly view the equal sign as synonymous with *the answer is* (i.e., McNeil & Alibali, 2005; Kieran, 1981), and seem to have difficulty with perceiving the equal sign as providing a definition or a statement of equality. Middle grades, students who were found to have a limited understanding of the equal sign tended to also perform poorly on relational thinking problems (Knuth, Stephens, McNeil, & Alibali, 2006). Limited interpretations of the equal sign have also been found to persist among some high school and college students (McNeil & Alibali, 2005). These findings highlight the challenges students encounter as they transition beyond arithmetic to more advanced mathematics, namely algebra, where relational thinking is critical for students to understand new concepts.

In our research and development of an elementary grades curriculum, we explore a different conceptualisation of mathematics topics from what most U.S. curricula assume. Using English translations of an experimental curriculum developed by V. V. Davydov (1975) and his team in Russia, we research the potential for this curriculum in the U.S. One of the reasons we were interested in the curriculum was that it introduced children to abstract representations and letter symbols in a manner consistent with algebraic concepts, and yet was presented in a developmentally appropriate fashion. Psychological, mathematical, and instructional considerations of Davydov's approach have led us to hypothesise that students are capable of developing robust understandings of symbols and flexibility in applying this understanding to different contexts (Venenciano & Dougherty, 2014). Furthermore, students learn about number in a more general way than do students who learn numbers via a counting sequence. The objectives for this study are to characterise how students from our curriculum project think about nonnumeric relationships and to examine the prevalence and possible endurance of this thinking.

Literature Review

Filloy, Puig, and Rojano (2010) discuss the nature of elementary mathematics concepts as a response to the experiences children have with the real world. Mathematics textbooks for elementary students often draw on the manipulation of objects and direct children's thinking toward capturing this manipulation with an expression or equation. Symbolisation is primarily used for communicating observed or hypothetical manipulation in a mathematical way (i.e., quantitative modelling). However, for students to develop skills needed in their later studies of algebra Filloy, Puig, and Rojano stress that a curriculum must be designed to also include inverse experiences, that is, going from an abstraction to a concrete example.

A didactic path discussed by Filloy, Puig, and Rojano (2010) for learning algebra via modelling, is a cycle of three processes; (a) translation of concrete situations to algebraic representation, (b) analysis of relations between variables, and as based on the manipulations of the algebraic expression, and (c) interpretation of the concrete situation as it relates to the work with algebraic syntax. This path allows for the emphasis on algebraic symbols and expressions as they correspond with real world phenomena. We believe this approach is also relevant for introducing mathematical symbols in arithmetic and propose this in the development of elementary grades curriculum.

Davydov (1975) discusses a similar learning cycle in young children's learning of mathematics. Davydov noted how young children spontaneously observed "relationships among definite quantities of objects" (Davydov, 1975, p. 90) in their everyday context and were capable of classifying and seriating as these were children's "practical methods of distinguishing and designating certain mathematical relationships" (Davydov, 1975, p. 90). Children observed and developed informal understandings of objects in their world and were capable of comparing objects, albeit in a relatively ad hoc manner, to test relative magnitudes and decide which was bigger, heavier, taller, etc. By using familiar contexts from children's experiences in their physical world, Davydov and his team of mathematicians, mathematics educators, and psychologists developed an experimental curriculum (Davydov et al., 1999). This curriculum, sometimes referred to as the El'konin-Davydov curriculum, was designed to support the development of formal knowledge of mathematics concepts and properties.

The El'konin-Davydov Curriculum

Rather than begin mathematics instruction with numbers and computation, Davydov (1975) and Minskaya (1975) proposed the starting point for mathematics education to reflect the fundamental structure of mathematics. Davydov believed that instruction needs to first focus on generalised properties, as opposed to numbers, because properties (e.g., equality, commutativity, transitivity) are invariant across the study of rational numbers. For example, if we know $A > B$ and $B > C$, then we also know $A > C$. This property endures far beyond elementary school curricula, as opposed to a rule such as, "adding results in a bigger number," that does not always hold when students first work with negative numbers.

The El'konin-Davydov curriculum embodies a Vygotskiian theory about the formation of spontaneous and scientific concepts. Vygotsky (1978) theorised that children develop spontaneous concepts when they abstract properties from (everyday) concrete experiences. Tharp and Gallimore (1995) described this as learning *upward* from sensory experience to generalisation, in contrast to learning *downward* from formal experiences with properties that are more general than what children may otherwise observe in their everyday interactions. Opportunities for learning downward are created by starting with scientific concepts, such as a generalisation, and then exposing children to a collection of concrete examples, which may not otherwise appear to be connected beyond a relatively ad hoc manner (Wells, 1994). By acknowledging the formation of spontaneous concepts that children bring from their everyday experiences, school curriculum can be designed to introduce scientific concepts in an appropriate way that makes the sophistication of mathematical properties accessible to young children.

The El'konin-Davydov curriculum structures learning activities with the use of measureable attributes (e.g., length, area). The activities allow students to compare generalised or non-numeric quantities, identify or create a unit, iterate a unit, and use the unit as a tool for reasoning about magnitude and numbers. Measurement activities are designed to serve as the primary means for developing number concepts such as base ten, place value, the four basic operations, and rational numbers.

The broader research questions we are interested in are if this approach prepares students to think differently about mathematics, and if so, how might this preparation contribute to algebra preparedness. In an attempt to address these questions, we experiment with an adaptation of the El'konin-Davydov curriculum in a project named *Measure Up*.

The Measure Up Curriculum Project

In 2001, the Curriculum Research & Development Group at the University of Hawai'i at Mānoa adapted and implemented the El'konin-Davydov curriculum and tested instructional strategies at a unique laboratory school in the U.S. This was the beginning of the Measure Up or MU project. The school serves students from Kindergarten (5 years old) through Grade 12 (17–18 years old) and is designed to be a fertile ground for education research. There are 10 students in each of the Grades K–5. At Grade 6, an additional group of 48 students are admitted to the school and are mixed with the 10 students from the elementary program, resulting in about 58 students who progress together through middle and high school.

To ensure a diverse student population for research purposes, the school has a policy for admitting a student body to reflect the state's public school population with respect to ethnicity, socioeconomic background, and academic achievement as measured by standardised test scores. Additionally, the school attempts to maintain an equal gender ratio. All students take the same set of courses with their grade level peers and are not separated by ability level for instruction. By carrying out our research efforts at this school, we are able to implement and study a program at the elementary level, and track students as they progress through the upper grades.

Against a backdrop of a growing body of early algebra research, the MU project applied design-based research methods to experiment with the Russian curriculum and to integrate opportunities for students to develop mathematical processes specified by the National Council of Teachers of Mathematics (e.g., communication, problem solving, and reasoning and proving). MU materials (CRDG, 2006) emphasise class discourse and small group interactions together with teacher-led class demonstrations.

MU focuses initial arithmetic instruction on nonnumeric relationships of measurable quantities. Minskaya (1975) argued that beginning mathematics instruction with numbers was complex and problematic. In addition to memorising number names and counting order, children need to learn that the last number they count, when counting items in a set, represents the total. Another difficulty children have is with understanding that every number represents a specific magnitude, and the magnitude is dependent on the size of the unit. Children need to able to distinguish the characteristic or quality that defines the unit in order to know what constitutes a valid count and a valid comparison between quantities. Moreover, Minskaya reported that when children do not have a full understanding of number, there are negative impacts on mastering operations on concrete numbers and understanding the connection between whole numbers and fractions.

MU delays the introduction of numbers in the curriculum until after students have developed fluency in measuring with a unit or creating a quantity, and identified a need to describe quantities more precisely. The measurement context with nonnumeric quantities builds on concepts children observe in the physical world. For example, students are able to verify that one lump of clay feels heavier than another lump of clay. Building from the spontaneous concepts students develop through their observations, MU instruction guides students to form scientific or mathematical concepts. Students begin to understand the structure in mathematics properties by using and linking multiple concurrent representations of quantitative relationships—such as holding up two paper strips of unequal lengths, two line segments of unequal lengths, and writing an inequality statement (Dougherty, 2008). The lessons emphasise the repeated use of these representations and link actions with its physical, iconic, and abstract representations. In relation to the processes discussed by Filloy, Puig, and Rojano (2010), MU lessons promote algebraic representation, albeit simple algebraic representations, as one of several representations that are used to capture relationships among quantities. Class discourse emphasises the translation of the algebraic representation, as well as the analysis and justification for why it gives the same information about the quantities as the other representations. Students learn to fluidly toggle between concrete, iconic, and abstract representations.

This approach was found to help children develop understandings of the relationship between generalised quantities given in a story problem (Dougherty & Slovin, 2004). Children were capable of using multiple representations to symbolise the quantities in the problem situation and the associated actions of the problem. Students used algebraic symbols and generalised diagrams for solving the problem. Other MU studies have shown that students who were instructed with the MU curriculum were likely to develop the skills needed to read, interpret, and work with variables in algebra (Slovin & Venenciano, 2008; Venenciano & Heck, 2015). Building on these findings from MU studies, this present study focuses on the possibility of an enduring MU perspective and method for using nonnumeric quantities to reason about relationships.

Method

The following research questions guide this study:

1. How do students relate an equation to an iconic representation of nonnumeric quantities?
2. What, if any, distinctive qualities exist in responses from former Measure Up students?

The first question helps us establish parameters for the range of responses given to the tasks, and the second question focuses our attention on identifying possible long-term effects that may be attributed to MU. This study aims to uncover how MU students interpret an algebraic equation with linear quantities in a measurement context, and if there is an MU method of thinking, does it endure over time and after exposure to middle and high school mathematics.

A qualitative comparative case study approach (Merriam, 2009) is used to address the research questions. To maximise the likelihood that we observe and understand the different approaches students use to think about the mathematics, we follow Stake's (1995) recommendation and include multiple cases. The data that contribute to the formulation of the cases are comprised of students' written work, audio recordings and transcripts of group interviews, and observation notes recorded by researchers during the interviews.

In order to address *how* students relate abstract and concrete representations, we first employ an assessment consisting of a set of mathematics tasks for students to work on independently. This is then followed by a group interview or group think-aloud for students to share their solutions and discuss their thinking with others from their MU or non-MU peer group. Multiple sources of data are needed to enhance the reliability and internal validity of the findings. The group interaction serves as a means for researchers to observe how students make sense of the context, to record the questions students ask of each other, and to assess the extent to which the group is able to agree on the reasonableness of their individual responses. The collected data are then analysed for indicators that contribute to formulating the cases.

The assessment problems were from MU Grades 4/5 project materials. Project records show that these problems were given to MU students during Grade 5 and that about half of the students produced a productive response. We posed the problems to four groups of former MU students during Grade 12 and to four groups of students who had not received MU instruction.

Sample and Participant Selection

A total of twenty-seven Grade 12 students participated in this study. The students were from the same laboratory school. The students had the same middle and high school mathematics courses and were accustomed to engaging in small group and whole class discourse in their mathematics classes as a means for developing their thinking about solution strategies.

Thirteen of these students attended the school during their elementary years and participated in the MU project; the remaining fourteen students comprised the non-MU group. These students were selected for the study based on two criteria. The first was on the basis of how well they mirrored the MU group in terms of ethnicity, Grade 7 achievement scores in mathematics and language arts, and social economic indicator as determined by school admissions data. Grade 7 achievement data served as a point to match students because it preceded their study of Algebra I and was not confounded by their common algebra experiences. All students successfully completed a course of algebra in Grade 8 and followed the same high school sequence of courses, which included a full year of geometry and three years of an integrated mathematics program. These curricula emphasise problem solving from an inquiry approach and class discourse as an everyday instructional practice.

The second criterion for selecting participants was Grade 12 teachers' assessments on students' willingness to engage in problem solving discussions and make their thinking visible. Students who were more vocal about their thinking were recommended over others with similar demographic backgrounds. Students selected for the non-MU group had attended different elementary schools in the U.S. At the time of this study, U.S. mathematics curricula were guided by national standards for school mathematics (NCTM, 2000). These standards showed the number being most heavily emphasised in the early grades and the algebra strand most heavily in the latter grades (see Figure 1). The measurement strand appears to be far less represented in the standards. Thus, we can assume that most non-MU students learned mathematics from a number-based approach and did not receive substantial instruction in measurement concepts.

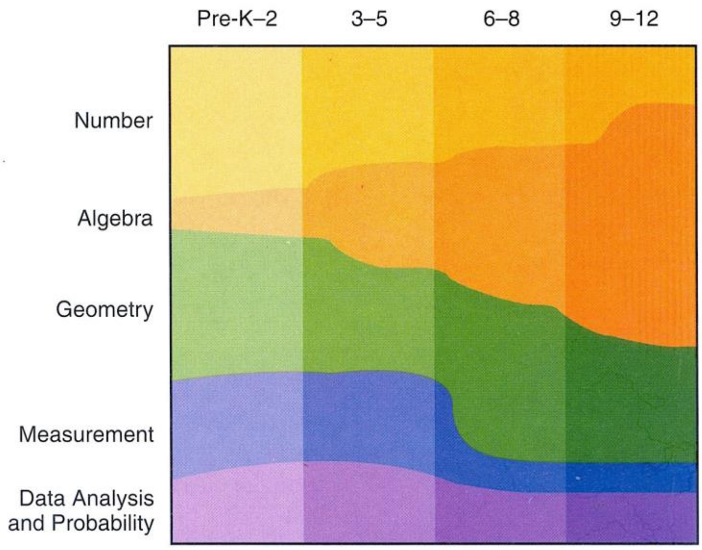


Figure 1. Suggested content emphasis across the grade bands from *Principles and Standards for School Mathematics* (National Council of Teachers' of Mathematics, 2000, p. 20).

Data Collection

The instrument used for this study is based on content from the MU program (see Figure 2). It is designed to assess relational thinking of nonnumeric quantities in a measurement context. For each task, nonnumeric relationships are given in equations and depictions of the lengths. Students are to notice the relationships and preserve them in the creation of a length representation in Problem 1. In Problem 2, students are asked to apply an understanding of the quantitative relationships to determine additional ones.

1. a. Mark and label each line segment with *X*, *A*, *B*, and *C* so that it is a representation of the equation.

   $X = A - (B - C)$ |————————————————|

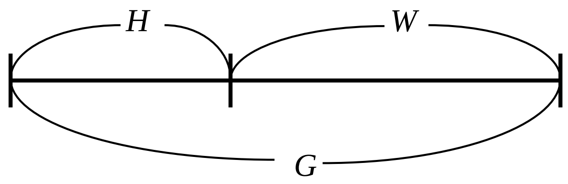
   Explain your reasoning for how you labelled the line segment.

   b. Mark and label each line segment with *X*, *A*, *B*, and *C* so that it is a representation of the equation.

   $X = (A - B) - C$ |————————————————|

   Explain your reasoning for how you labelled the line segment.

2. a. What is $\dfrac{W}{D}$ if $\dfrac{H}{D} = y$ and $\dfrac{G}{D} = r$?



b. How many *E*s are in *W* if $\dfrac{D}{E} = t$?

Figure 2. Assessment problems adapted from the MU elementary program.

The assessment protocol is designed for students to grapple with an algebraic expression and to link the expression with a representation in the context of length quantities. Students were placed in interview groups of 3–4 students, based on whether or not they had prior experience in the MU program. This resulted in four MU groups and four non-MU groups. Groups were interviewed in separate sessions with each session lasting for approximately 20–30 minutes.

For the first ten minutes of each session, students were asked to work on the problems independently. Then immediately following the independent work, students were instructed to sit together as a group for an additional ten minutes to share and discuss their thinking, discuss alternate answers, ask questions of each other, and to make notes or revise their answers if they had changed their response. A researcher facilitated the discussion in a non-evaluative manner.

To differentiate independent work from the group's influence, students wrote in ink during the first half of the session and in pencil for the second. The opportunity for students to change their initial response was an indication of their willingness to make sense of the mathematics and to reconsider the reasonableness of their own thinking. Two additional researchers were present during the interviews and recorded observation notes during the group discussions. These notes were also used to interpret students' confidence in their responses.

For the first pass through the data, a team of four researchers analysed the written response to each problem. The team discussed each solution and if or how the student considered measurement concepts in their solution strategy. Short descriptors were used to note the prominent traits of each response. While some students questioned the possibility of representations with a line segment, others responded thoughtfully. Analysis of the written work resulted in the following set of response characteristics–

- applied mathematical properties to manipulate the equation, and then addressed the line segment representation,
- attended to the part-whole concept in subtraction and treated letters as generalised quantities,
- focused on using the letters to label points along the segment, and
- modelled the mathematics coherently using the line segment and explanation, and in a manner that was consistent with the equation.

For the second data pass, the identified response characteristics were analysed together with transcriptions from the group interviews. The list of characteristics was then refined to three essential features of productive responses,

- consistently used a method for marking and labelling quantities along the line segment,
- justified relationships given by the equation, and
- used nonnumeric quantities in their representation.

If a student gave a productive response it typically contained more than one of the three traits. This pass through the data informed the validity of findings from the first pass. To reduce bias in the analysis process and identification of potential themes, response sheets were de-identified to conceal student identity and group membership. Similar solution strategies were coded and clustered for further examination and to define the characteristic. Prominent characteristics emerged from the analyses and were used to identify similarities and contrast differences in the responses. Following this coding process, written responses were compared together with transcriptions of the group interviews and re-identified as either an MU or a non-MU Grade 12.

Grade 5 assessment data were collected from the MU project files. MU students responded to the tasks, as they were part of a summative assessment. These tasks, shown in Figure 1, were part of a five-problem assessment.

Findings

The analyses of responses to Problem 1 led to the identification of prominent characteristics of students' perceptions and solution methods. We attempted to analyse the responses to Problem 2 with this set of characteristics, but there were fewer productive responses to the Problem 2 and we were not able to discern prominent traits of solution method. Problem 2 posed a more complex situation and students needed to negotiate information between the equation and the line segment in their response. Nevertheless, the findings offer insights to how students relate an equation to an iconic representation and if there is an MU method that endures through the end of students' high school education.

Prominent Traits of Relating an Equation to a Representation

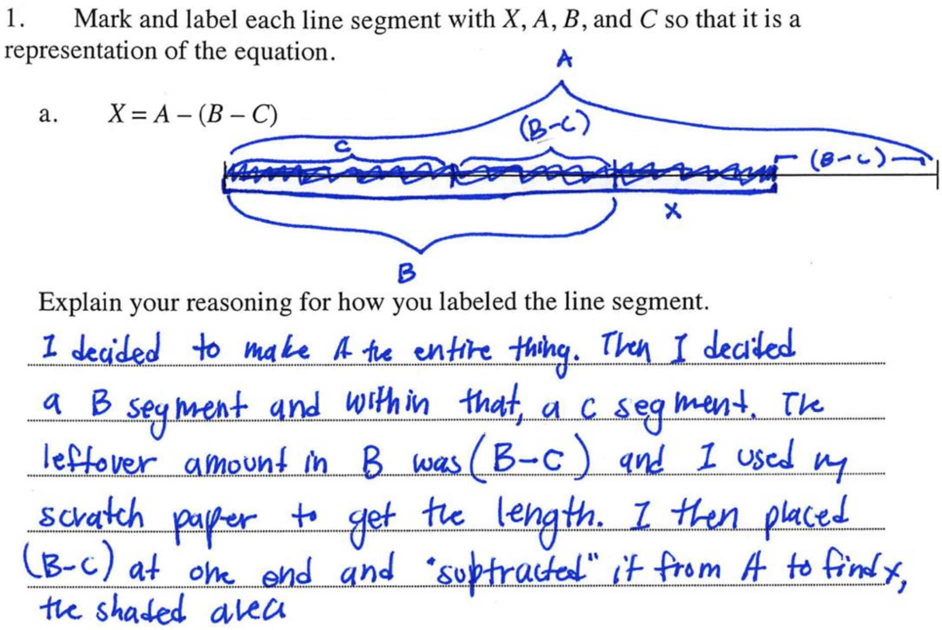
To address our first research question regarding how students relate an equation to a representation of nonnumeric quantities, we note the emergence of three characteristics in productive responses. The first characteristic relates to students' methodical marking and labelling quantities of lengths along the segment. This often appeared in the data as a bracket above or below the segment with a letter label that corresponded with a letter from the equation. This response indicates reasoning about amounts of length. Some responses exhibiting this reasoning showed an accurate modelling of the length quantities and addressed how the quantities related to each other, as indicated by the equation.

The second characteristic is students' justification of the preservation of relationships in their diagram. These responses highlighted an interpretation of the equal sign as a relational symbol, rather than an operator, and intent to preserve the relationships given by the equation. The third characteristic that emerged from the data was the use of nonnumeric quantities representations. Nonnumeric responses treated the quantities as if they had a life of their own and were not complicated by numbers and computations used to evaluate the equation.

Responses showing these characteristics are depicted below in the discussion of sample student work.

Responses Characteristic of MU Students

Following the initial set of passes through the data to code for characteristics, the data were analysed to address our second research question, what, if any, distinctive qualities exist in responses from former MU students? At Grade 12 nearly all responses from former MU students exhibited characteristics of productive responses. These responses showed careful articulation of length quantities *A*, *B*, *C*, *B* – *C*, and *X*, often with complex arrangements depicting relationships among the length quantities. For example in the group's discussion captured below, Elle began the discussion by sharing her work (see Figure 3) and engaging her peers as they considered how to interpret and use the line segment. An excerpt from the interview transcript showcases Elle's scheme for depicting the quantities. Elle was apparently using the line segment as a tool for reasoning about the relative quantities.


1. Mark and label each line segment with X, A, B, and C so that it is a representation of the equation.

a. $X = A - (B - C)$

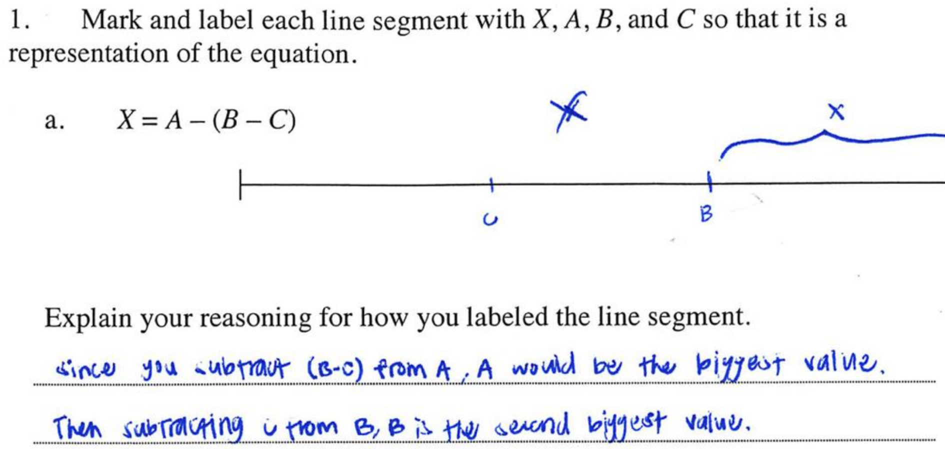
Explain your reasoning for how you labeled the line segment.

I decided to make A the entire thing. Then I decided a B segment and within that, a C segment. The leftover amount in B was (B-C) and I used my scratch paper to get the length. I then placed (B-C) at one end and "subtracted" it from A to find x, the shaded area

Figure 3. Elle, a former MU student, depicts and explains her perception of length quantities based on the relationships given in the equation.

Elle:	I actually thought that since the whole segment was *A*, I randomly picked a point, or length to represent *B*… and within *B* I found *C* and whatever was left of *B*… that had to be *B* – *C*. So you take the length of *B* – *C*, and put it to the other side of *A* and that much had to be *X*.
Alu:	That's when I got confused.
Brayden:	Yeah, I know I totally forgot. I totally forgot that we label gaps until the second one…
R:	What did you say? You label gaps?
Brayden:	Yeah, I forgot that we label gaps. I kind of did that in [problem] b, but we are in [problem] a, …
R:	And Elle's depiction where is there a label of a gap?

Elle: Gap?
R: The way that Brayden…
Alu: It's not on the line, like you don't label this [pointed to the line]; it's the whole segment.
Brayden: Yeah, its a certain point on the line segment, its the distance between a point…
Alu: The length of the…
R: Okay, so does that mean you agree with what Elle is saying?
Brayden: I agree, yeah.
R: Alu, anything more to add about your solution?
Alu: No, I basically just did what Elle did but I just didn't know how to find *X*. Like you find a random point of *B* and within that is *C* because you are subtracting *B* from *C* … so *B* would be bigger than *C* and *A* would be bigger than *B* – *C* because you are subtracting *B* – *C* from *A*.


1. Mark and label each line segment with X, A, B, and C so that it is a representation of the equation.

a. $X = A - (B - C)$

~~X~~ X

C B A

Explain your reasoning for how you labeled the line segment.

since you subtract (B-C) from A, A would be the biggest value.

Then subtracting C from B, B is the second biggest value.

Figure 4. Alu, former MU student, responses with a combination of a nonnumeric quantity and markings to show relative magnitudes.

During the group interview, Elle explained and gestured how she used the segment in a dynamic manner. She first drew a representation for the length *B* – *C*, then used her thumb and index fingers to measure the length *B* – *C*, and finally marked and labelled this quantity at the end of the segment. Although Brayden was not able to successfully mark and label the segment, as indicated in the transcript of the group interview, upon discussing the problem with his group, he immediately noticed length quantities and was able to engage in the discussion with his peers. Alu's response was less clear. She seemed to agree with Elle, however she referenced "a random point of *B*" and her explanation focused on magnitudes based on a quantity being a subtrahend or a minuend. We also noted how Alu labelled quantities *A*, *B*, and *C* differently from how she labelled *X*. Although Alu did not seem to consistently use a method for marking and labelling quantities, she had a conception about length quantities, as indicated by the bracket of length marked between *B* and *A* in Figure 4. Alu's written response seems to emphasise the relative size of the quantities as based on the parts-whole relationships in the equation. Thus, although Alu shows that she is able to notice magnitudes among nonnumeric quantities, she did not accurately create a length illustration to represent the equation.

Response from a Grade 5 MU student. Responses from students while they were receiving MU instruction were retrieved from project files to establish a perspective of the immediate effect of instruction. All MU Grade 5 responses to Problem 1 exhibited the three characteristics of productive responses. Variations noted among the Grade 5 responses stemmed from attempts to mark and label the relationship between each quantitative pairing. Although these responses were considered productive, working with five quantities (i.e., A, B, C, $B - C$, and X) may have been overly cumbersome for students as it led to misrepresentations with the segment.

One student overcame the obstacle by reducing his representation to three quantities. Jordan (see Figure 5) represented $B - C$ as a quantity of length that was marked as half of the given line segment. He identified the entire length as "A" and labelled the remaining half of the segment as "X." Although the quantities $B - C$ and X did not need to be equal, Jordan acknowledged that it was "handy." Furthermore, this did not violate relationships given in the equation and shows elegance to how he thought about the mathematics.

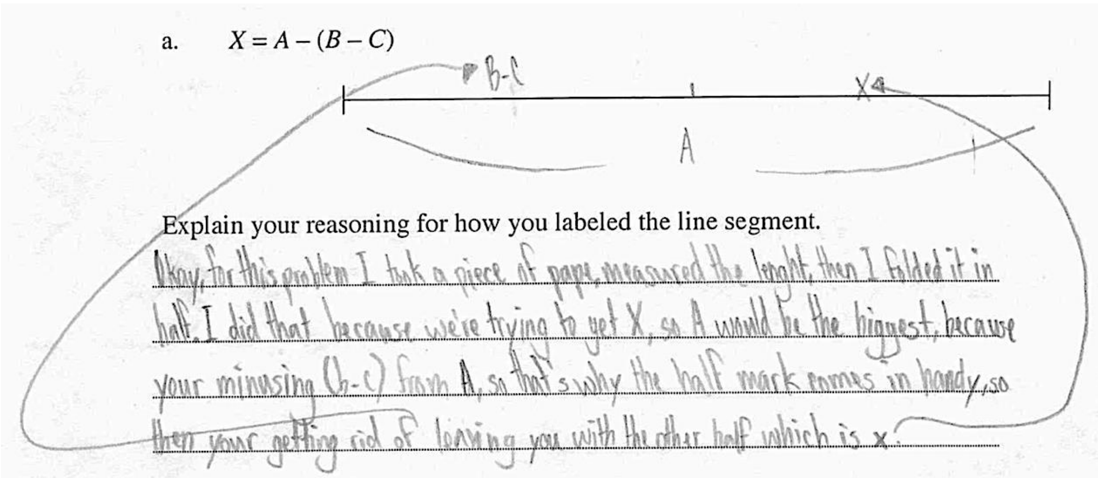


Figure 5. Jordan, as a Grade 5 MU student, marks the length to represent two parts and one whole.

Jordan applied the same reasoning to solve the second part of the problem, with the equation, $X = (A - B) - C$. He labelled the entire segment $A - B$, about half of the segment C, and the other half of the segment was labelled X. He explained that this problem was harder because he had to think of subtracting from a difference and that difference ($A - B$) had to be "the biggest because you to have to minus C from that."

Jordan's response shows that he is able to use the line segment to represent quantities of length, reason about quantitative relationships, and reason with generalised, nonnumeric quantities. Jordan's ability to think theoretically is evidenced by his application of this method for representing and reasoning.

Responses Characteristic of Non-MU Students

Written and interview responses from non-MU groups were examined to help distinguish responses given by the MU group. Of the four non-MU groups, one group discussed responses with the productive characteristics noted earlier. The group discussed the line segment in a method for marking and labelling quantities of lengths on the segment, justifications were based on relationships given by the equation, and generalised quantities were used in the representations. Marcus initiated the group discussion by sharing his thinking. His work is shown in Figure 6 and his group member's, Allie's, work is shown in Figure 7.

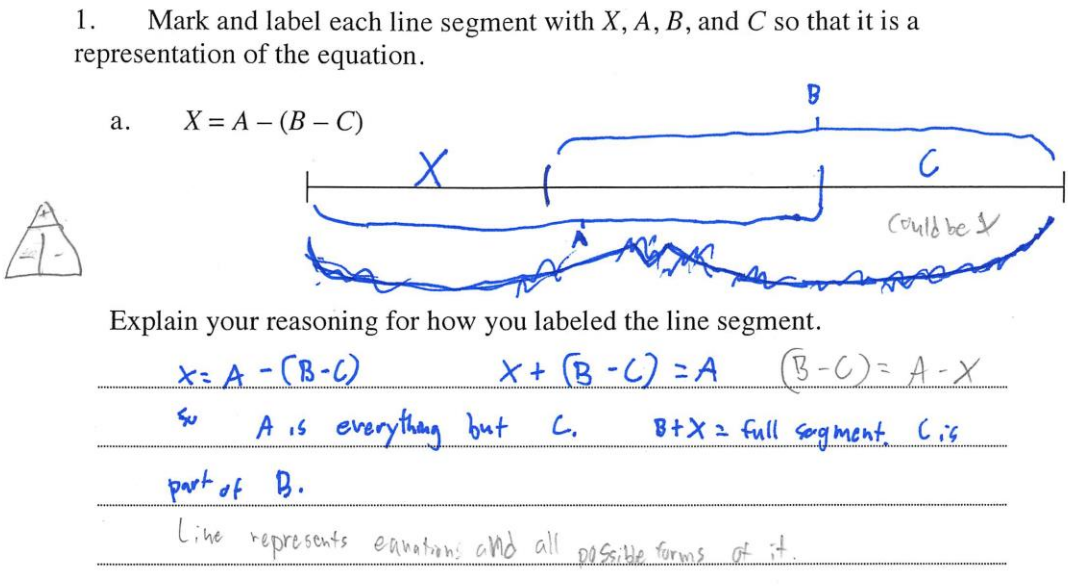


Figure 6. Non-MU student, Marcus, responds with a diagram indicating length quantities and transformations of the equation.

Marcus: Well, a line segment would represent a whole, or at least what could possibly be a whole …. I just used it to represent everything … because there aren't really any wholes in this situation. … Well there is kind of a whole in *B*. So in *B*, everything comes together to equal *A*. And so that would be a whole, I just used it to represent the entire equation, to try to make it so that all situations [in the equation work]. For example, … the addition subtraction thing you can change around the first one $X = A - (B - C)$ into $X + (B - C) = A$ … so $A = X + (B - C)$, so no *C* is located in *A*. But, otherwise *A* includes that part of *B* and *X*. $X = A - (B - C)$ so *X* is just that part of *A* that doesn't include *B* and also does not include *C*.…

R: You folks are writing some things down. OK, well let's just take a look at [Problem] 1a. Are you agreeing?

Jake: Yeah.

Marcus: I guess the line represents the equation and all possible forms of the equation.

Jake: … because it's saying *X* would equal or be part of *A* that doesn't include *B* and *C*. So you can, yeah, oh, that makes sense. So *X* is a part of *A*.

R: Does that make sense? Jake, how did you think about it before Marcus started talking; what were you thinking?

Jake: I was thinking that the line would equal *X*, but after that it didn't make any sense because if you did make the line equal *X*, you cannot increase the size of the line, or because when you subtract you are taking away so I didn't think of that *X* could be a part, could be represented as a part of something….

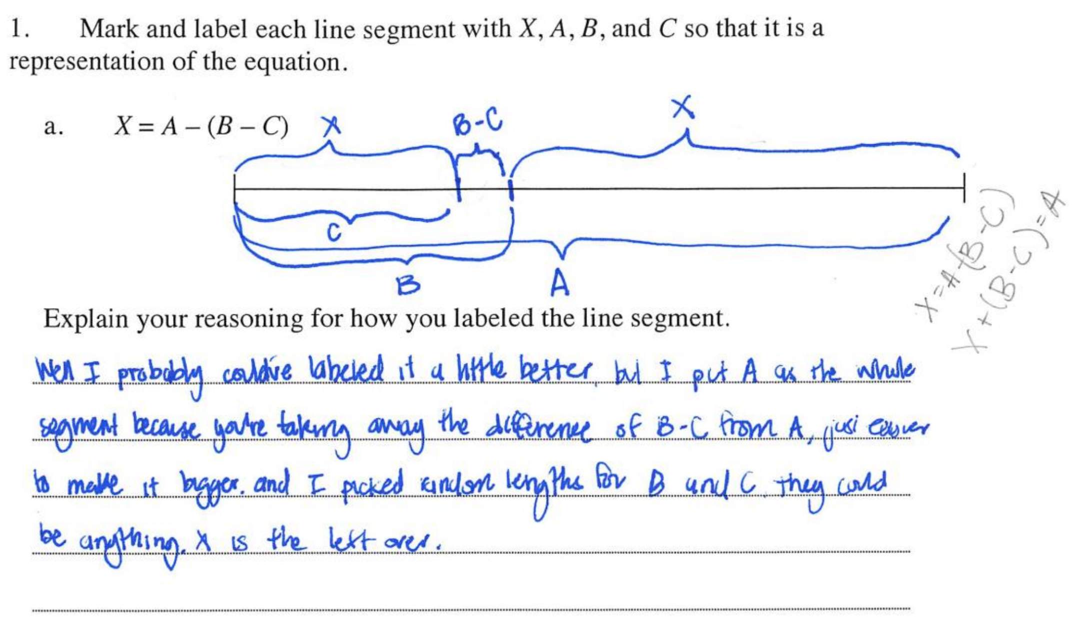


Figure 7. Productive characteristics in the response from non-MU student, Allie.

Allie: I don't even know if I did it right. I made A the whole thing because you are subtracting from it and then I made random values for B and C. Could do what Marcus did I guess, but then X would basically be what, A… I don't know. I just made it random so then I made B this line and C this line and then when you subtract B from C you are going to end up with this amount of space. So this is $B - C$ and then X is going to be this value plus this value; what's left over. Or the opposite way, I don't know.

R: You are saying X is in two different places? Is that what you just said? X is this value plus this value?

Allie: Yeah, well both… yeah cause there is a space in between… yeah.

R: I see what you are showing. There is some space in between that is not part of the X that you want to show.

Allie: Yeah, the rest of it is X, but…

Marcus: It would make sense, actually, because if it was $A - B - C$ then X might actually include C as well. … because you are just subtracting the part of B that doesn't have C from A, so X could equal that part of A that doesn't include B and that part of B that included C. But, would that work with $X + B - C = ...$ But, then that doesn't seem to work for $X - B - C = A$...

R: Clearly there is more that you are thinking about… Kama, what were you thinking?

Kama: First I was thinking what Jake thought because I thought that this was X, but then when you look at it, it's $X = A$ minus a part, so A is the whole. But, then like I didn't like B and C was confusing to me because I guess I didn't really know um, yeah B and C just threw me off.

The group discusses quantities in relation to each other and refers to part-to-whole relationships in their reasoning. Although students appear to be comfortable with equivalent forms of the equation to, they seem to struggle with applying the modified equation to the line segment representation.

Another non-MU group expressed hesitation and confusion in their responses. Students seemed to think about letters as variables that could represent positive or negative values.

Jason: I got confused because you have to minus and then you'd actually have to graph negative *B* plus *C*, right? So you'd actually be graphing like that. And then I didn't know how to do like where the middle would be so you could actually just determine where a negative side was... and then well I don't know if you just draw an arrow showing that you have to go back cause its minus *B*, or that it is minus *C*. So, I wasn't really sure....

Tara: That kind of makes more sense … I don't know about the *A*, *B*, and *C*, but the *X* makes sense because *X* equals …

…

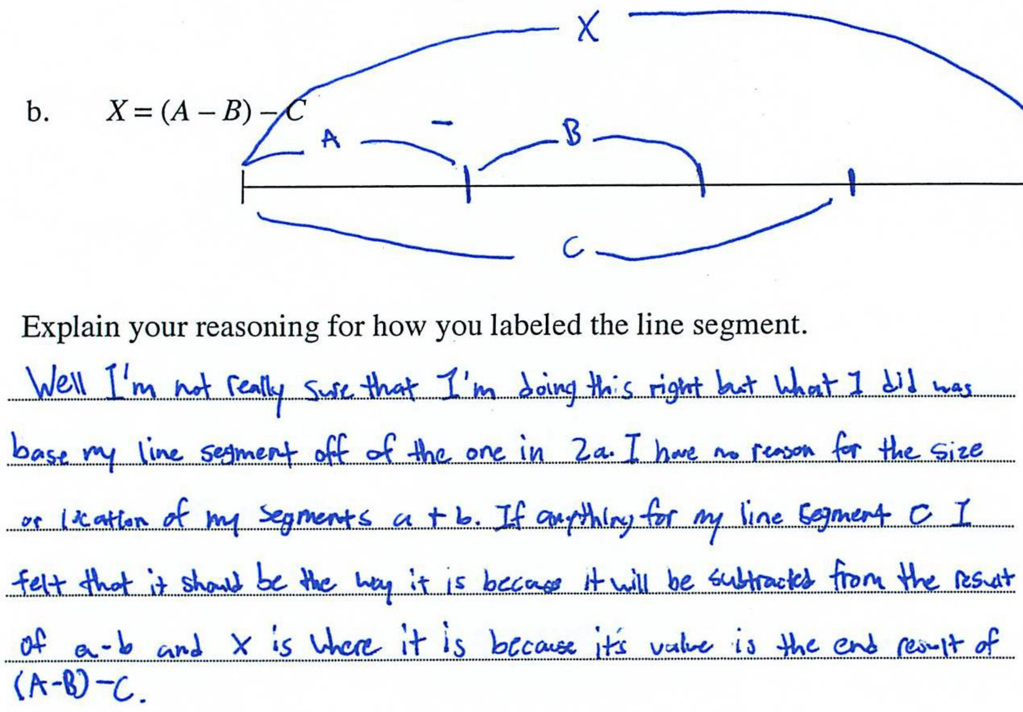


Figure 8. Non-MU student, Jason, indicates quantities of length, but misinterprets the part-to-whole relationship in the subtraction problem.

Cate: … my thinking here was more for [part] b, because since you are subtracting and the *A* is here, the *A* would be the biggest. … I still think *A* is the biggest here, but I still did this wrong.… Because I am subtracting and this would be more like adding.

…

Jason: It kind of depends on what you assign the values and how you do it.

Keisha: Yeah, I tried assigning values and that just doesn't work out.

…

Tara: I said there wasn't enough evidence to prove anything. Because *A*, *B*, and *C* could equal anything…

Keisha: I mean it doesn't say that it has to be like positive or negative so you don't really know like anything.

When asked what the segment meant, members of this group responded that it was something given in the problem. When asked what the letters represented, students responded that the letters represented numbers. This topic was revisited when students discussed the letters as variables that can have positive or negative values, but they were unable to connect this to the line segment representation. As with Jason's work (Figure 8), two other group members decided that *X* would be greater than the other quantities. When asked what the line segment represented, the group did not offer an explanation.

For Problem 2, in addition to noticing the part-to-whole relationship indicated by the line segment representation, students needed to make sense of how these quantities were related in the equations. Students had to consider how the given ratio quantities coordinated with relationships among length quantities indicated in the diagram. Although no prominent characteristics emerged from the analyses, we were able to assess perceptions of the problem and solution strategies.

Responses from Former MU Students to Ratio Quantities

Responses from former MU Grade 12 students on this problem varied widely. In addition to the types of responses used by the non-MU students, approaches used by former MU students were to change the format of the equations by manipulating algebraically or rewriting the ratio with a division symbol, and then making the representation reflect the new equation.

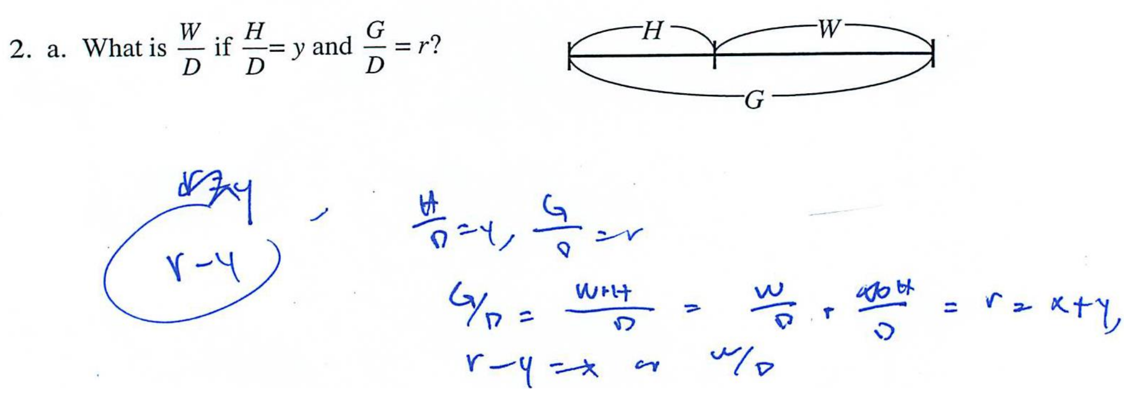


Figure 9. Former MU student, Reyn, who began his solution with information from the line segment.

During one MU group interview, students arrived at the conclusion that $\frac{W}{D} = r - y$, and discussed their thinking. One student said that she discounted the *D*'s and just looked at the line segment to decide that $W = G - H$, then reintroduced the *D* to present her answer. Another student, Reyn, also focused on the line segment and reasoned,

> If we divide *G* by *D* then you are dividing the value of *H* plus *W* by *D*. That's equivalent to *W* divided by *D* plus *H* divide by *D*. And since W divide by, *H* divide by *D* is *y* and *H* divided by *D* is *r* then *W* divided by *D* would be $r - y$. (Reyn, Grade 12, former MU student)

Reyn's work (see Figure 9) verifies that he has considered *D* in the relationship among quantities. The group continued to discuss the problem and lingered over the issue of the form of the solution as either $r - y = W$ or $r - y = \frac{W}{D}$. One student believed either equation was acceptable. When asked what they thought the letters represented, the students responded that they represented algebraic symbols and not numbers.

When asked if the line segment was necessary to solve the problem, the group agreed that it was helpful to see quantities *H* plus *W* making *G*, but that an equation, like $H + W = G$, would have been just as effective at conveying the information. These responses show thinking about relational thinking and algebraic manipulation as strategies for solving this problem.

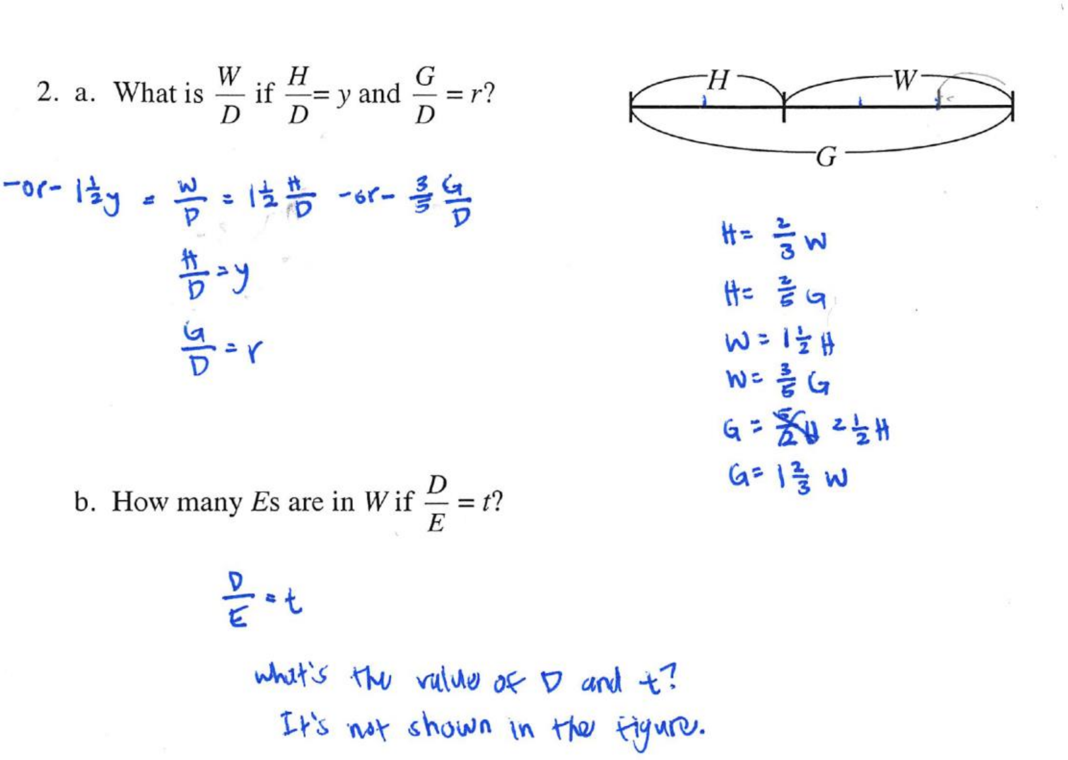


Figure 10. Response from former MU student, Alu, that shows the creation of a unit, measurement of the segment as 5 units long, and a set of relationships based on her unit.

Alu defined and marked a unit and quantified the lengths with her unit. This resulted in a marked segment from which she could then create a set of numeric length relationships (see Figure 10). Alu's work shows that she divided length *G* into five equal length units. During the group discussion she explained how this mark-up enabled her to then make numerical conclusions about several other relationships among the quantities. Alu did not discuss if or how she thought about quantities *D*, *y*, and *r* in the problem.

Responses from Non-MU Grade 12 and MU Grade 5 Students to Ratio Quantities

About half of the students in the non-MU Grade 12 and the MU Grade 5 groups wrote that $\frac{W}{D} = r - y$. Among these responses, some showed use of the parts-to-whole relationships from the line segment illustration together with the given equations. Other students simply wrote the solution and seemed to dismiss *D* altogether, such as the response in Figure 11 where the placement of the subtraction and equation symbols are aligned with the numerators and not the ratios.

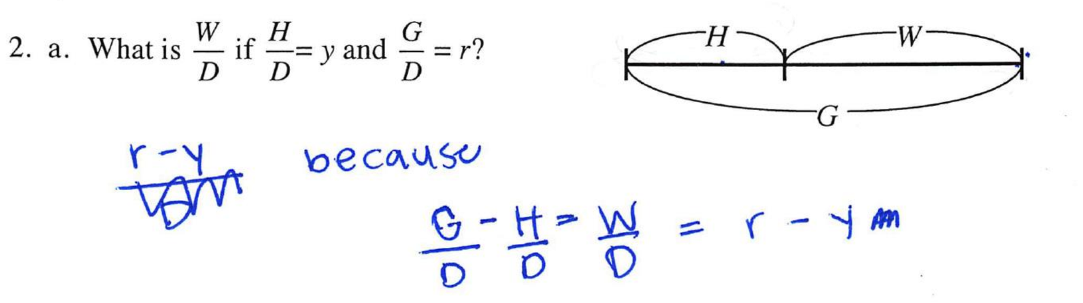


Figure 11. Response from non-MU student that appeared to disregard the role of D.

A cursory look at answers to the question "What is $\frac{W}{D}$?" might lead us to think that non-MU Grade 12 students were as successful on this problem as MU Grade 5 students. A more careful examination of their written work shows that Grade 5 students paid greater attention to the quantitative relationships than non-MU students. This was evidenced by the solution strategies and explanations written by Grade 5 students as well as their engagement in attempting a solution to the second part of the problem.

For example the response from Brayden (see Figure 12) lacks an explanation or illustration about his thinking, leaving no indication that he considered length quantities or measurement by a unit. However on part b, there is an illustrated length *E* within length *D*, all within length *W*. This indicates recognition of relationships among the lengths expressed in the equations. Although he was not able to provide a complete answer, as also indicated by the question mark and comment "ther[e] is nothing to represe[n]t it," this work shows consideration of ratio relationships and not mere treatment of the letters as representations disconnected from the length quantities.

In the interview with a non-MU group, one student shared how he thought about the letter representations in this problem.

Seth: … all the variables are over *D*, so I can just cancel out *D* basically, since it's all over the same.

Fran: Where did *D* come from?

Seth: Yeah, I don't know where *D* came from. So I basically like just looked at the top ones [numerators].

The researcher then asked Seth's group what the letters represented and they responded that the numerators were parts or segments from the line. Their discussion did not reference *D*. When asked if they thought differently about the use of upper and lower case letters, the students responded that the upper case letters represented something on the number line and the lower case letters were number values for the quantities.

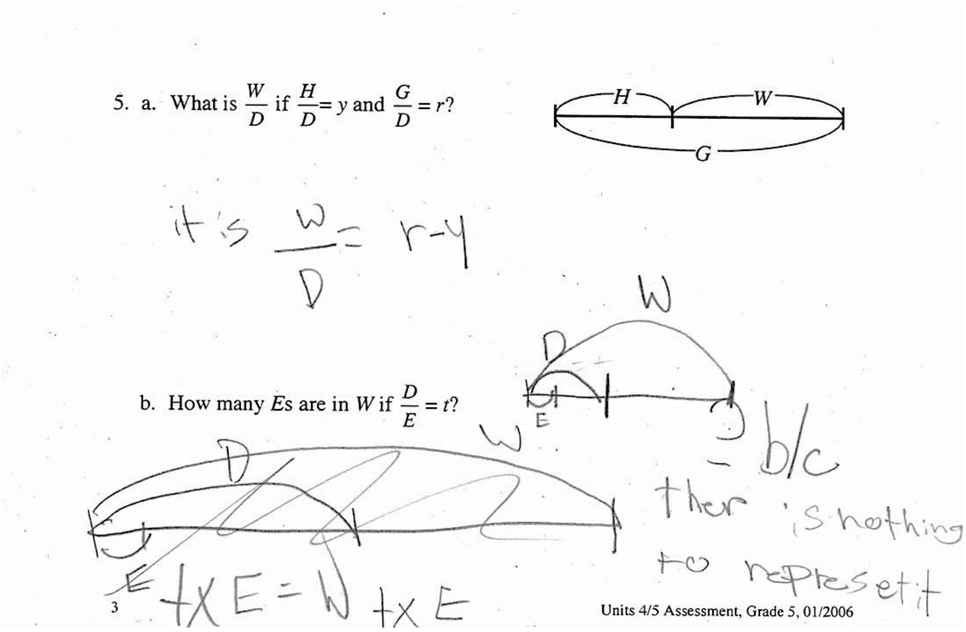


Figure 12. MU Grade 5 student's (Brayden's) response indicating relationships, as represented by the letters, as quantities of length.

Discussion

The MU group discussions clearly show attention to seeing and reasoning with nonnumeric quantities represented by letters. Although not all MU students were able to completely or coherently respond to the problems, overall, the MU groups understood that relationships given in one format should be preserved when represented in another. A few non-MU students responded similarly in their independent response, but expressed some uncertainty during the group discussion. These findings show that by the end of high school, students develop a strong urge to work with the letters in equations, but they might not be able to successfully model the given relationships in a length representation. This has implications for algebra instruction and connections with other mathematics, specifically in this case, measurement. Students may be engaged in the work of simplifying equations and expressions without making connections to conjectured quantities represented by variables. By seeing quantities that have a life of their own, misconceptions such as generalising $3a + 5b = 8ab$, may be avoided.

The line segments were not marked with an origin, a unit, or any numerical values. Some students accepted the nonnumeric context, while others tried to convert the segment to a number line by creating tick marks and assigning numbers along the segment. This leads us to question how non-MU students perceived drawn length quantities. Students seemed to exhibit some angst when working with nonnumeric quantities (i.e., variables), yet as Davydov's work reminds us, conceptualisation of such quantities occurs in everyday activities. Perhaps if early mathematics instruction focused on relating letter representations to tangible quantities found in everyday experiences, such as length, a wider range of students would find the study of algebra accessible and pragmatic for working with more complex mathematics.

Findings about how MU students at Grades 5 and 12 relate an equation to an iconic representation of nonnumeric quantities suggest that their understandings enable them to use length as a tool for noticing and reasoning about mathematical relationships. Such abilities were most evident in the Grade 5 responses. At Grade 12, although students seemed to apply the MU perspective with some success, responses indicate that they had accommodated additional perspectives for interpreting equations. Algebraic procedures from middle and high school curricula were apparent in the solution methods from the former MU and non-MU groups. This was evidenced in the attention students gave to the equations relative to the line segments. In middle school, students learn that a ratio is a comparison of one quantity to another and all ratios have a similar appearance, even though they may be used to represent different relationships, such as part-to-part, part-to-whole, and dividend-divisor. These conceptions broaden the ideas students consider to employ when solving problems.

From the MU curriculum, students learn that a quantity, such as length *H*, measured by a unit, such as length-unit *D*, can be quantified as some amount and that this amount could be represented by a number, a letter, or an expression. To the question about what $\frac{W}{D}$ was, one Grade 5 student wrote, "It would be $r - y$ because *r* is the same length as *G*. So you subtract *y* from *r* and get *W*." Although the student's response does not include an explanation of *D*, it illustrates reasoning about nonnumeric quantities of length and shows that she can think flexibly about multiple representations for the same quantity.

Success in algebra continues to strongly influence opportunities students have for advanced high school mathematics courses as well as their options for postsecondary education (Hein, Smerdon, & Sambolt, 2013). In their work to define a unified framework of algebra for all, Eddy et al. (2015) identified six key ideas for algebra from their analysis of various national and international policy documents and standardised assessments. Among these ideas are variables, functions, patterns, modelling, technology, and multiple representations. Our study of long-term effects of MU instruction shows how early experiences with measurement concepts can be used to support the development of the pillars in this algebra framework, particularly with regard to variables and multiple representations. Letters are used to represent unknown quantities of a measureable attribute throughout the MU curriculum. Students learn that a nonnumeric quantity can be measured by identifying and iterating a unit in order to arrive at a number value. MU students also learn that a quantity can be expressed more generally with a letter and that generalised notations can be used to represent mathematical relationships.

Our MU research illuminates an approach that uses measurement to introduce mathematics concepts in the early grades and produces a long-term residual effect in students' later mathematical conceptions of variables and representations. Based on findings from this study, MU students connect various representations, consider the dynamic state of measuring and the static state of representing a measurement, and are able to represent and make sense of quantitative relationships.

Limitations

Although the assessment tasks were atypical of what most U.S. students encounter in elementary or high school mathematics, interpreting equations, representing quantities, and modelling relationships are processes students developed throughout the grades. During the interviews, a few individuals from both the MU and non-MU groups shared their strategy of using the diagram in the ratio problem as a model for representing the quantities in the length problem. This strategy seemed to lead students to interpret the segment as a quantity of length and to focus on marking quantities in relation to each other. Some of the former MU students may have drawn on their work from their earlier school years, but this was not always apparent in their explanations. In either case, it is not clear if the context impeded students from responding to the full extent of their capabilities. This leaves us with some uncertainty about what is known of students' general capabilities for working with equations, representations, and relationships among quantities.

Future studies might address this concern by assessing students with a set of tasks more typical of mainstream algebra, in addition to the MU tasks. These responses could provide information about students' general capabilities for making sense of algebraic relationships given in equation and diagrammatic forms. Responses to the MU tasks and the algebraic tasks could then generate richer cases of students' capabilities.

Conclusion and Implications

Davydov's work in developing an elementary grades mathematics curriculum has inspired the work of the MU program. Translations, iterations, testing, and research of the curriculum have led to a nonnumeric alternative approach for the early study of mathematics. Through our research, we have found that numbers and computation often distract students to the extent that they overlook generalisations of relationships and structures. MU is designed for instruction to focus learning activities on relationships and representations in multiple, concurrent ways. The approach used in our experimental curriculum follows a didactic path outlined by Filloy, Puig, and Rojano (2010) for learning algebra. Heavy emphases are placed on translating, analysing, and interpreting concrete situations to and from algebraic representations. We believe these experiences help students make meaning of symbols, expressions, and properties in the study of algebra. Through MU instruction, young children learn to look for and make use of connections among concrete, iconic, and abstract representations to carry out operations and reason with quantities.

The objectives for this study were to assess how students related an equation to an iconic representation of nonnumeric quantities and to see what, if any, qualities in responses from former Measure Up students endured. Our findings show that MU students can perceive lengths as quantities, apply a method for marking and labelling quantities, reason with quantities in a generalised way, and justify representations of relationships given by an equation. These findings addresses the concerns raised at the beginning of this paper, with regard to the appropriate interpretation of symbols in relational contexts. The MU experimental curriculum appears to support students' understanding of nonnumeric expressions with generalised quantities, and their capabilities for modelling expressions with length.

This study could inform broader efforts in early mathematics research as we aspire to create effective curriculum for instruction that lead students to understanding fundamental symbols and expressions needed in the preparation for algebra and the study of more advanced mathematics. Where the vast majority of contemporary U.S. elementary mathematics curricula and instruction begin with numbers, children tend to memorise what they believe to be rules in mathematics. Unfortunately, some of these rules are based on their work with whole numbers and therefore expire with the introduction of more advanced topics (Karp, Bush, & Dougherty, 2014). This, unfortunately, perpetuates demands for reteaching elementary topics in the higher grades.

A nonnumeric measurement perspective could help students develop an enduring understanding of properties that do not expire with the study of later mathematics. This could potentially address the issues currently found in the transition from arithmetic to algebra. If this perspective and Davydov's general-to-specific approach to curriculum were to be applied to the development of curriculum for older students, sophisticated concepts (e.g., complex quantities, such as speed) and representations (e.g., coordinate graphing) may be more meaningful to students and create greater access for more students to continue studying.

Acknowledgements

This research was supported by the Curriculum Research & Development Group. The author would like to thank the students who participated in the study, their teachers, and the school for their enthusiastic cooperation. The author would also like to acknowledge her colleagues from the Measure Up project, Ms. F. Zenigami, Dr. H. Slovin, Dr. M. Olson, Dr. J. Olson, and Dr. B. Dougherty whose suggestion set us on this path.

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