One is Not Born a Mathematician: In Conversation with Vasily Davydov

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That mathematics education has been one of the central concerns of educational systems worldwide is no secret. It is also an established consensus that as far back as eighty years ago, Russian psychologists such as Vygotsky, Luria, Meshcheryakov, and Davydov have pioneered work that contributed to the understanding of teaching and learning and shifted the trajectory of schooling in the Western World. Looking back, we feel that there is more to this legacy that awaits to be further explained and extended to potentially address some of the pressing issues in mathematics education. To bring forth this legacy, the authors engage in an imaginary conversation with Vasily Davydov to tease out notions that include, inter alia, language and interaction, learning and teaching, and empirical and theoretical thinking. The utilisation of a conversation as a method of inquiry for the purpose of this paper was intentional as it not only encompasses the very method of teaching advocated by Davydov, thus conveying that the means is the message, but it is also conducive to the exploration of simultaneously surfacing ideas and phenomena. The questions asked, clarifications provided, and contradictions that still remain chart a map that displays theoretical vistas and empirical landscapes drawn and inspired by Davydov's legacy. Specifically, Fellus and Biton bring forth citations from Davydov's works that are used as signposts in the conversation that unfolds.

Conversation as Methodology

Conversation as methodology has been used for research purposes to blaze "collaborative paths to knowledge" (Josselson, Lieblich, Sharabany, & Wiseman, 1997, p. vii). It has recently won its own nomenclature as a *duoethnography* (Norris, Sawyer, & Lund, 2012), a term that underscores a process of collaborative meaning making. Because the word *conversation* etymologically means *to get acquainted with* or *familiar with* (Klein, 1966) and because we wish to get acquainted with the legacy of Davydov, we choose to use the term *conversation as methodology* rather than *duoethnography* or *trioethnography*. In principle, the term *conversation as method* reflects both manner (together with) and purpose (get acquainted with) and brings forth the dynamic nature of the process of meaning making embedded in the activity of conversation. Recently employed by Zazkis and Koichu (2015), conversation as methodology is used to clarify, exemplify, and explicate perceptions, concepts, and sense making. In this paper, we employ the methodology of conversation to bring forward the voice of Vasily Davydov to collaboratively make sense of some of his not-yet-fully-applied notions. To ensure a close representation of Davydov's contribution, we cite relevant quotations in full (including the original punctuation) and use square brackets to enclose transitions between paragraphs. We do not use ellipses but rather begin a new set of quotation to indicate that there is a stretch of text in the original source in between the quotations. What was omitted (once) was a connective, the omission of which did not change the meaning of Davydov's words.

We aim to achieve two objectives by using the conversation methodology in this form. First, we aim to bring forth Davydov's voice, perspectives, and understandings and draw

connecting lines to issues in mathematics education. Second, we wish to put forth a methodology that manifests the very teaching of Davydov - that of collaboration and interaction in learning and development. To keep the conversational sound of the paper and in order to harness the advantages of the spoken word through the written mode, we note the source of the citations within the text in our turns.

Three overarching questions unfold in the ensuing conversation: What are some of Davydov's key tenets? How does Davydov conceptualise them? And why should we treat these as a legacy to be further explored? We now invite the reader to imagine the three speakers sitting in wingback burgundy armchairs set in a classically designed library where the walls are furnished with floor-to-ceiling red-cherry shelves heavily loaded with books and collected works that we turn to, pull out, and cite from when the need arises.

In Conversation with Vasily Davydov

Osnat: The Russian scholarly work had a substantial impact in the field of education. I would like to begin with discussing some of the key tenets in the learning theory you are advocating for, Professor Davydov. I am referring to the socio-cultural theory that was formulated by Vygotsky and further developed by his disciples and followers. This theory contends that we develop in interaction using language, which includes, in its broader sense, not only speech but other meaning-carrying tools such as gestures, for example. Some of the seminal work that unpacks this theory and its key tenets is Bakhtin's 1981 *The Dialogic Imagination*, Vygotsky's work such as his 1978 *Mind in Society*, and his 1986 *Thought and Language*, and Wertsch's 1991 *Voices of the Mind*. The theory identifies teaching and learning, or as you call it *upbringing and instruction*, as a dominant and determining factor in a child's development and frames interaction as ontologically relevant to learning and development. Let us then begin with language because it is through language, in its broader sense, that interaction and thinking take place, it is through language that teaching and learning is mediated, and it is through language that humans develop.

Vasily: [I agree] "the fundamental thesis of this theory is that the child's psychical development is mediated by upbringing and instruction from the very start" (Davydov, 1986/2008, p. 40). "As recently as the start of the 20th century, children's development was regarded by psychologists on a global plane, relatively independent of the concrete and variable conditions of upbringing and instruction. However, the spread of public education has shown that upbringing and instruction have the social and psychological effect of deepening people's intellectual development. This, in turn, suggests that there is an internal relation between a given system of upbringing and instruction and the ontogenesis of a person's psyche" (Davydov, 1986/2008, p. 35).

Yaniv: Clearly, upbringing and instruction is carried out by interaction and in order to understand how learning happens, we need to understand the relationship between learning and interaction. On page 285 in the 1987 first volume of *The collected works of L. S. Vygotsky*, he writes: "Thinking and speech are the key to understanding the nature of human consciousness." He then explains:

> Consciousness is reflected in the word like the sun is reflected in a droplet of water. The word is a microcosm of consciousness, related to consciousness like a living cell is related to an organism, like an atom is related to the cosmos. The meaningful word is a microcosm of human consciousness.

The metaphors Vygotsky uses speak volumes of the nature of language as an organic part of a much larger dynamic learning system we call consciousness.

Osnat: This quotation from Vygotsky is helpful indeed because it underscores the importance of language in the development of consciousness. But the association between language and consciousness deserves further exploration. In fact, scholars in various fields addressed this association from different perspectives. Vygotsky, for example, treated it from the point of view of psychology. To him, this association between language and consciousness is manifested in goal-oriented actions that humans perform with the help of objects and tools around them. It is important to note that, for Vygotsky, language is not a static, unidirectional reflection of one's consciousness. It is, in fact, a constantly changing, context-dependent, dynamic, multidirectional learning system through which one acts and reacts in the world.

John Austin, a philosopher, promoted a similar notion of using language to carry out actions in his seminal work *How to do things with words*. The book, which is a collection of lectures given by Austin at Harvard, was originally published in 1962. The title of the book speaks volumes of the action-embedded nature of language. Austin shows how words perform multiple types of actions and are, in fact, never a mere expression of true or false statements. I will not go into Austin's fascinating examples that show how different types of actions are carried out by what seemingly sound like flat statements. Suffice it to say that Austin's work was a harbinger of our understanding of language as a tool that is culturally embedded and that shapes - and is shaped by, one's consciousness.

Rom Harré and Grant Gillett, the former a philosopher; the latter, a neuroscientist, have shed more light on this intersection point of language and consciousness in their 1994 book, *The Discursive Mind.* They draw our attention to the etymology of the word *consciousness* and explain on page 172: "*Consciousness* is a word derived from the Latin roots *con* and *scio* - 'with' and 'I know' indicating that, also etymologically, the concept of consciousness involves interactive and discursive activity."

Consider the work of Pierre Bourdieu - a sociologist, anthropologist, and philosopher - who wrote on page 502 of his 1986 paper *The forms of capital*: "[U]tterances are not only (save in exceptional circumstances) signs to be understood and deciphered; they are also *signs of wealth,* intended to be evaluated and appreciated, and *signs of Authority,* intended to be believed or obeyed." Bourdieu refers to language as a system of signs thus making it a tool in the formation and reformation - read learning - of one's consciousness. This is just a taste of the plethora of literature on the association between language and consciousness. Professor Davydov, how does this relationship look like in your theory of upbringing and instruction?

Vasily: [Well,] "The essence of man is the set of all social relations. Consequently, man relates to his social relations as to his own essence, and thereby relates also to himself as to a species-being." "The universality of real social relations can be represented in the individual's consciousness (thinking) by means of the ideal nature of consciousness" (Davydov, 1986/2008, p. 29). "A person's consciousness is the set of his real social relations (his real being) as represented to himself ideally, subjectively. The holistic nature (totality) of the individual's real being is given to him in ideal form. The ideal as the basis of consciousness emerges, as was noted above, by means of people's speech communication, which is related to linguistic meanings. These meanings are based on socially developed methods of acting, wherein the ideal form of the essential links and

relations of the social world and the world of object, as revealed in joint social practice, are represented and 'bound up' in the stuff of language" (Davydov, 1986/2008, p. 30). One of the central problems in general, developmental and pedagogical psychology is to reveal the links between a child's psychical development and his instruction and upbringing" (Davydov, 1986/2008, p. 35).

Yaniv: I think we all acknowledge the strong association between language and consciousness and the role of language as ontologically related to the work of education. I wonder whether we can safely argue that these notions are also relevant to mathematics education. That is, if we develop through interaction, as the socio-cultural theory contends, and if we strive to achieve development in mathematics education, then interaction - in all its forms, domains, dimensions, and divisions, should be the focus of our investigation of mathematics education. Would you agree?

Vasily: "In Western psychology, there are several different theories of human psychical development (the theories of Arnold Gesell, Sigmund Freud, and Jean Piaget, among others). In these theories, development is regarded as a wholly independent process having its own internal laws; the process of development does not depend on the children's instruction and upbringing. (More correctly, instruction and upbringing have only a small effect on psychical development, acceleration or sometimes even delaying it" (Davydov, 1986/2008, p. 36).

Osnat: What you are pointing out, Professor Davydov, provides an important distinction that positions a child's development as dependent on instruction and upbringing - as Yaniv just noted, rather than as an independent process operating on its own internal rules - as you claim Western psychology tends to see it. The notion of instruction and upbringing is an important tenet in your theory because you see it as directly shaping development. I wonder what else is at play in this relationship? Assuming that learning and development are a matter of occasioned opportunities of instruction and upbringing that are generated by chance or causation - the former is dynamic and fluid, the latter is predictable and fixed, how can we better understand how teaching and learning happen using your notion of instruction and upbringing?

Vasily: "Descartes created a machine theory of animal behaviour, according to which it is fully predestined and predictable. However, he immediately ran into a paradox when analysing human behaviour. It turned out that his strict causal predestination of behaviour could not explain the universal nature of human actions. A person can do any of a number of things in a given situation; the person's actions are not programmed and cannot be deduced from past events alone. It turned out that the cause-and-effect relationships set forth by Descartes could not explain the links between 'universality,' 'goal-setting' and 'soul'" (Davydov, 1986/2008, p. 25).

Yaniv: I do not disagree with the basic assertion that chance plays an important part in human development. In fact, chance, which I define here as unpredictable events, may be constructive for humans but potentially destructive for animals. That humans adjust to chance constructively can be observed by the rapid adoption of and adaptation to the ever-changing technology that changes the way we live. With animals, chance can be destructive as shown in Ewers and Didham's work from 2006 where they describe how changes in animals' spatial arrangement are destructive to them. This is not to say that animals do not learn and adjust. They do!

Take for example, the classic 1962 work of Kawai Masao, who observed the behaviour of the wild Japanese monkeys on Koshima Island, Miyazaki Prefecture. He noted that after being provided with food for about one year, a female monkey started to wash the sweet potatoes that were provided to the monkeys. It took her mother and three of her playmates five months to start copying her action, and about three years for half of the troop of monkeys to follow suit. It took the rest of the monkey troop that counted 35 members, 10 years to imitate the process of washing the sweet potatoes. The only ones who did not copy this behaviour were the newborns, the one-year olds, and the more-than-12-year olds. All other things being equal, humans would most probably begin washing those potatoes within a few hours from first noticing the action. Animals take longer than humans to learn and adjust to chance. I wonder how these notions of learning generated by chance and causation apply, if at all, to your learning theory, Professor Davydov?

Vasily: "Cultural-historical theory stands opposed to theories that hold to a natural-science or naturalistic understanding of human psychical development. The latter theories interpret psychical development as based on the features of the separate human individual and on the 'spontaneity' and 'internal factors' of his psychical development, thus ignoring the initial sociocultural, semiotic, and 'long-term investment' nature of the human psyche. (Such theories include ones that deny the possibility of developmental instruction)." "This theory really does not admit any immanent development of the separate individual detached from sociocultural values, from communication and cooperation with other individuals, from instruction and upbringing. On the contrary, many theories admit and maintain the presence of precisely this sort of immanence. But, at the same time, cultural-historical theory admits the immanence and the presence of an internal logic of development of each individual, who from the moment of birth and throughout his life constantly communicates and cooperates with other individuals (either directly or in 'ideal form'). This immanence of development is inherent in the social individual, who is situated in interaction with other people" (Davydov, 1986/2008, pp. 198-199).

"For representatives of Vygotsky's school, all of a person's psychical functions that direct his activity have their deep roots not within the separate human individual, not within his organism and personality, but outside of him - in the social interactions of individuals, in their relations with each other, in their joint (or collective) activity. And the social interaction of human individuals is realised with the use of extremely diverse forms of signs. Even a person's most elementary psychical functions, which emerge during the early stages of his life, have this mediated or social structure. The development of this structure in individuals occurs in the process of their appropriation of the achievements of material and spiritual culture. In a broad sense, a person's instruction and upbringing can be interpreted as this process of assimilation, and therefore instruction and upbringing are the essential and necessary forms of the person psychical development. In the process of assimilating culture, each individual, on the one hand, continually interacts with other individuals and, on the other hand, reproduces, in a special form, those capacities by means of which people created culture in the course of history" (Davydov, 1986/2008, p. 197). "One of the most critical moments in the history of psychology was Vygotsky's conception that specific functions are not given to a person at birth but are only provided as cultural and social patterns" (Davydov, 1995, p. 18).

"In Piaget's theory, upbringing and instruction are regarded as the condition for the *accommodation* of the pedagogical process to the child's psychical development. The pedagogical process follows development, as it were. As applied to mathematics

instruction, Piaget has formulated this idea [in his 1972 paper on page 136] as follows: 'it is on the progressive organisation of the operational structures that mathematical teaching should be based.' [In the first volume of his collected works published in 1987,] Vygotsky remarked ironically that such a system of upbringing and instruction 'does not lead the child's development forward but rides its tail' focusing not on the tomorrow but on the yesterday of development" (Davydov, 1986/2008, p. 36).

Osnat: So if we assume that chance is more dominant in human development than causation, then Vygotsky's focusing on the tomorrow rather than on the yesterday of development is, to me, a powerful observation because it highlights the role of education or, as you call it, upbringing and instruction in a child's development. In trying to better understand the different perspectives between Piaget's and Vygotsky's learning theories, I wonder, what else did Vygotsky criticise in Piaget's perceived perspective on learning?

Vasily: "Vygotsky criticised the point of view that instruction should focus on those features of the child's thinking that have already matured - on that which the child can do independently. He asserted that this is a focus on 'the path of least resistance, toward the child's weakness rather than his [sic] strength'" (Davydov, 1986/2008, p. 37).

Yaniv: It seems to me that focusing on what the child already knows has been the lynchpin around which many educational systems are structured. One of the criticisms about schooling in general is that it essentially functions like an egg-grading-and-sorting machine that determines what students already know or can do and places them into ability grouping. This practice is prevalent in mathematics education worldwide.

Osnat: You are making a valid point, Yaniv. To me, it seems that the concept of *development* plays an important role in the difference between Piaget's and Vygotsky's perspectives. For Piaget, it seems that development *precedes* learning; for Vygotsky, development is the *result* of learning. I wonder what empirical verifications Vygotsky put forth to support his criticism of Piaget's notion of learning.

Vasily: "It is no coincidence that the 'old educational psychology' should agree with the latest genetic psychology, in particular that of Piaget. First of all, in both cases psychical development is regarded as essentially a process of maturation, if not on an organic basis, then based on the 'coordination and organisation' of actions that nonetheless have organic sources. Given this understanding of the developmental process, instruction can have no other function than to apply and exercise the forms and capacities that have appeared in the child prior to and outside of instruction. Second, on a methodological level, both the 'old psychology' and Piaget stand on identical naturalistic positions. The child's development is regarded as the development of the 'natural' individual - under social conditions, it is true, but lacking any idea that it is human beings, together with things that have been 'humanised' by their activity, that are the mediating link in the development of all forms of the child's psyche from the very start. Third - and this is the most profound basis for the agreements - in both cases a conception of development and instruction has been constructed based directly or indirectly on a wholly definite, historically evolved pedagogical practice that possessed no real means for the purposeful guidance of children's development. For the most part, that development occurred spontaneously. Piaget's "new" approach is the unfolded theoretical psychological expression of a pedagogical practice whose principles were formed in the European-American education of a bourgeois society. For example, one of these pedagogical principles is the requirement that instruction be "developmentally appropriate." This means that at each stage of instruction the child

should only be offered knowledge that he can understand right away, i.e., in psychological terms, he should have already developed to the corresponding level of thinking in order to master the knowledge" (Davydov, 1986/2008, p. 39).

"This theory was held by psychologists such as A. Gesell, Sigmund Freud, and others. The views of the eminent psychologist Jean Piaget on the intellectual development of children were in complete accord with this theory and reinforced it" (Davydov, 1998a, p. 13). "Many believe that the theory of pedagogy itself and the many years of practical experience accumulated in education strongly support such propositions, for, after all, the famous didactic principle, the principle of accessibility, wholly accords with this psychological theory (according to this principle, children can learn and need to learn only what they can "understand," only those things for which they have already acquired the cognitive capacities to learn" (Davydov, 1998a, p. 13). "Such a theory cannot provide the scientific basis for the practical support of the developmental role of children's upbringing and instruction, which is becoming increasingly relevant to solving the problems of improving public education" (Davydov, 1986/2008, p. 40).

Yaniv: All this talk about 'genetic psychology,' 'maturation,' 'organic sources,' 'cognitive capacities,' and 'developmentally appropriate' makes me think of the practice of ability grouping in mathematics that to me seems to be a manifestation of how Piaget's learning theory was interpreted and translated into curricular structures and practices where in order to learn higher level mathematics, more often than not, students need to demonstrate that they are 'capable,' that they 'already know' math, that they somehow have a genetic, organic mathematical disposition. There is substantial literature such as Devlin's 2005 book: *The math instinct: Why you are a mathematical genius*, or McCrink and Wynn's 2004 paper on *Large-number addition and subtraction by 9-month-old infants*, to mention just two of many that provide evidence to show that the ability to mathematise is biologically innate. Yes, animals mathematise too! Unfortunately, a substantial part of the experience of school mathematics centres on the notion that you either have a knack for mathematics or not and so the whole notion that we all, in fact, mathematise is hijacked and kept hidden by policies and practices that continually reproduce the myth that mathematics is talent-specific thus bucketing people into those who can and those who cannot do mathematics. Jo Boaler's extensive work on this very issue of ability-based grouping is fundamentally relevant here. In a paper she published together with Wiliam and Brown in 2000 titled *Students' experiences of ability grouping - disaffection, polarisation and the construction of failure*, she provides evidence from a longitudinal study in six UK schools. They detail several apt observations on pages 633-634:

- Approximately one-third of the students taught in the highest ability groups were disadvantaged by their placement in these groups because of high expectations, fast-paced lessons, and pressure to succeed. This particularly affected the most able girls.
- Students from a range of groups were severely disaffected by the limits placed upon their attainment. Students reported that they gave up on mathematics when they discovered their teachers had been preparing them for examinations that gave access to only the lowest grades.
- Social class had influenced setting decisions, resulting in disproportionate numbers of working-class students being allocated to low sets (even after 'ability' was taken into account).
- Significant numbers of students experienced difficulties working at the pace of the particular set in which they were placed. For some students the pace was too slow, resulting in disaffection, while for others it was too fast, resulting in anxiety. Both responses led to lower

> levels of achievement than would have been expected, given the students' attainment on entry to the school.

In her 2013 paper, *Ability and Mathematics: the mindset revolution that is reshaping education,* Boaler draws on empirical research to substantiate the understanding that ability groupings in mathematics do, in fact, hinder learning of mathematics. Looking back, in 2004, Wiliam and Bartholomew have shared similar insights in their paper titled *It's not which school but which set you're in that matters: The influence of ability grouping practices on student progress in mathematics.* Interestingly, the work of Linchevski and Kutscher from 1998 titled *Tell me with whom you're learning and I'll tell you how much you have learned,* also draws attention to the phenomenon of students who previously had lower achievements, achieve higher academically in mixed-ability mathematics classes. They also found that high-achieving students still do as well as before in such classes. In light of these studies and others, if high-achieving students do as well or even better in mixed classes as they do in streamed classes and if mathematically low-achieving students learn more in mixed classes, we can no longer afford to make do with sorting students by what they seemingly already know and doom them to ability-based grouping.

Vasily: "[Human psyche] is inherent only at the philosophical level of psychological knowledge which makes it possible to use the categories of relations between 'object' and 'subject', 'matter' and 'consciousness', and consequently revealing the specificity of the 'psyche', 'consciousness', 'the soul', and their genuine set - the subject of activity. You may well ask what are the unique features of these objects of study? The long history of philosophy and psychology (which is closely related to it) identifies that special trait as follows: human activity is *goal-oriented* activity, i.e., man possesses a special capacity for setting and achieving goals corresponding to particular needs." "There is every ground for believing that the degree of development of the capacity for setting and achieving goals is the chief feature inherent in the life activity of creatures endowed with a psyche" (Levitin & Davydov, 1982, p. 312). "One of the main tasks of psychology consists in developing methods of investigating human activity, consciousness, and personality. Psychologists have notched up some impressive successes in the study of the process of goal orientation, the building of sensuous and intellectual images, and the interconnection between the needs, tasks and actions of the person emerging within various forms of life activity" (Levitin & Davydov, 1982, p. 313).

Osnat: Indeed, this relationship between *object, subject, matter,* and *consciousness* and the notion of the human-specific goal-oriented activity may explain the growing empirical evidence about the dissatisfaction with the practice of ability-based grouping. Let me share just a few recent studies with you.

Take, for example, the 2010 work of Daniel Muijs and Mairead Dunne titled *Setting by ability - or is it?* The paper brings forth the phenomenon of ability-based grouping formed not on past academic achievements but on students' ethnicity, socio-economic status, gender and other readily observable characteristics. Similarly, Katherine Degner's 2012 work titled *Demography as destiny* also provides empirical support that leaves little room for interpretation in regard to the socio-cultural mechanisms behind the practice of ability-based grouping. Placing students in ability-based grouping has shown to be restrictive in regards to students' choice and possibilities of *goal-oriented* activities. This practice has shown to put "weaker" students at a disadvantage because they associate their placement at a lower-level mathematics class with their identity as learners, and users, of mathematics.

A case in point is Ralf Mason and Janelle McFeetors' 2005 paper *Voice and success in non-academic mathematics courses* and their 2007 paper *Student trajectories in high school mathematics*, where they show how students in a Canadian high school who were assigned to the applied-level mathematics simply did not believe they could do the math at the applied level even though they could do 'higher-level' math before. The context for this work may be better understood if we read the 2014 *Manitoba Education and Advanced Learning*, a document created in concerted collaboration among seven education ministries - the four Canadian western provinces: Alberta, British Columbia, Manitoba and Saskatchewan and the three northern territories: Northern Territories, Nunavut, and Yukon Territory. On page 17, *applied math* is defined as:

> This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for post-secondary studies in programs that do not require the study of theoretical calculus. Topics include financial mathematics, geometry, logical reasoning, measurement, number, relations and functions, and statistics and probability.

Whereas the Academic and Applied levels in mathematics are not officially defined as tracking in Manitoba or in any other province or state, research such as McFeetor and Mason's suggests that socio-cultural dynamics that are not - and cannot be - controlled through definitions and guidelines frame students' experiences. I wonder to what extent they are experienced as such, what might be some implications of such experiences, and how can these experiences be changed.

You may think that tracking students might put only "weaker" learners at a disadvantage but, as a matter of fact, Boaler's 1997 work *When even the winners are losers* provides evidence to show how even 'top' students lose in ability-based mathematics education because they mostly learn that mathematics is all about doing it fast and right and if they do not, then they cannot do math. The 2010 paper of Franz Preckel, Thomas Gotz, and Anne Frenzel titled *Ability grouping of gifted students: Effects on academic self-concept and boredom* shows how high achievers too lose in ability-based grouping. These are just a few among many scholars who have continually called for the much-needed transition from tracking to mixed-ability teaching. Such a transition is a necessary, however insufficient, condition for the type of interaction that is conducive to a child's development.

This call for transition lends particular significance to Jeannie Oakes' classic 2005 *Keeping track: How schools structure inequality* that highlights the inefficacious and counterproductive nature of tracking. In particular, her 1990 work *Multiplying inequalities* sheds light on the debilitating effects of ability-based instruction on students by showing how tracking in actual fact instils societal inequalities - the very problem the idea of schooling is overtly designed to overcome. To draw on your theory of instruction and upbringing, Professor Davydov, schools that still practice an ability-based mathematics education may benefit from a reform in mathematics education that builds on the notions of the human-specific goal-oriented activity that allows all children, regardless of their "already acquired" knowledge to learn and develop by focusing on the tomorrow rather than on the yesterday of development.

Such a reform would most probably have profound implications on the architecture of mathematics education but it should not be treated as a panacea. It is the first step that may gradually mitigate some of the side effects of - to use Boaler's words in her 2005 paper - the 'psychological prison' generations of learners have been confined within believing they cannot do math. Sorting learners by what they already know as an entry point to the

content and method of teaching is, in turn, oriented towards a positivistic approach to instruction as it essentially views learners in flat terms of either 'able' or 'unable' to do mathematics. Refraining from ability-based thinking of teaching mathematics by focusing on the tomorrow of development rather than on the yesterday may liberate our thinking of mathematics education and allow us to depolarise the mathematics curriculum. Doing otherwise, will reproduce the positivistic mindset that shrivels possibilities of growth.

Vasily: "Positivism is a bad theory for all the natural sciences, but it is simply disastrous for psychology, for positivism induces it to study the psyche in terms of the concepts of physics, chemistry and physiology and thus leads it away from revealing the essence of things fixed in such concepts as 'activity,' 'subject,' and 'goal orientation.' That is why overcoming positivist trends and using the rich arsenal of philosophical categories and notions from the humanities is one of the current tasks for our psychology. It is now clear that the view of the human psyche as presenting physical, chemical or physiological problems obscures rather than elucidates the basic questions of antiquity. The natural sciences approach, owing to the successes it has made possible in the study of inanimate objects, creates the illusion that the problems of psychology, too, can be tackled in the terms, say, of biochemistry and physiology" (Levitin & Davydov, 1982, p. 313). "Let me stress that Western (chiefly American) psychology is dominated by positivism, which is in principle ill-equipped to discuss fundamental problems of science. One of the tenets of positivism is that 'science is its own philosophy.' On the theoretical plane, such a tenet is unacceptable for Soviet science. The trouble is that we, too, are not without sin: although we are aware of the snares of positivism and its wingless and utilitarian nature, in our practical research we sometimes succumb to this approach which has about it the appealing simplicity of common sense" (Levitin & Davydov, 1982, p. 310-311).

"What should one do in this situation? It would be very useful, among other things, to turn to the sources of psychology, as science which was born from the bosom of philosophy; but we must not try to go back" (Levitin & Davydov, 1982, p. 314). "One must stress the great contribution to the development of that method made by the humanities which grapple with the key problems of the personality, in particular, the problem of choice. Choice exists only where there are possibilities. And it is only when there is choice that one can talk about will. Without *will*, there is no *subject*, and it is only the *subject* that possesses "soul" and consciousness. In the absence of this approach to reality and in the absence of these categories, one cannot get at the foundations of human activity, consciousness or personality" (Levitin & Davydov, 1982, p. 315).

"By no means! The need for a precise understanding of the nature of the psyche is prompted by earthly reasons. Here is a vivid example - and I will quote from the work of Ilyenkov, from an article devoted to the successful experiment in educating the deaf, dumb [sic], and blind students who graduated from Moscow University and are now on the staff of our Institute. Ilyenkov writes: 'Any animal forms the trajectory of its movement in accordance with the shape and position of external bodies, with the geometry of the environment. A person born deaf, dumb [sic], and blind must be taught that. Here, one can discern the first stage in the solution of the task: to form the child's need and ability to move in space on its own initiative toward food, adjusting the direction in accordance with the shape and position of external bodies – the obstacles in its way. The ability to construct a trajectory in accord with the geometry of the external world, changing it every time there is a new 'geometrical' situation, unexpected and unforeseen (and therefore incapable of being recorded by any genes) must be developed'" (Levitin & Davydov, 1982, p. 317-

318). "A person's basic want as a personality is the want to create the world and himself" (Davydov, 1986/2008, p. 64).

Yaniv: Ilyenkov was referring to the phenomenal work done at the Zagorsk school, a boarding school set up in 1963 for blind, deaf, and mute children. The work done at the Zagorsk school led to having four graduates pursuing their higher education under the supervision of Professor Alexander Meshcheryakov. One of these graduates was Olga Skorokhodova who became a writer in her own right as Levitin and Davydov tell us in their 1982 book on page 88. The work at the Zagorsk school epitomises the notion of focusing on the tomorrow of development rather than on the yesterday, to allude to what you said earlier, Professor Davydov. I think we need to ponder some more on how this theory of instruction and upbringing translates into practice.

Osnat: Before we do, though, I would like to summarise what we have discussed up to this point. We began with a description of the theoretical framework of instruction and upbringing, we then positioned language as the mediating tool in instruction and upbringing. Following Professor Davydov's reference to Piaget's work - that underscores what the child already knows as a factor in shaping teaching and learning - we brought forth the practice of tracking that stands in stark opposition to focusing on the tomorrow of development. We then provided substantiated research that shows tracking is counterproductive to the theory of instruction and upbringing and by extension to what we wish schooling to be for. These are important elements in Professor Davydov's perception of teaching and learning. Going forward, I ask how, indeed, does your instruction and upbringing theory translate into practice?

Vasily: "[Vygotsky once said]: 'Every function in the cultural development of the child appears on the stage twice, on two planes. First, on the social plane, and then on the psychological; first, between people, and then, inside the child" (Vygotsky, 1987b, p. 145). "All forms of human intellectual activity, including scientific activity, are realised by social processes rather than by isolated individuals" (Davydov, 1986/2008, p. 94). "Developmental upbringing and instruction deal with the whole child, with his holistic activity, which reproduces, in the individual, socially evolved wants, capacities, knowledge, and modes of behaviour. This activity, regarded as a special object of study, contains a unity of many aspects, including social, logical, pedagogical, psychological and physiological aspects" (Davydov, 1986/2008, p. 140). "We have full reason to suggest that these tenets of Vygotsky are the theoretical and psychological base for his idea of collaboration between adults and children, and of collaboration among children in the process of their upbringing and teaching. And at the same time, these tenets are one of the theoretical sources for the concept of the 'zone of proximate development,' which was introduced into science by Vygotsky himself" (Davydov, 1995, p. 16).

"[According to Vygotsky], 'the teacher must orient his work not on yesterday's development in the child but on tomorrow's. Only then will he be able to use instruction to bring out those processes of development that now lie in the zone of proximal development.' He introduced into psychological science the concept of the 'zone of proximal development,' which expresses the internal link between instruction and development. The existence of this zone presupposes the child's formation of psychical features that he does not yet possess" (Davydov, 1986/2008, p. 40).

Yaniv: Can you elaborate on the notion of the zone of proximal development?

Vasily: "We found that traditional primary education does not guarantee full-fledged development in the majority of schoolchildren" (Davydov, 1998a, p. 24). "[It] has as its basic preposition the idea that child development is independent of teaching and learning processes. In this case teaching is regarded, as Vygotsky wrote, as a "purely external process that must, in one way or another, be brought into line with the course of a child's development, but does not itself participate actively in child development, alters nothing in it, and uses the achievements of ontogeny rather than promotes its course or alters its direction" (Vygotsky, 1991, p. 375 in Davydov, 1998a, p. 12). "This means that traditional primary education does not create in its work with children the necessary zones of proximate development, but rather trains and strengthens those psychological functions that basically came into being and began to develop already at preschool age (these functions are essentially observation, empirical thought, utilitarian memory, etc.)" (Davydov, 1998a, p. 24).

"Vygotsky introduced the extremely important concept of the 'zone of proximate development' into psychology. The sense of this concept is the following: As a child imitates peers and adults in collective activity with them, he is able to do it with understanding. What a child does today with the help of others he will do independently tomorrow. This 'what' is, in fact, the content of the zone of proximate development - it 'helps us determine the child's tomorrow, the dynamic state of the child's development, which takes into account not only what has already been achieved but also the process of maturation" (Davydov, 1998a, p. 34, n. 2). "The tasks and actions that the child performs initially with guidance and in collaboration comprise the 'zone' of his proximal development, since later on he will perform them independently" (Davydov, 1986/2008, p. 40). "The child is able to do something new independently after having done that thing in joint activity with others" (Davydov, 1998a, p. 16).

"What the child is initially able to do only together with adults and peers, and then can do independently, lies exactly in the zone of proximal psychological development" (Davydov, 1995, p. 18). "This is the essence of Vygotsky's original hypothesis about the sources and psychological laws of child development: the source of development is the formal learning plus communication and cooperation with adults and peers; the basic principles are proper organisation of communication and collaboration, i.e., the creation of zones of approximate development and the "transfer" of collective performance of some mental function to the level of individual, independent realisation of that function" (Davydov, 1998a, p. 17).

Osnat: Indeed! I think it is worth noting at this point that even though you speak mostly of children learning, we can safely extend the referent to include other age groups whose members learn and develop through interaction with sentient, or insentient, entities. Those 'Aha!' moments - moments of development that emerge when learners can do something they could not do before - are occasioned by 'the zone of proximal development' where learners require the guidance of another to carry out a task until they can perform the task independently.

Having said that, I think the term 'zone of proximal development' requires further discussion. What do we mean when we say, 'zone' or 'proximal' or 'development?' How can we operationalise each of the concepts in order to better understand, and potentially create, conducive conditions for a learner's *zone of proximal development*? But wait! Let us take a step back and ask ourselves, do we want to operationalise the term? Should we?

What may the field of education gain, and what might it lose, in transitioning the term from the theoretical realm into the operationalised, measureable realm?

Jaan Valsiner has written extensively about these issues and other questions related to the work of Vygotsky. Of particular relevance is Valsiner and van der Veer's 1993 book chapter titled: *The encoding of distance: The concept of the zone of proximal development and its interpretations*. In this chapter, the authors provide an overview of the historical and logical evolution of the concept of *zone of proximal development* describing the use and treatment of the term from its inception in the early 1930s to its becoming the lifeblood of long lines of research and scholarship. In addition, Valsiner and van der Veer bring forth a detailed discussion of the difficulty to reify and operationalise the theoretical idea of the *zone of proximal development* because of issues such as *time irreversibility* that preclude the measurement of linear processes of learning. In short, Valsiner and van der Veer laid out some of the paths through which Vygotsky's term of the *zone of proximal development* has reached new dimensions that depict how humans develop.

Vasily: "It is well-known that every human being is appropriating knowledge and skills throughout his/her entire life. Particularly during childhood the process of learning is rather intensive: through communication with other people, through play, through games and sport, and through obligations to work" (Davydov, 1999, p. 123).

Yaniv: Indeed, learning is a social activity, we do it with others and through others; it is continuous and thus never ends; it is a part of everyday life and never limited to formal schooling and teaching; it is, actually an innate ability that allows us to change and develop.

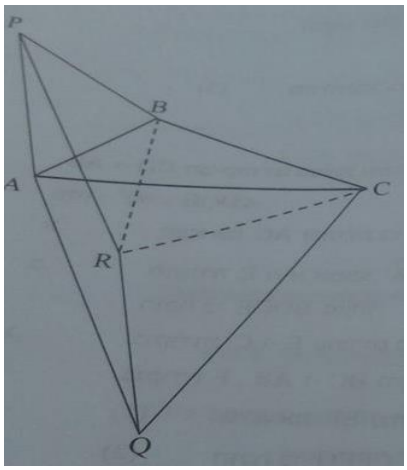
Osnat: That is true! And that applies to learning mathematical concepts in particular, which is not confined to institutionally established roles, nor to culturally or socially privileged ways of learning. Anyone, anywhere, can function as the guide, the teacher, the mentor. The works of Jean Lave, a social anthropologist, and of Paulus Gerdes, a researcher of ethnomathematics, are revealing in this respect. Jean Lave documented in her 1997 work how tailors in the Vai and Gola tribes of Liberia learn the craft of measurement from more experienced tailors. Paulus Gerdes' 1988 work shows how mathematics is not, in actual fact, a culture-free practice. Both have published extensively in this area. Wenger, Lave's disciple, writes on page 3 of his introduction to his 1998 *Communities of Practice: Learning, Meaning, and Identity:*

> [What if we placed] learning in the context of our lived experience of participation in the world? What if we assumed that learning is as much a part of our human nature as eating or sleeping, that it is both life-sustaining and inevitable, and that - given a chance - we are quite good at it? And what if, in addition, we assumed that learning is, in its essence, a fundamentally social phenomenon, reflecting our own deeply social nature as human beings capable of knowing? What kind of understanding would such a perspective yield on how learning takes place and on what is required to support it?

What is important to note is that these works and others have substantiated our understanding that the kind of mathematics that is privileged in school is only one kind among many in the use of mathematics in everyday life.

Yaniv: Absolutely! We do learn from each other and the context of mathematics education is replete with examples. A case in point can be considered from a project conducted by the Centre for Educational Technology in Israel. I describe this project in a paper that I

wrote in collaboration with Osnat Fellus, Sara Hershkovitz, and Maureen Hoch, where 614 high school students were assigned to eight different Facebook groups - four in Hebrew and four in Arabic - to prepare for a repeat exam in mathematics. What we did not really expect is the extent of student participation and agency in working and learning together. Through the use of the Facebook platform, space and time for students' mathematising were extended and the Facebook page showed student engagement in mathematics at the wee hours of the night. Looking closely at student interaction yielded numerous examples of between-student guidance, that is, students' zones of proximal development. Let me show you what I mean. Here you can see a problem given to the students. The problem shows three similar isosceles triangles whose bases are the sides of another triangle. The students were asked to prove that the two angles ∡ACB and ∡QCR are equal. In the example that follows, you can see how two students, Shiran and Achinoam, work together on Shiran's solution. Shiran started off with a few questions to see whether her method of solution is right. She asks Achinoam to go over her solution, which she posts on the Facebook page. Achinoam then draws Shiran's attention to a couple mistakes.



Triangles APB, ACQ, BCR are similar isosceles triangles whose bases are the sides of triangle ABC.

Prove: ∡ACB = ∡QCR

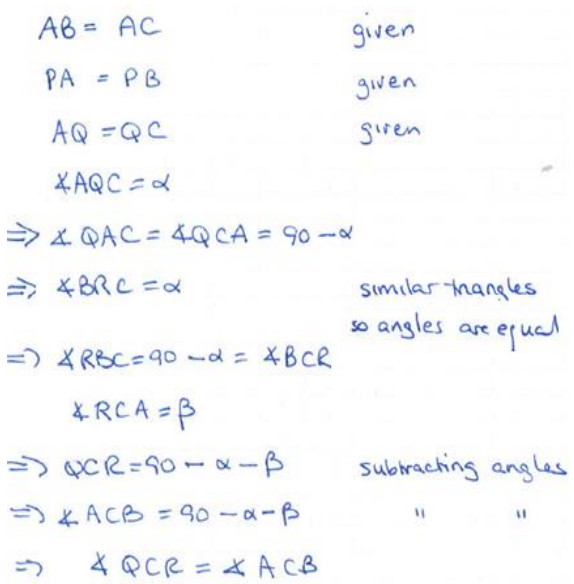


Figure 1. Geometric problem and a student's solution posted on Facebook

Achinoam:	I think you made a mistake with the angle QAC. Shouldn't it be 90 minus half alpha?
Shiran:	Yes you're right. And that changes them all to 90 minus half alpha…So it is the same proof, I simply need to change the alpha to half alpha…right?
Achinoam:	Oh, okay, I see. Thanks! ☺ But I think that generally you can't say that AC=AB is given. Right? Shouldn't it be 90 minus half alpha?
Shiran:	I meant BR=RC
Shiran:	I've got too many mistakes ☹

Achinoam: Aaah. Now it all makes sense! It is really not so bad. It is a mini mistake! ☺

In her solution, Shiran demonstrates that she knows the properties of similar isosceles triangles. She also demonstrates she knows that having three similar isosceles triangles means that the angles in the three triangles are congruent, and that the median of a triangle can help find the measurements of the angles. She uses the triangle sum theorem. We know this because she notes that the two vertex angles in the similar triangles AQC and BRC are equal: $\measuredangle$AQC=α, $\measuredangle$BRC=α and that the two base angles are equal $\measuredangle$QAC=$\measuredangle$QCA. At this point, however, she notes: 90-α demonstrating that she knows that the median of a triangle creates a right angled triangle and that the remaining 90° divide between the remaining two base angles.

This, again, demonstrates that she knows all three interior angles in a triangle add up to 180°. However, it is Achinoam who draws Shiran's attention to two mistakes that Shiran had made. One is Shiran's misrepresentation of the two sides AC and AB as equal; the other is Shiran not noting that the median of the triangle AQC drawn from the vertex Q creates two right-angled triangles and that $\measuredangle$QCA, in fact, equals 90°-1/2α. Each of the base angles of an isosceles triangle can be obtained from the vertex angle α also as half of the difference 180° – α, that is 90° minus half alpha. It may be, of course, that the students used the property of the median that occurs here, but there is, in fact, no evidence for that.

Whichever way they chose to solve the problem, Shiran acknowledges the mistakes she made and Achinoam admits that now she understands what is going on in Shiran's solution. What we see here is that two classmates, Achinoam and Shiran engage in a deeper exploration of the properties of triangles and gain deeper understandings, by way of their own admission, thus working their zone of proximal development as each gains new insights into the problem. The collaboration between these two students and their expressed thoughts on the problem was indeed intriguing.

Vasily: "[A]t the basis of all our mental functions, including thought, lies the real object-oriented activity of collaborating people" (Davydov, 1988, p. 180). "[Indeed,] [a]ppropriation is not the individual's passive accommodation to the existing conditions of social life. It is the result of the active reproducing activity of a child who is mastering historically evolved methods of orientation in the object-related world and the means for its transformation, and which gradually become the forms of his self-activity" (Davydov, 1986/2008, p. 51). "In order to learn while in class, the children should learn and appropriate knowledge and skills in the process of full-fledged learning activity; such activity should be *properly arranged*" (Davydov, 1999, p. 125).

Osnat: This is an instructive example, indeed! Achinoam and Shiran were working together on the relationship of the different qualities of triangles and angles thus engaging in acts of appropriation, which is a social act. What I like about this example too is the setting of Facebook, which, in this case, is conducive to students' interaction and process of appropriation of mathematical ideas. The episode from Facebook reflects the students being on-task, using terms and concepts in mathematics in a meaningful way, thus engaging in an activity that is *properly arranged*, to use your words, Professor Davydov. Achinoam and Shiran mathematise through Facebook, which I see as a setting that epitomises a *properly arranged* learning activity that brings forth a snippet of the students' process of appropriation. They use an available technology to engage with mathematics, free of time constraints, which are associated with brick-and-mortar schools.

The question that arises now is what happens next? That is, how do we examine whether Shiran and Achinoam really understood how things work in isosceles triangles? To answer this question, I want to bring in the concept of *appropriation.* I use the term appropriation to refer to learners' gradual process of manifestations of fluent, and correct, use of the skills and content they know. This is, obviously, a complex process rather than a straightforward one. Bakhtin writes in his 1981 work on pages 293 and 294:

> As a living, socio-ideological concrete thing, as heteroglot opinion, language, for the individual consciousness, lies on the borderline between oneself and the other. The word in language is half someone else's. It becomes 'one's own' only when the speaker populates it with his own intention, his own accent' when he appropriates the word, adapting it to his own semantic and expressive intention. Prior to this moment of appropriation, the word does not exist in a neutral and impersonal language (it is not, after all, out of a dictionary that the speaker gets his words!), but rather it exists in other people's mouths, in other people's contexts, serving other people's intentions: it is from there that one must take the word, and make it one's own.

Now, the question is how does such an activity look like? What do we mean by *appropriation?* What do students need to do in order to be engaged in the learning activity of appropriation? How can learners of mathematics populate the words with their own intentions, their own accents?

Paul Lockhart, writes on page 126 in his 2009 *A Mathematician's Lament:* "Mathematics is fundamentally an act of communication." Through communication and interaction we can allow for appropriation but there is more to it than simply using the words. Appropriation is also about a manifestation of the ability to connect between ideas and concepts, to construct logical networked systems of concepts and relationships.

Vasily: "[As far back as two-and-a-half-thousand years ago, Heraclitus noted:] 'Much learning does not teach understanding.' The twenty-five centuries that have sped by have convinced people that it is not enough, it is in fact very little, to be knowledgeable, literate, or even educated. One must also be able to think" (Levitin & Davydov, 1982, p. 307). "The content and methods of traditional instruction are primarily aimed at forming the foundations of empirical consciousness and thinking in primary school age children. This is an important - but at present not the most effective - way of developing children's minds" (Davydov, 1986/2008, p. 73). "The task of bringing our education system up to the level demanded by the scientific-technological revolution requires not just an ordinary change in the content and methods of teaching but the complete replacement of the traditional principles for selecting subjects of instruction and for developing didactic materials. Up to now, these principles had mainly to do with forming the students' empirical judgment. The new principles should facilitate the formation of scientific [theoretical] thought" (Davydov, 1988, p. 169).

Osnat: You are suggesting that to be knowledgeable, literate, and educated is not enough to facilitate the formation of what you call *theoretical thought.* I wonder what conditions can be considered as conducive in facilitating theoretical thinking.

Vasily: "Nobody can force a small child to play. The child should feel the need to play. Equally, we cannot force school children to enter into learning activity if they do not have the need to do so. Although school children, without having such needs, can in fact learn and appropriate various knowledge (and indeed appropriate very well), they are not able to transform learning material creatively because they do not have the ability to formulate such critical vital questions that can be answered only by means of secret-revealing experiments. The second condition needed to arrange proper learning activity is the

formulation of learning tasks for pupils which would require their making experiments with the material to be appropriated. The learning tasks should be formulated in such a way that transformation is required. For example, in a lesson on mathematics in primary school, the following learning task was formulated for the pupils (of course, this task included sequences of minor exercises): *If we want to measure a very large object and only have a small measuring device, how could we then reduce the time of measuring and yet present the result by means of this device?*" (Davydov, 1999, p. 127).

"To solve this task, the children should make a series of experiments. In particular, they needed to be introduced to a wide range of large measurement exercises as part of the major task. In short, learners should act in the following order: First, they realise that to reduce the measuring time some type of large-scale measuring device would be required; then (with the help of their teacher) they understand the need to know the relationship between small and large measurements. Finally, the children having understood this connection and working with a large measuring device, can quickly measure the large object and present the result in the units of the small measuring device. Although the children solve the task with the help of their teacher, they actually discover for themselves the need to use the mathematical action of multiplying in order to find out the answer to a practical question (getting this kind of answer means orienting children towards the relationship between different scales of measurement)" (Davydov, 1999, p. 128). "The process of generalisation consists in the child's comparing a group of objects and singling out certain recurring properties (qualities) of the objects." (Davydov, 1986/2008, p. 74). "In reality, behind each generalisation there is a special activity, an activity of transition, a movement of thought from one content to another" (Davydov, 1988, p. 181).

Yaniv: Are you suggesting then that the mental activity of identifying connections and relationships between ideas is an example for generalisation?

Vasily: "Generalisation involves searching for some invariant property within a class of objects and denoting that invariant by a word. The invariant is then used to identify other objects as belonging to the given class" (Davydov, 1986/2008, p. 74). "The formation of children's conceptual generalisations is supposed to be one of the main goals of school instruction. In school textbooks on various subjects, the material, as a rule, is arranged so that the students' work with it leads to the formation of the corresponding generalisations and concepts. Teacher manuals give detailed instructions on how to guide this process and on how to assess the level of conceptual generalisation attained by the children" (Davydov, 1986/2008, p. 74). [But more often than not,] [i]n the present system of teaching mathematics, children are trained to solve problems of various types. The teacher wants them to solve as many stereotyped problems as possible. And that is why one often hears the pupils say: 'We haven't solved problems like this before.' Traditional education is oriented toward developing empirical thinking alone" (Levitin & Davydov, 1982, p. 320).

Osnat: It seems to me that the way you use the term *empirical* suggests a particular type of teaching and learning. In what sense do *you* use the term *empirical thinking?*

Vasily: "Empirical thinking is developed in humans *without* any schooling; schooling only supports the further utilisation and cultivation of this form of thinking. Therefore, however paradoxical this may seem, schooling that follows traditional programs *does not promote the development* of the learners" (Davydov, 1999, p. 134). "At first, humans possess empirical-rational thinking allowing them to group and classify things and phenomena of the surrounding world by comparing and pointing out the interrelations of genus and kind.

This type of thinking makes it possible to solve the tasks of relating things to a certain class (genus) or, vice versa, divide a class into certain subclasses or kinds" (Davydov, 1999, p. 131).

"Quite clearly the phenomenon and core (particular and general, etc.) of a system are not similar. They characterise different contents of that system and are *contradictory* to each other. In a lot of different life situations it is necessary to find out and to cope with such leading contradictory characteristics of systems of objects. For instance, ordinary (and often observed) bulbs are particular and concrete phenomena of electricity, but the essence of that system is the 'movement of electric power' between the poles 'plus' and 'minus,' which is *not* visible to the human eye" (Davydov, 1999, p. 132). "Analysing the mutual connections and transformations of phenomenon and essence is, at the same time, the process of studying their *development.* Tasks of this type cannot be solved by [empirical] thinking" (Davydov, 1999, p. 132).

Yaniv: So when you speak of this non-empirical thinking, you speak of the activity of making connections and establishing relationships between nodes in a system - that is, understanding how things work beyond what is readily observable. Can you elaborate on this?

Vasily: "Many people believe that dialectical [or theoretical] thinking is a quality known only in a small group of particularly bright humans. This is *not* the case: A large part of the population have spontaneously acquired this type of thinking through difficult life practices (e.g., by solving complicated moral problems). However, a certain percentage of the population has not acquired the capacity for this kind of thinking at all (which is caused, above all, by the education they received or by a 'conflictless' life) or they have to a very small extent demonstrated this kind of thinking and are not theoretically oriented in complicated situations" (Davydov, 1999, p. 132). "Dialectical thinking can and should be formed in school children at all educational stages. Their development of creative abilities, initiatives, self-understanding, and, finally, the development of their personality depends on it. A proper organisation and implementation of learning activity contributes to this aim" (Davydov, 1999, p. 132).

"Usually, teachers believe that theoretical consciousness and thinking represent an abstract attitude to real life and are often connected to verbal knowledge only. This is clearly a misunderstanding, undoubtedly related to obsolete philosophical and psychological views. First of all, theoretical consciousness and thinking is realised in an action-performance-visual, visual-image, and verbal-discursive form. Second, it is represented in the sciences and arts, moral practices and law. Third, it consists in humans' rational attitudes to real life, in humans' ability to resolve rationally both abstract and practical everyday tasks" (Davydov, 1999, pp. 130-131).

Osnat: Can we argue that dialectical or theoretical thinking can still be developed on its own regardless to pedagogical practices?

Vasily: "Yes, it can. But it will be formed spontaneously, not to a full scale, and not necessarily for all children. And when it happens, it happens *in spite of* the many recommendations of commonly accepted didactic and methodical principles which varies within the real practice of different teachers" (Davydov, 1999, p. 133). "[Theoretical] thinking helps students to orient among general relationships and allows them to derive from these relationships various specific consequences. Such thinking does not remove the

need for empirical thinking, this kind of thinking should be seen as a kind of thinking directed at other kinds of tasks. Theoretical thinking does not originate and does not develop in people's everyday life. It develops only in the kind of teaching that uses a curriculum based on dialectical concepts of thinking" (Davydov, 1999, p. 136). "One may ask whether pupils who study the usual textbooks and procedures can thus acquire theoretical consciousness and theoretical thought. The answer is an unhesitating yes: they do acquire theoretical consciousness and thought spontaneously, but not in full measure; moreover, by no means every schoolchild does so" (Davydov, 1998b, p. 44).

Yaniv: That is an explanation, but if we want to refine this framework, how would you bring about theoretical thinking?

Vasily: "Reach the real connection of things, i.e., their "logic", their *dialectical logic*, which takes the form of [studying the] mathematical, physical and other relations [between things] (Davydov, 1988, p. 174). "[For example, Aristotle has already established] that the triangle is the simplest figure, to which one may reduce [other figures] and from which the other [figures] may be derived" (Davydov, 1988, p. 177). "The triangle is thus a *reduction* of *every* figure to simple determinacy. This is why it is the first, truly universal, figure. 'Thus', writes Hegel, 'the triangle, on the one hand, stands alongside the square, pentagon, etc., while, on the other - and here lies the greatness of Aristotle - it is the basically universal figure'" (Davydov, 1988, p. 176).

Yaniv: This view of the *real connection of things* that you are advocating for, Professor Davydov, resonates with me. I have an example that helps to better understand the distinction between empirical and theoretical thinking. I bring this example from a 2016 paper that I wrote together with Osnat Fellus and Sara Hershkovitz, where we describe an episode from a class of prospective mathematics teachers. I call it, *The circle episode*. It took place when teaching a group of prospective teachers using an elementary-school digital textbook in geometry. Each prospective teacher used their own login name and password to answer a series of questions from the digital textbook. One of the questions was about the possible number of points of intersection in two circles with different radii. As the results from the automated marking system were fed into my screen, I noticed that one of the prospective teachers, whom we shall call Esther, got a wrong answer filling in: "Two circles with different radii can intersect in endless number of points." I turned to Esther and asked her to explain what she wrote and to show me an example of what she meant. Esther showed me this drawing:

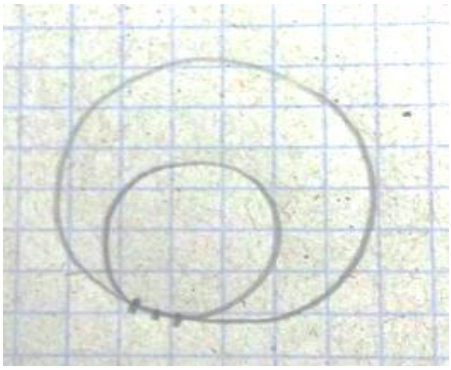


Figure 2. Esther's drawing of the points of intersection

She explained that the number of intersecting points can be "20, 30, and endless." As Esther was speaking, others in the room, who were sitting close by, joined in the discussion and were convinced by what Esther was showing in her drawing and by her reasoning. It seemed to me that her empirical reasoning, i.e., her drawing, convinced them to think like she did. I then asked, "If your pencil was sharper, do you think you would have a different answer?" She said, "No!" I then drew two circles on the whiteboard using GeoGebra, see Figure 3, and asked the students' opinion about the number of points of intersection between the two different circles. I had the impression that quite a few students sided with Esther saying that the number of points of intersection between two different circles can be "20, 30, and endless." Others, were still arguing that there cannot be more than two points of intersection but it seemed to me that they relied on their rote memory and that their confidence in their own answer was waning as a result of the discussion in class. Using GeoGebra, I zoomed in and out of the "problem area," see Figure 4 and Figure 5, so the students could see that something that initially looked like "endless" points of intersection, has in fact only two points of intersection! Changing the position of the circles showed that two circles with different radii, may have zero, one, or two points of intersection at the most. A heated discussion ensued about whether it is an invariable property, to use your term, Professor Davydov, and whether this will be the case at all times and how this can be taught to the school children without necessarily using GeoGebra.

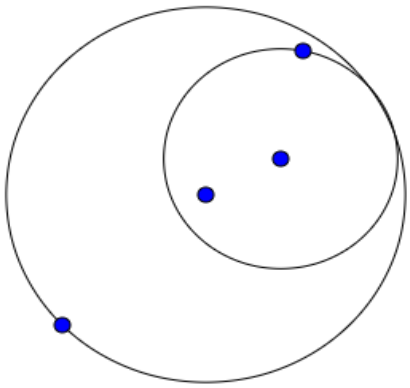


Figure 3. Instructor's drawing of two intersecting circles using GeoGebra

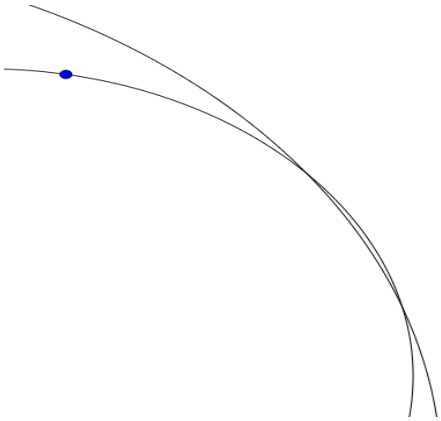


Figure 4. Zooming out of the intersecting points using GeoGebra

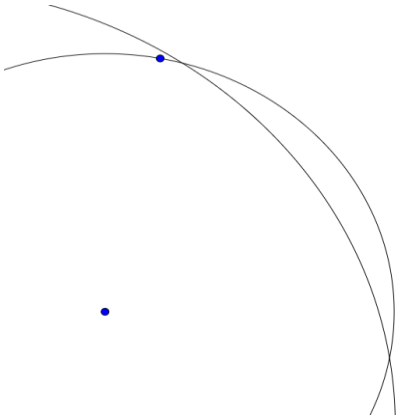


Figure 4. Zooming in of the intersecting points using GeoGebra

This circle episode demonstrates how a learning moment of generalisation emerged by instruction, and technology, first through empirical thinking, that is, the ability to see one example of two intersecting circles and then through transitioning this knowledge to theoretical thinking that allowed the prospective teachers to see the invariant property of the relations of two different circles. The prospective teachers could not have learned about the theoretical or dialectical relationship between two intersecting circles had I not been attentive to their empirical thinking, that is, to their actually seeing the circles and to what seemed to them like endless points of intersection. Had I not demonstrated to them that their hand drawing the circles created a visual illusion, I do not think we could have engaged in the ensuing discussion that brought forth their theoretical thinking about points of intersections in circles and about how to effectively teach it to their own students.

Vasily: "Empirical knowledge, which is based on observation, reflects the external properties of objects and rests on visual notions. Theoretical knowledge arising on the basis of a mental transformation of objects reflects their internal relations and connections and thus goes beyond the bounds of sensory representation" (Davydov, 1998a, p. 25-26). "'To accomplish a task theoretically is to accomplish it not only for a given case but also for all similar cases'" (Rubinstein 1957 in Davydov & Kudriavtsev, 1998, p. 61).

Yaniv: Thank you for the clarification in regard to the distinction between empirical and theoretical thinking, Professor Davydov. I would like now to discuss another important observation you made in your work. In your paper *The Psychological Conditions for the Origination of Ideal Actions*, you provide a fascinating account of your investigation of addition tasks among young children. You describe how four- to seven-year-old children who are presented with a series of addition tasks with or without objects, use gestures or movements of fingers, head, or body to help them do the addition tasks. For example, children were observed pointing to objects or moving the hand to mark the outline of the objects before them. What I found fascinating was the fact that when the children were told not to move or use their fingers while counting, their counting was interrupted, which points to a strong association between children's performance of addition tasks and body movements.

This idea of the strong, and necessary, relationship between body movement, including use of fingers, and calculating or making sense of mathematical ideas such as subitising, addition, and subtraction gained empirical support in research published by Badets and Pesenti in 2010, Newman in 2016, Noël in 2008, and Garcia-Bafalluy and Noël in 2008, to mention just a few. Brian Butterworth, a neuropsychologist, who has published his book *The Mathematical Brain* in 1999, concluded as far back as two decades ago that learning about numbers, if carried out successfully, has to be done by thinking through our fingers.

It was only recently, though, that research in neuroscience has provided evidence through neuroimaging that shows we make sense of mathematical ideas through our sensory perceptions thus buttressing what you have long ago pointed at. Specifically, Ilaria Berteletti and James Booth, educational neuroscientists, found that there is direct relationship between the activation of the parietal lobe in the brain, the very region that integrates all *sensory* information fired into the brain, and one's work with numbers. They published two important papers in 2015 that show that the very use of fingers when working with numbers is essential in the development of number sense because it helps children "see" the numbers.

But going back to your work, Professor Davydov, you also detail how children demonstrate difference between actions of additions and actions of counting. In actions of addition, for example, children subitise and see groups of objects as symbols of numbers; in counting, they sometimes use numbers to represent objects and movements. You then conclude, on page 25 of your and Andronov's 1981 work, that a development is manifested when "[t]he movement, in an abbreviated, reduced form, then itself becomes the symbol of a number." I find this observations instructive not only because the insights that you have formulated almost four decades ago are now corroborated by research in brain imaging but also because now, with the help of technology that was not available before, we can understand the developmental relationship between "seeing" numbers and using them.

Osnat: I agree. Another important however not-yet-adequately-understood idea you have articulated, Professor Davydov, is related to the issue of fragmentation and compartmentalisation of mathematical ideas. Can you elaborate on it?

Vasily: "any attempt to design the content of education and project age-group norms of mastering the material to be learned will be doomed to failure" (Davydov & Kudriavtsev, 1998, p. 38). "This method of 'continuity' (rather, the disruption of continuity) projects an abstract model of the child that is unrelated to his age, a model from which the idea of development has been cut out. The child's development continues in chaotic subcultural forms of assimilation of social experience, which is to say, by-passing and outside of its organised assimilation within the educational system. It would be simplest to perceive such a state of affairs as resulting from particular 'scientific miscalculations' or 'methodological failures.' And of course, such things do happen. However, just how it is possible to cultivate the self-worth of each age of childhood while simultaneously ensuring ongoing progress in age-phase development, how to ensure in practice a productive interaction between the ideal and the real forms of that development, remains an open question for the entire adult community involved in furthering educational processes" (Davydov & Kudriavtsev, 1998, p. 41).

Yaniv: Professor Davydov, you were the guest of honour in this conversation. Would you like to say a few concluding remarks?

Vasily: "[M]uch in practice depends on what theoretical positions teachers and scholars are able to take" (Davydov, 1995, p. 18). "The question arises whether the 'theory' of teaching methods should be brought into accord with the content of an operation which is found in a psychological analysis of it. This question can be stated in a broader context: must we impart to the student the object operations that lead to notation of a product? Suppose that children master them. What value will this have for their future study of mathematics or for their orientation in everyday situations?" (Davydov, 1991, p. 31) "The basic principles are proper organisation of communication and collaboration, i.e., the creation of zones of proximate development and the 'transfer' of collective performance of some mental function to the level of individual, independent realisation of that function" (Davydov, 1998a, p.17).

Osnat: Thank you, Professor Davydov for your inspiring contribution to this conversation. We began this conversation with asking ourselves what were some important tenets in your work, how you present them, and why these can potentially guide current policies and practices in mathematics education. To answer the first two questions, we have discussed important tenets that were derived directly from your work. These include the zone of proximal development, teaching and learning with a focus on the tomorrow of development, embodied mathematics, and empirical and theoretical thinking. We brought in research to buttress your work and legacy and included two examples from work with prospective teachers and high school students.

If you ask what the main message of this conversation is, I would choose the title of this paper as an answer. Here is why. We were debating what would best reflect your perspective about teaching and learning mathematics. After considering different options, we decided to allude to the book you published together with Karl Levitin, *One is not born a personality* changing the word *personality* with *mathematician*. This allusion to your book, we hope, will not only spark interest among our readers to read the book itself but also convey the very message you advocated for: that teaching and learning mathematics concerns important concepts that go beyond any artificially organised mathematical content. One, indeed, is not born a mathematician. It takes a knowledgeable other to engage in conversation with the learner, to activate the zone of proximal development, and to nurture a solid transition from empirical thinking to theoretical thinking. I hope that many more research studies will build upon your legacy to improve mathematics education for all.

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