How the Relational Paradigm Can Transform the Teaching and Learning of Mathematics: Experiment in Quebec

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The main goal of this paper is to show how Vasily Davydov's powerful ideas about the nature of mathematical thinking and learning can transform the teaching and learning of additive word problem solving. The name Vasily Davydov is well known in the field of mathematics education in Russia. However, the transformative value of Davydov's theoretical work in this field has not yet been fully recognised by the larger international community. In this article, I use Davydov's vision of mathematics as the study of quantitative relationships - the vision underlying the relational paradigm in teaching and learning of mathematics. I use the example of one research project to demonstrate how the relational paradigm changes and reforms teaching practices. I discuss the teaching approach, developed within this paradigm, as well as the learning outcomes demonstrating the transformative power of Davydov's ideas.

About the research project (introduction)

Solving arithmetic word problems unmistakably represents an important part of any elementary school mathematics curriculum. It is not surprising then that the questions related to teaching mathematical problem solving have been continually studied by researchers from all over the world. Students begin to solve arithmetic word problems very early, when they first learn about addition and subtraction. In Quebec, the mathematics curriculum (MELS 2001) has traditionally required the development of profound understanding of arithmetic operations in students before teaching problem solving. However, in 2001, in light of recent research highlighting the importance of mastering the concept of additive and multiplicative structures of a problem, the Quebec ministry of education adopted a new elementary school math curriculum (Ministère de l'Éducation, Enseignement superieur et Recherche). In classrooms, students usually begin to solve arithmetic word problems of addition and subtraction in their first year of compulsory education, i.e., when they are about six years of age. In order to help students deepen their understanding of arithmetic operations, the Quebec new curriculum guides teachers to engage students in solving more complex problem-situations. Nonetheless, many teachers share concerns about how to efficiently guide and effectively support students in the development of profound understanding of addition and subtraction and what the place of additive structures should be in this process (Gervais et al., 2013). It goes without saying that despite the teachers' efforts, students often get stuck or choose the incorrect operation while solving word problems.

In order to improve students' reasoning development for problem-solving, the research team, in which I participated as one of the principal investigators, devoted three years of work to help teachers revise and redesign their teaching practices to fully meet the curriculum requirements. We began with a thorough literature review examining multiple empirical studies related to problem solving, additive structures, mathematical understanding and mathematical thinking, algebraic and arithmetic reasoning, and neuroeducation. This examination of the literature guided us to identify a learning theory that can be used as a framework to reform practitioners' understanding of what an additive word problem solving is and how the development of related knowledge can be approached. Such a framework was identified in Davydov's work on the theory of developmental instruction (Davydov 2008), which played the main role in our work with teachers in this study. Later in this paper, I will describe the paradigmatic change we identified in more detail.

Within the scope of the research project, we studied current teaching practices and examined students' solving strategies. We then formulated and proposed an original teaching approach, the Equilibrated Development Approach (EDA), to teach additive problem solving (I describe the theoretical basis of this approach later in this paper). Throughout the research, we designed, tested and adjusted many teaching/learning activities and didactic scenarios to nourish the teaching practice. To attain a deeper understanding of the processes of change and reform in the teaching practices, we set up experimental groups and control groups and regularly gauged students' problem-solving performance. The study lasted for three years working with experimental groups that consisted of 18 teachers of Grade 1 and 2 classes (N= 203 students of 6 to 8 years of age of whom 96 were girls and 108 were boys), who participated in implementing and testing the new teaching practices. The control group consisted of 208 other students (between 7 to 8 years of age of whom 86 were girls and 122 were boys.) Some of the results of this study are documented in previous publications (Savard et al., 2013; Savard & Polotskaia 2014; Savard & Polotskaia 2013; Polotskaia et al., 2015a; Polotskaia et al., 2015b; Polotskaia 2015).

In this paper, I discuss the transformative role that Davydov's insights and teachings played throughout this project and delineate successful transitions to more relational-directed teaching practices. Firstly, I compare the traditional theoretical perspective of teaching arithmetic word problems with the relational framework inspired by Davydov's theory. Secondly, I analyse how Davydov's relational paradigm transforms our vision of students' difficulties in solving problems. Thirdly, I show how the relational paradigm informs the design of new problem-solving tasks. Fourthly, I identify several principles inspired by Davydov's tenets and suggest changes in teaching practices. Finally, I present results to demonstrate the differences between the experimental and the control groups.

Traditional theoretical framework

Schoolchildren's difficulties in solving arithmetic word problems are well documented in the literature (e. g., Vergnaud, 1982; Barrouillet & Camos, 2002; Checkley, 2006; Riley & Greeno, 1988) and confirmed by international studies such as the PISA (Artigue 2011). These difficulties are usually described as an incorrect choice of arithmetic operation or inability to solve a problem. Thus, the knowledge of operations is implicitly, and sometimes explicitly, recognised as the basis for successful problem solving. For example, Barrouillet and Camos (2003) describe the following sequence of knowledge development in early elementary years: pre-mathematic concepts (such as classification, order, naming), concept of number, counting, and addition. Similarly, Riley and Greeno (Riley & Greeno 1988) also speak about numbers and operations:

> Successful performance [in problem solving] requires both the general ability to represent quantitative information and the specific knowledge required for understanding the terms in problem texts and carrying out operations (p. 84).

Furthermore, researchers suggest that basic knowledge about operations should be further developed into a conceptual understanding through the acquisition of various meanings of each operation. For instance, with regard to the subtraction operation, Brissiaud (2010) explains that to have a conceptual knowledge of subtraction means having different senses (meanings) of the subtraction operation, such as finding the difference, finding the complement, and taking away. We can propose that number and operation are the most important concepts within the scientific discourse about problem-solving knowledge development in young children.

Today, the great majority of researchers also recognise the importance of mathematical structures in additive problems. Vergnaud (1982) defines a conceptual field where addition and subtraction belong as a conceptual field of additive structures.

> I believe that the concepts of measure (of discrete sets and of other magnitudes), addition, subtraction, time transformation, comparison relationship, displacement and abscissa on an axis, and natural and directed number are also elements of one single conceptual field, the field of "additive structures" (Vergnaud, 1982, p.40).

To better understand students' difficulties, researchers classified problems according to their semantic structures. The most known classifications of additive problems (Vergnaud, 1982; Nesher et al., 1982; Riley et al., 1984; Carpenter et al., 1999) are based on the semantic meaning of the actions of adding, retrieving, and comparing. These classifications propose multiple structures based on the semantic meaning of the action (relation) and the place of unknown element within the semantic structure. For example, the Separate category (Carpenter et al. 1993) comprises situations where something is retrieved, removed, eaten, etc. The following problem is a Separate problem with the result unknown: "Paco had 13 cookies. He ate 6 of them. How many cookies does Paco have left?" (Carpenter et al. 1993, p. 434).

Researchers (Pape 2003; Thevenot 2010; Coquin-Viennot & Moreau 2007) acknowledge that the most difficult problems are those whose solution requires an operation that is inconsistent with the action (relation) described in the problem. In the following problem, the expression "7 **more** apples than pears" is inconsistent with the operation of subtraction needed for solution.

> There are 25 apples and some pears in a basket. There are 7 more apples than pears. How many pears are there in the basket?

In relation to the consistent and inconsistent language of the problems, it was recognised (Nesher et al. 1982; Vergnaud 1982) that there is also a distinction between a numerical sequential understanding of operations supporting the solving of consistent problems and a flexible relational understanding of problems' structures required for inconsistent problems to "invert the operation." Vergnaud (1982) gives two different names for such reasoning: numerical calculus and relational calculus.

> To interpret the behaviour of children faced with elementary arithmetic problems, I find it essential to distinguish two sorts of calculus: "relational calculus" and "numerical calculus." By "numerical calculus" I mean ordinary operations of addition, subtraction, multiplication, and division. By "relational calculus" I mean the operations of thought that are necessary to handle the relationships involved in the situation (Vergnaud 1982, p. 40).

Multiple empirical studies (Beckmann 2004; Lesh & Zawojewski 2007; Novotná 1998; Lingefjärd 2011; Xin 2012; Ng & Lee 2009; Xin et al. 2008) on the use of representations in solving word problems confirm, without qualification, the importance of the relational understanding of additive problems by showing the positive impact that teaching structural representations of problems has on students' learning. As soon as a model or representation describes the quantitative relationship involved in a situation, it can potentially support the relational reasoning required to solve the problem. Nonetheless, the representations and models are mostly seen by researchers and practitioners as a means to facilitating the problem-solving process for students, and the understanding of structures is seen mostly as knowledge that can be developed based on already-attained knowledge about numbers and arithmetic operations.

Examining the research literature on the subject of arithmetic problem solving mentioned above, we identify a governing paradigm that we dub the Operational paradigm. Within this paradigm, the knowledge development process is expected to begin from the number concept and arithmetic operations and grow towards a flexible understanding of additive structures. Various researchers propose different stages (three or four) of such development (e.g., Nesher et al. 1982; Riley et al. 1984). Figure 1 schematically presents the knowledge development process seen from the point of view of the Operational paradigm.

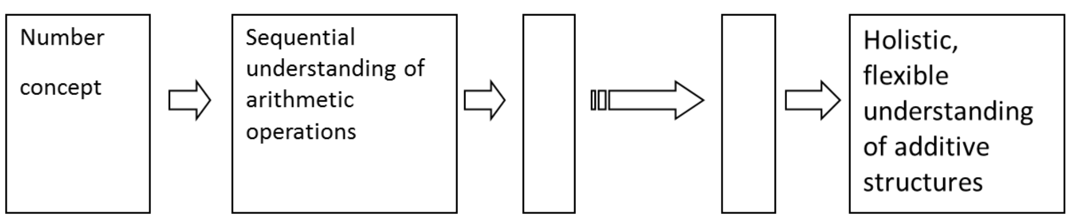

Figure 1. Problem-solving knowledge development within the Operational paradigm

Shift of paradigm (new theoretical framework)

Reading works of Davydov (Davydov 1982; Davydov 1990; Davydov 2008) yields a completely different view on the number concept and additive problem-solving knowledge development. Firstly, Davydov proposes that the concept of number be seen as a multiplicative relationship between two quantities. Thus, to produce a number, one quantity can be measured by using the second quantity as a unit of measurement. According to Davydov, the development of the number concept should be based on the understanding of the measurement principle and the idea of unit of measurement. Several studies (Gerhard 2009; Mellone & Tortora 2012; Schmittau n.d.; Dougherty & Slovin

2004; Iannece et al. 2010; Iannece et al. 2009; Schmittau 2005; Schmittau & Morris 2004) have explored this perspective and have empirically substantiated Davydov's ideas.

When speaking about addition and subtraction, Davydov (1982) defines a concept of additive relationship as "the law of composition by which the relation between two elements determines a unique third element as a function" (p. 229). Davydov (1982) argues that the additive relationship is the basis for learning addition and subtraction and should be a part of mathematics curriculum preceding the study of these operations on numbers. From this point of view, addition and subtraction operations are not the means to understanding a situation (problem), but serve as tools to modify the situation, once understood as additive relationship. In the problem-solving context, this means that one should first grasp the additive relationship described in the problem and then derive from this relationship the arithmetic operation needed to find the unknown element. For example, the Paco problem should first be interpreted as an initial number of cookies composed of the number of cookies eaten and the cookies left. The apples and pears problem should be interpreted as a number of apples composed of a number equal to pears and some extra apples.

What makes this approach special is that the additive relationship does not include any action, but only describes the relationship between elements and how they comprise the whole and the parts. This fundamental principle of composition can be understood by children very early on and, according to Davydov, should develop into a flexible understanding of additive structures. In his experiments, Davydov proposed students first to play with physical objects, such as water, ropes and paper strips, to help them understand and master the composition principle. Only when students were able to describe and operate upon additive relationships of continuous quantities in a general form (A=B+C) did they begin to solve additive problems.

Summarising this short overview, we dub Davydov's theoretical approach as the Relational paradigm. Figure 2 schematically presents the order of knowledge development within the Relational paradigm.

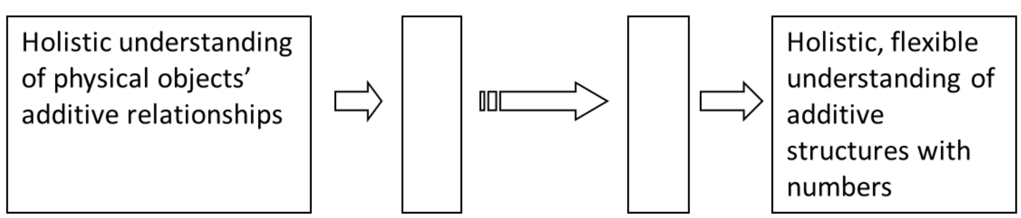

Figure 2. Knowledge development within the Relational paradigm

Working within the context of Quebec's compulsory education with a well-established mathematics curriculum, it was not possible to drastically change the order of teaching/learning mathematics in elementary school that mostly draws on teaching mathematics within the Operational paradigm. At the same time, research (Okamoto 1996; Nesher et al. 1982; Riley et al. 1984) has long substantiated the important role that number knowledge plays in supporting successful problem solving. In order to reconcile these differences, we considered both numerical reasoning (addition and subtraction of numbers) and holistic reasoning about additive relationships as two necessary parts of problem-solving knowledge development. We proposed to organise teaching so that students have the opportunity to develop holistic structural reasoning and numerical reasoning in harmony and coherence with each other. To reflect this balanced view of the two interlaced objectives, we named the teaching approach the Equilibrated Development Approach (EDA). Thus, within the EDA, knowledge development is grounded in both the holistic understanding of additive relationships of physical objects and sequential understanding of arithmetic operations (Figure 3).

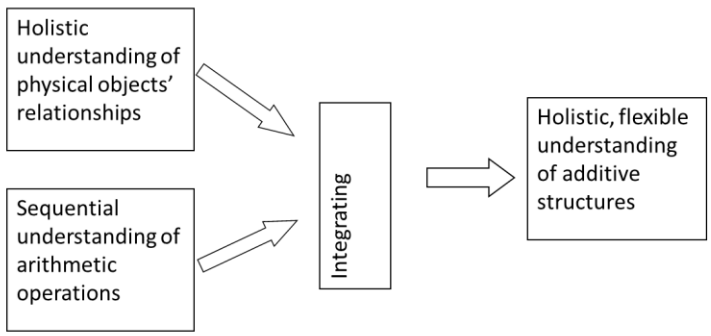

Figure 3. Knowledge development within the Equilibrated Development Approach

While developing new teaching/learning activities, we tried to complement existing teaching practices with some special activities focused on the development of relational reasoning within the context of solving additive word problems. For this reason, we mainly used the Relational paradigm as the theoretical framework for the research project. As a result, the Relational paradigm profoundly shaped our vision of the field of additive word problem solving and guided our reformulations and reconceptualisation of problem-solving difficulties, problem-solving tasks, and problem-solving teaching practices.

Questioning problem-solving difficulties

The process of solving an additive word problem looks different within the two paradigms. The Operational paradigm proposes to see this process as the transformation of a word description containing numbers and some semantic meaning into an arithmetic operation.

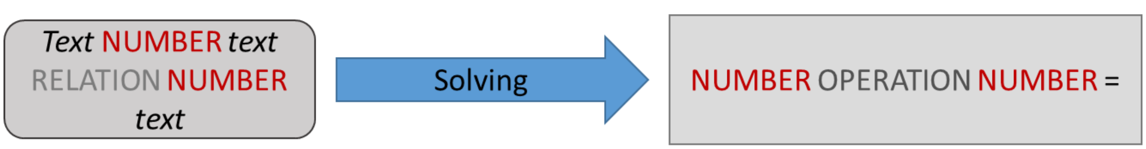

Figure 4. Problem-solving within the Operational paradigm

Accordingly, additive word problem-solving difficulties are seen as an inability to figure out the appropriate arithmetic operation.

Researchers adopting the Operational paradigm claim that the easiest problems are those with an unknown final state (e.g., Nesher et al. 1982; Riley et al. 1984; Carpenter & Moser 1982). The semantic meaning of the verb in these problems can be directly interpreted as the arithmetic operation to use. For example, the Paco problem can be interpreted as 6 cookies being subtracted from the initial number of 13 cookies.

The difficulty arises when this direct interpretation is no longer possible. Thus, the semantic meaning of the relation more in the apples and pears problem cannot be directly interpreted as subtraction, which is required. Within the Operational paradigm, problem-solving difficulty can be described as an inability to transform one semantic meaning into another (ex. adding into removing).

However, from the perspective of the Relational paradigm, the problem-solving process includes the phase of analysis and identification of the additive relationship described in the word problem, as well as the phase of planning or constructing the operation to calculate the unknown element.

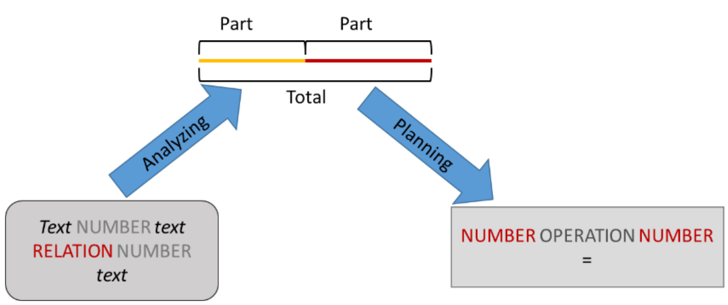

Figure 5. Problem-solving within the Relational paradigm

From this point of view, the problem-solving difficulty looks different. The student, who cannot solve a problem, cannot do so because of not seeing the additive relationship present in the word problem and constructing the operation while considering the relationship.

What is special about this approach is that, theoretically, all additive word problems should be equally difficult if solved in two steps: transforming the semantic meaning of the text into an additive relationship, which is followed by transforming the relationship into the operation. At least one of these steps, if not both of them, requires some transformation of meaning. Thus, to solve the Paco problem, one should first transform the eating action into a part-part-whole relationship, then transform the relationship back into the subtraction operation. To solve the apples and pears problem, one should transform the more relationship into a part-part-whole relationship and then transform the latter into an operation.

At first glance, it seems as though the Operational paradigm is better grounded in students' natural understanding of the semantic meaning of situations. At least, some problems are less difficult (not at all difficult) and learning can begin at this point. More importantly, the semantic meanings of real life situations, such as receiving or removing, better support the basic meanings of arithmetic operations (addition and subtraction). Thus, the learning of operations can begin from these basic meanings.

A second glance, however, reveals some disadvantages of the Operational paradigm in favour of the Relational paradigm. First, the excessive use of additive problems with the final state unknown, which can be directly interpreted into arithmetic operations, produces an overgeneralisation effect. The students begin to think that more always corresponds to addition and remove always corresponds to subtraction. Instead of developing flexible reasoning, they limit themselves to the direct translation of keywords into operations (Hegarty et al. 1995). As a result, the Operational approach can potentially reinforce students' difficulties instead of remediate them. In contrast, the Relational paradigm requires two steps of transformations of meaning to solve any problem. This means that solving any problem will help students to develop the flexibility of reasoning.

Second, within the Operational paradigm students sometimes confront cases with no difficulty (such as the Paco problem) and others with great difficulty (such as the apples and pears problem), the latter requiring a more complex process of transformation of meaning. The Relational approach proposes a process of transformation of meaning in two steps - each of which can potentially be of moderate difficulty.

The third argument in favour of the Relational approach is the presence of the intermediate step of representation. Apparently, the additive relationship, identified in the word problem, can be easily represented in a graphical way (Vergnaud 1982; Cai et al. 2005; Schmittau 2005; Kaur 2008; Davydov 1982). I described one such representation (Polotskaia 2010) using line segments through the technique of Arrange-All diagrams (A-A diagram). These diagrams represent the relationships between quantities rather than the numbers per se. What makes this technique effective in teaching additive word problems is that even unknown elements can be represented by line segments (see Figure 6 and Figure 7 for examples of representations) thus enabling a more and deeper understanding of the relationship between the elements. The diagram can then serve as a model in the same way that models are usually used in solving more complex mathematical problems (Lesh & Zawojewski 2007) to support deeper and efficient mathematical reasoning.

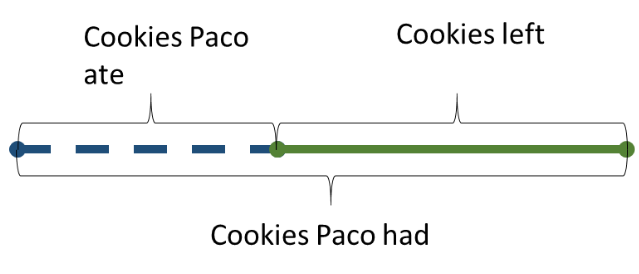

Figure 6. Representation-model of the Paco problem

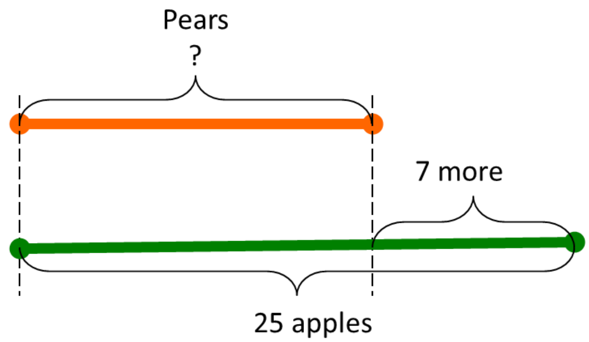

Figure 7. Representation model of the apples and pears problem

The Operational approach, that is more often than not, implemented in today's classrooms usually requires students to represent numbers aiming to the calculation process. The need to represent numbers often makes representing the relationship impossible, because unknown numbers cannot be represented. For example, if in the Paco problem the initial number of cookies were unknown, it would be impossible to represent the initial number of cookies and the process of eating.

Summarising the three arguments that support the use of the Relational paradigm in teaching additive word problems, it becomes clear that the Relational approach provides stronger support for flexible holistic reasoning development, potentially helping to avoid overgeneralisation of keywords. Furthermore, it allows the allocation of time and space to construct an explicit representation and modelling of the mathematical relationship between known and unknown elements. Another important potential benefit of the Relational approach is the type of reasoning it promotes. Understanding a situation in a relational way and operating on known and unknown elements of the relationship requires a type of reasoning similar to the algebraic thinking (Squalli 2007). Therefore, students can gradually develop conceptual understanding and skills to support the further development of algebraic thinking.

The use of Arrange-All diagram representations within the Relational approach to problem-solving opens new horizons in designing and implementing problem-solving tasks.

Reconsidering problem-solving tasks

Traditional problem-solving tasks in elementary school mathematics, more often than not, tend to present word problems as situations and questions, which can be answered by applying some arithmetic operations to numerical data from the text. The explicit goal of such tasks is finding the numerical answer. In many cases of simple additive problems (such as the Paco problem, for example), this goal can be attained without any relational analysis of the situation at hand (Nesher et al. 1982; Vergnaud 1982). Thus, finding a numerical solution does not directly support the elicitation of holistic relational thinking. Furthermore, analyses of students' difficulties within the Relational paradigm (Polotskaia 2015) reveal two possible issues. It can be difficult to see the situation as an additive relationship and model it, and it can sometimes be difficult to manipulate this model to figure out the required arithmetic operation. In light of these issues and within the context of a curriculum favouring numerical sequential reasoning development, the relational holistic vision of a situation can indeed be difficult for students (Polotskaia 2015; Malara & Navarra 2002). The traditional formulation of problem-solving tasks that explicitly requires a numerical answer also contributes to this difficulty.

How can the traditional task of word problem solving be re-designed to strongly elicit working with relationship instead of immediately working with number? The Relational paradigm helps demonstrate that the first tasks that students encounter relative to problem solving should not be about solving, but about analysing and modelling the situation in terms of relationship. So, the explicit goal of the task should be reformulated from finding a solution to finding the relationship.

In the literature, there are several non-traditional ways to formulate a word-problem task. School textbooks, that draw on Davydov's ideas, require students to examine models (similar to the Arrange-All diagrams technique), construct models, compare models, and compose a problem using model (Alexandrova 2004). Bartolini Bussi et al. (2011) describe "problems with variations" used in Chinese textbooks where three problems are formulated within the same context, each using a different element as unknown. For example the Paco problem formulated with variations would be associated with two "sibling" problems:

> Paco had some cookies. He ate 6 of them. Paco has 5 cookies left. How many cookies did he begin with?
>
> Paco had 13 cookies. He ate some of them. Paco has 5 cookies left. How many cookies did he eat?

The Relational paradigm helps explain the positive effects of using these types of tasks of using different elements as unknown. Instead of just calculating the required number, students are involved in discussion about the problem's structure. In Russian textbooks, the use of (structural) models is explicitly required. In Chinese textbooks, the comparison of situations having similar (in some sense, identical) structures makes the structure more visible for students.

Further development in this direction led us toward the formulation of other tasks which strongly elicit the analysis of the relationships between quantities involved in a word problem (Polotskaia et al. 2013; Ducharme & Polotskaia 2010; Savard et al. 2013). One such task (Polotskaia et al. 2013; Savard et al. 2013) presents students with a situation where the numerical data is incompatible with the semantic meaning - creating a Mathematically Impossible Situation (MIS) which, in turn, creates the conditions for students to look for the relationship between the elements. Any traditional word problem-solving task can be transformed into an MIS. The following is the example based on the Paco problem:

> Paco had 10 cookies. He ate 6 of them. Paco has 5 cookies left.

The students realise that the situation is not possible, so they need to explain why not. Analysing the relationship between the quantities involved, they discover that the quantity of cookies at the beginning should necessarily comprise two parts: cookies eaten and cookies left. Students, then, should find different ways to "correct" the situation (one of the three numbers), each time using an appropriate arithmetic operation. To correct the impossible Paco situation, one can propose the following three solutions:

1. 4 instead of 5 cookies left (10-6=4)
2. 5 instead of 6 cookies eaten (10-5=5)
3. 11 instead of 10 cookies at the beginning (6+5=11)

Another task (Ducharme & Polotskaia 2010) is a game which can also be based on any traditional word problem. The game requires the communication of the mathematical structure of a problem and is called the Mathematical Structure Communication Game. A team of 3–4 students chooses its captain, who then leaves the room. The rest of the team analyses a word problem and tries to represent it graphically while respecting the following rules:

- They may not write words or letters
- They may not use mathematical or other symbols for operations.
- Only the numbers appearing in the problem can be used.
- They can also use the symbols "=" and "?", as well as any drawings.

This representation should then serve the captain as a message describing the problem. The captain, who did not read the word problem, uses the visually represented message to make sense of it mathematically by proposing arithmetic operations with appropriate numbers to calculate the numerical answer to the initial problem. The team that wins the game (against other teams) is the one whose message clearly represents the problem and the captain is able to obtain a correct numerical answer.

The MIS task helps to explicitly attract students' attention to the existence of a stable relationship between numbers/quantities used in word problems. The game of representations helps students realise that the ideal way to win the game is to first represent the structure of the problem, and not numbers. Thus, both the MIS task and the Mathematical Structure Communication Game activate students' analysis of mathematical relationship and mathematical structure in word problems.

Both tasks were created with respect to the following proscriptive design principles (Savard et al. 2013):

- The task should be based on a situation involving a simple additive relationship between three quantities.
- The task should involve students' mathematical analysis of the described relationship as a whole. It should help students discover different properties of the relationship and see how different arithmetic operations can be used in the described situation for different purposes.
- The task should not contain any explicit and immediate questions that could be answered by finding one particular number. This criterion is followed to prevent students from immediately calculating the answer. However, the task should include an intriguing element, which would support their natural interest and commitment.
- The goal of the task, which is learning to analyse the situation, should be explicitly communicated to students (Savard et al. 2013, pp. 273-274).

The task does not become a learning activity without the appropriate teaching/learning process. Introducing the Relational paradigm into traditional curriculum requires significant adaptation of teaching practices.

Reconsidering teaching practices

Usually, word problem-solving tasks are implemented in a classroom in a straightforward way. The following four steps proposed by Polya (1945) have been adopted and used in many countries: understand the problem; create a plan; execute the plan; and look back to check the results. Unfortunately, the initial idea of rounding up the analysing-solving process ("look back to check the results") often becomes a simple verification of calculations or just vanishes from the teaching practice (Gervais et al. 2013). Many researchers (Mellone et al. 2014; Leung 2009; De Corte 2012; Savard 2008) stress the importance of reviewing one's reasoning and of much re-thinking within a problem-solving activity. Within the Relational paradigm, the explicit attention to the organisation of students' reasoning becomes vital because it potentially supports the transformation of a sequential, movie-like mental representation of the situation into a holistic and systemic one.

Below, I describe how the modified teaching practice of work on a traditional problem-solving task can promote holistic and systemic analysis in students. The following is a short description of a problem-solving activity for Grade 2 elementary school class. The teacher implements the ADE framework working with the following traditionally presented problem:

> There are 187 adults in the dining room of a sugar shack. There are also 74 kids. How many people are there in the sugar shack?
>
> Before proposing the problem, the teacher asks students about two types of representations they used to analyse word problems, and the cases each of them should be used. She announces that today they will continue learning how to solve problems. After careful reading of the problem, the teacher proposes that students work in teams of two to choose the appropriate diagram and represent the problem.
>
> After a short period of time, the whole-group discussion begins with the re-reading of the problem. The teacher then asks what type of diagram should be used. One student correctly proposes the one-line diagram. The teacher asks whether anybody used the other type and why. The students explain that the other type is used when something is compared, which is not the case in this problem. The teacher asks students to invent a question to make this problem a comparison one. The students propose "How many more adults are there than kids?" and "How many less kids are there than adults?"
>
> Having agreed upon the diagram type for the original problem, the teacher asks the students to complete the representation. The students propose two different ways to represent the situation (Figure 8).

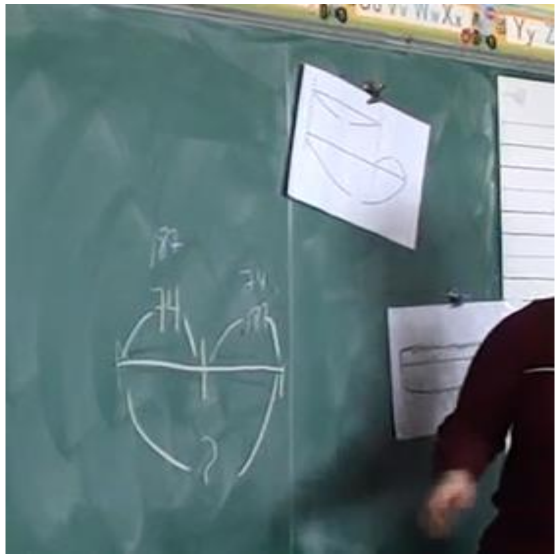

Figure 8. Two ways to represent the sugar shack problem. 74+187=? and 187+74=?

The teacher asks: “Are the two representations correct, and why?” The students explain that the order does not matter within the total. Then, the teacher asks the students to look at the diagram and to formulate a mathematical expression to find the total number of people in the dining room. Students propose: 187+74=. The teacher shows the operation on the diagram, follows the 187 part with her finger, and then the 74 part. Another student proposes: 74+187=. The teacher also shows this solution on the diagram.

One of the students writes a calculation process on the blackboard to obtain the numerical answer 261.

The teacher then erases all except the diagram on the blackboard and asks students to think a bit further (Figure 9). She says: “You know that there are 74 kids in the dining room, and you also know that there are a total of 261 people. If you want to know how many adults there are, what mathematical expression should you use? Look at the representation we have.”

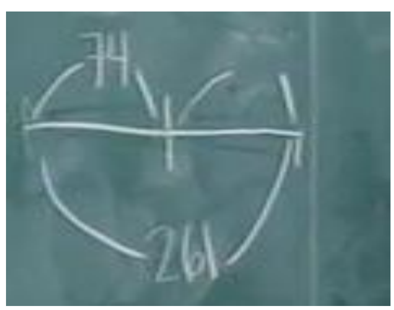

Figure 9. Representation of the modified problem. The number of adults in unknown.

One student proposes 74-261=. The teacher tries to show this operation on the diagram and asks whether it works to find the number of adults. The students agree that if they remove all the people, no adults will remain. The teacher repeats the question while hiding the part of the diagram representing the 74 kids with a piece of paper to make only the adults part visible. A student proposes: 261-74=. The teacher comments: “Imagine all the people in the dining room. I am asking that all kids move to a corner. So, who will remain in front of me?” Students: “Adults.”

In this episode, the word problem in a traditional form is used when students are familiar with the idea of an additive relationship and with the A-A diagram representation/modelling technique. The teacher's explicit request to construct a representation transforms the traditionally implicit step "understand the problem" into an analysis and modelling process. Instead of finding known data and keywords in the text of the problem, the teacher invites students to think about the type of situation at hand (comparison or not) and to represent all the important elements - known and unknown - as line segments of a part-whole relationship. At this point, the numbers play a secondary role: they identify the parts of the relationship. The attention is focused on the role each element plays in the story. The discussion of the different possibilities of representation reveals the commutative property of the relationship.

The teacher insists on using the constructed diagram to find the mathematical expression. Thus, the "create a plan" step, traditionally consisting of proposing an operation and representing tens and units of given numbers, is transformed into manipulation with the constructed model and the use of its properties to figure out the unknown element. The "execute the plan" step, which is the calculation of the mathematical expression, does not have much importance in the discussion because, in line with the Relational paradigm, the calculation is not the learning target of this problem-solving activity. In contrast, within the traditional approach, the calculation is often the main goal of the problem-solving activity. The "look back to check the results" step, within the Relational approach, is replaced with a holistic analysis of the situation. The teacher modifies what is known and what is unknown in the situation while keeping the same situation (relationship) in place. This final discussion explicitly invites students to consider the situation as a whole and see it from different perspectives. The use of the same diagram "confirms" that the situation is the same and just the unknown is different. The diagram-model of the situation becomes the centre of the activity. The whole discussion "buckles up" around the model, returns to it many times, questions, and re-questions it.

The classroom example described above shows how the teaching practice, developed within the EDA, transforms the traditional word problem solving task into an activity of analysing and modelling an additive relationship. The calculation (working with concrete numbers), which usually represents a main part of the problem-solving activity in early grades, plays a secondary role in this new activity and is not the learning target. It does not mean that working with numbers is not important in the early-grade mathematics - it is. However, calculations with numbers can be worked on separately within other activities. Within the Relational paradigm, the relational aspect of problem solving is the focus in order to better balance numerical and relational reasoning development.

The analysed episode shows that the new teaching practice does not make the solving process easier for students. In the traditional classroom, this problem could probably be solved much faster by just adding two numbers. However, the new approach elicited students' active discussion of a problem's mathematical structure, thus supporting desired thinking development.

Learning results

According to Davydov's theories and in line with the Relational paradigm, the development of the holistic relational thinking should provide students with stronger support in solving additive word problems. Other researchers (Vergnaud 1982; Riley et al. 1984; Nesher et al. 1982) confirm that relatively difficult problems are those requiring holistic flexible thinking for their solution and that easier problems can be solved by using sequential numerical thinking. Therefore, in order to evaluate the effectiveness of the new teaching approach, one should consider at least two characteristics of students' performance: the average success level in solving problems and the type of reasoning. The first characteristic is easy to obtain, but the second is a bit trickier. I used the difficulty gap between different types of problems, well observed by researchers (e.g. Nesher et al., 1982), to create a problem-solving test containing two "easier" and two "more difficult" problems (see Annex 1, page 180). The difficulty gap within the test was then confirmed by a statistical experiment (Sig. <=0.005): within the control group the average success level in solving the "easier" Homework and Marbles problems (0.68 and 0.59, respectively) was significantly higher than that of the "more difficult" Ladybug and Rolls problems (0.34 and 0.36, respectively). As researchers explain (e.g. Nesher et al., 1982), this difficulty gap is the characterisation of the difference in thinking: sequential numerical versus relational. Thus, the difference of success in solving "easier" and "more difficult" problems should characterise the level of relational thinking development in students.

Over four years, I administered the developed test to students at the end of the Grade 2 school year. In total, 203 students learning within the experimental approach (EDA) and 208 students learning in the traditional way wrote the test. Before beginning the test, I explained to students that the desired solution is the mathematical expression, which explains how to calculate the answer for the problem ("2 + 3 = ?" being acceptable, "2 + ? = 5" being inacceptable). I presented them with two problems solved on the blackboard as examples. Students could use tokens, drawings, and other manipulatives or techniques they usually used in class to solve problems. However, I insisted on them giving a mathematical expression, emphasising that the final number (the answer) was not so important. The students were allowed to request reading assistance, which was provided by the researcher, and to ask questions in case they did not understand words or expressions. There was no time limit set to work with the problems and students could take as much time as they wanted to write the test.

The obtained results show that the average success level in solving all four problems (shown in the appendix) is significantly higher (Sig. <.000) in the EDA group (0.62) than in the control group (0.49). At the same time, the average difficulty gap in the EDA group is close to 0 (0.037) while in the control group it is quite significant (0.286).

These results can be interpreted as follows. The students in the control group developed more sequential numerical thinking than holistic relational thinking. This disequilibrium makes the problems that require holistic thinking and inversion of semantic structure much more difficult for these students. The students in the EDA group, on the other hand, developed their mathematical thinking in better equilibrium. Holistic Relational thinking allowed these students to be as effective in solving "difficult" problems as in solving "easy" problems and contributed to the higher overall success (more detailed results will be published soon (Polotskaia & Savard, n.d.)).

Conclusion

The main goal of this paper is to show how the powerful ideas that Vasily Davydov developed about the nature of mathematical thinking and learning have transformed the teaching and learning of additive word problem solving. My analysis of multiple aspects of one particular research project shows several important shifts in theory and practice of teaching of problem solving, as well as in students' success levels. Table 1 presents the summary of two paradigms in relation to teaching and learning of additive problem solving.

Table 1
Summary of Relational and Operational paradigms

Aspect	Relational paradigm	Operational paradigm
Concepts	Additive relationship	Not recognised as a concept
	Addition, subtraction	Addition, subtraction
Reasoning processes in problem-solving	Holistic AND sequential	Holistic OR sequential
Representation	Model of the relationship	Models of numbers/objects
Learning target of the problem-solving activity	Development of the holistic relational reasoning	Application of arithmetic operations
Process of the problem-solving	Cycling around the relationship	Linear, straight forward
Students' success results	Better general results	Lower success level
	No difference as for "easy" and "difficult" problems	Significant difference between solving "easy" and "difficult" problems

The importance of relational thinking and the use of representations in solving problems have been long studied and discussed by researchers (Bodanskii 1991; Christou & Philippou 1998; DeBlois 1997; Elia et al. 2007; Fagnant 2005; Fuson & Willis 1989; Gamo et al. 2009; Kintsch & Greeno 1985; Lesh & Zawojewski 2007; Ng & Lee 2009; Novotná 1998 to name few). Though, in light of Davydov's work, the Relational paradigm and the developmental approach (Davydov 2008) help to take these ideas to a new level as it allows us to rethink paths of mathematical reasoning development, educational targets, and teaching strategies.

The learning outcomes obtained in the discussed study show the power of the new paradigm and underscore Davydov's stance for the developmental role of education. The positive results are not due to the immediate facilitation of the problem-solving process for students. As described earlier, solving through analysis and representation takes more time and effort, even for “easy” problems. However, this longer and more complex process of solving facilitates the development of relational reasoning for students and more effectively ensures higher quality of learning.

The discussed results were obtained in the domain of simple additive word problems that comprise the very first step in the problem-solving curriculum in elementary school. The Relational paradigm certainly applies to many other areas of mathematics education. Much more work is needed to further develop the extremely rich theoretical knowledge Vasily Davydov left us as a legacy.

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Annex 1 Problem-solving test

***Ladybug**: A 3-year old ladybug has 17 black dots on her back. Annie counted 8 dots on the right side of the ladybug's back. How many dots are on the left side of the ladybug's back?*

***Rolls**: Mom is making cinnamon rolls. She plans to make 36 rolls. After 25 minutes of work, she has made 28 rolls. How many rolls does she need to make to complete her work?*

***Homework**: Danielle has to solve a number of problems for the homework for this week. After 3 days of work, she has already solved 15 problems. She has to solve 7 problems to finish the homework. How many problems are there in the homework?*

***Marbles**: There are some marbles and 36 small cubes in the box. Fourteen marbles are red. The 27 other marbles are green. How many marbles are there in the box?*