

Unit 4 *Solving Inequalities*

Introduction

Learning objectives

This unit introduces the topic of *solving inequalities*, which is an important aspect of mathematics and a natural extension to solving equations. After completing this unit you should

- be able to illustrate inequalities on a number line
- be able to solve linear inequalities
- be able to solve quadratic inequalities
- understand how to illustrate linear inequalities in two variables
- be able to illustrate and solve linear programming problems in two variables.

Notes

It may seem strange to be dealing with inequalities towards the end of the Algebra units, since the idea of an inequality is fundamental to the whole of mathematics, i.e. $1 > 0$, $2 > 1$, etc.

On the other hand, you have met inequalities in a variety of forms earlier in your studies but it is this unit which picks up all the threads. Apart from the obvious trivial inequalities (which are seen clearly on a number line) the first important inequality was derived by *Archimedes of Syracuse* (287–212BC), who used 96-sided regular polygons (one inside and one outside a circle) to obtain bounds for π :

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

Of course, with calculators and computers we can dramatically narrow this inequality – indeed, to any desired degree of accuracy – but remember that *Archimedes* had none of this technology to help him!

In addition to his work on estimating π , Archimedes' great claim to fame arises from his theorem which gives the weight of a body immersed in a liquid (*Archimedes' Principal*). Unfortunately he was killed during the capture of Syracuse by the Romans; his death is recounted by *Plutarch*:

As fate would have it, Archimedes was intent on working out some problem by a diagram, and having fixed both his mind and eyes upon the subject of his speculation, he did not notice the entry of the Romans nor that the city was taken. In this transport of study a soldier unexpectedly came up to him and commanded that he accompany him. When he declined to do this before he had finished his problem, the enraged soldier drew his sword and ran him through.

Inequalities are also used in scientific notation, where numbers are expressed in the form

$$a \times 10^n$$

where n is an integer and $1 \leq a < 10$.

Another use of inequalities is in quantifying the accuracy of a measurement; for example, a height measured as 176 cm to the nearest cm really means that the height is somewhere in the range

$$175.5 \leq \text{height} < 176.5$$

Unit 4 *Solving Inequalities*

Introduction

There are also some classic inequalities which have been around for some time. The idea of isoperimetric inequalities has been derived from the Greek legend of *Princess Dido*; and the inequality

$$A \geq G \geq H$$

where the A is the arithmetic mean of a set of positive numbers, G the geometric mean and H the harmonic mean, has been known for some time although it is difficult to prove in general.

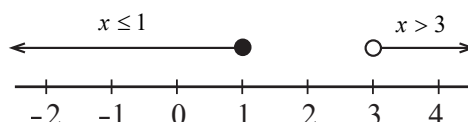
A more recent application of solving sets of inequalities is that of *linear programming*. This technique was developed during the 2nd World War by the mathematicians, *John von Neuman* (1905–1957) and *George Dantzig* (1914–2005), in order to solve problems of optimising convoys across the Atlantic. Since then, the technique has been expanded and used in a variety of contexts – essentially when you are trying to maximise or minimise a quantity (e.g. profit) subject to a number of inequality constraints.

In summary, inequalities are a fundamental building block in mathematics and although this unit deals with particular applications, it should be stressed that this *is* mainstream mathematics.

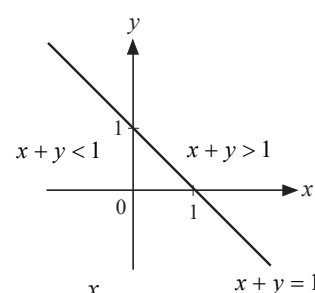
Key points

- You must be careful to differentiate between $<$ and $\leq$.
- Representing $<$ and $\leq$ on a number line must be made clear,

i.e. use small unshaded circle, $\bigcirc$, for the end point of $<$
use small shaded circle, $\bullet$, for the end point of $\leq$.

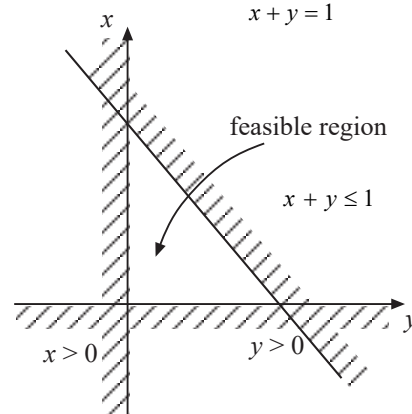


- A set of inequalities is solved by finding the values of the variables which satisfy all the inequalities.
- The equation $ax + by = c$ defines a line; one side of the line satisfies $ax + by > c$ and the other side satisfies $ax + by < c$. For example, $x + y = 1$.



- In linear programming, the feasible region is the region in which all the inequalities are satisfied and the solution to the optimisation problem is found at one of the vertices.

For example, for the set of inequalities $x + y \geq 1$, $x \geq 0$, $y \geq 0$, the feasible region, is shown opposite.



Unit 4 *Solving Inequalities*

Introduction

Facts to remember

- $x \geq a$ means that either $x > a$ or $x = a$
- Multiplying an inequality by a negative number reverses the inequality sign; for example, $x \geq 4 \Rightarrow -x \leq -4$.

Glossary of terms

- $>$ means 'greater than'; for example, $5 > 3$.
- $\geq$ means 'greater than or equal to'; for example, $5 \geq 3$, $5 \geq 5$.
- $<$ means 'less than'; for example, $3 < 5$.
- $\leq$ means 'less than or equal to'; for example, $3 \leq 5$, $5 \leq 5$.
- $\neq$ means 'not equal to'; for example, $3 \neq 5$.

Linear inequalities are of the form

$$ax + b \leq c \text{ or } ax + b < c, \text{ etc.}$$

For example, $2x + 3 \leq 7$

Solving linear inequalities means manipulating the inequality to give $x \geq \dots$ or $x \leq \dots$

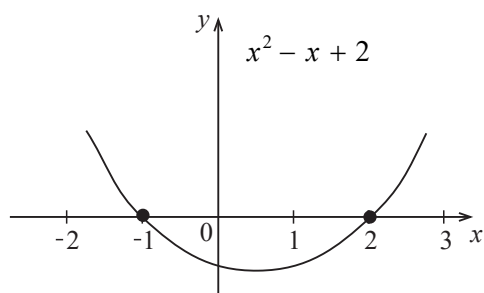
For example, $2x + 3 \leq 7 \Rightarrow 2x \leq 4 \Rightarrow x \leq 2$

Quadratic inequalities are of the form

$$ax^2 + bx + c \leq 0, \text{ etc.}$$

For example, $x^2 - x + 2 \geq 0 \Rightarrow (x - 2)(x + 1) \geq 0$

and illustrating on a graph,



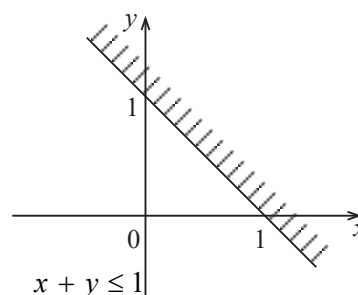
shows that the inequality is true for $x \geq 2$ or $x \leq -1$.

Unit 4 *Solving Inequalities*

Introduction

General linear inequality in two variables is of the form $ax + by \geq c$ and can be illustrated graphically.

For example, $x + y \leq 1$



Linear programming For example, maximum $P = 2x + y$ subject to

$$x + y \leq 5$$

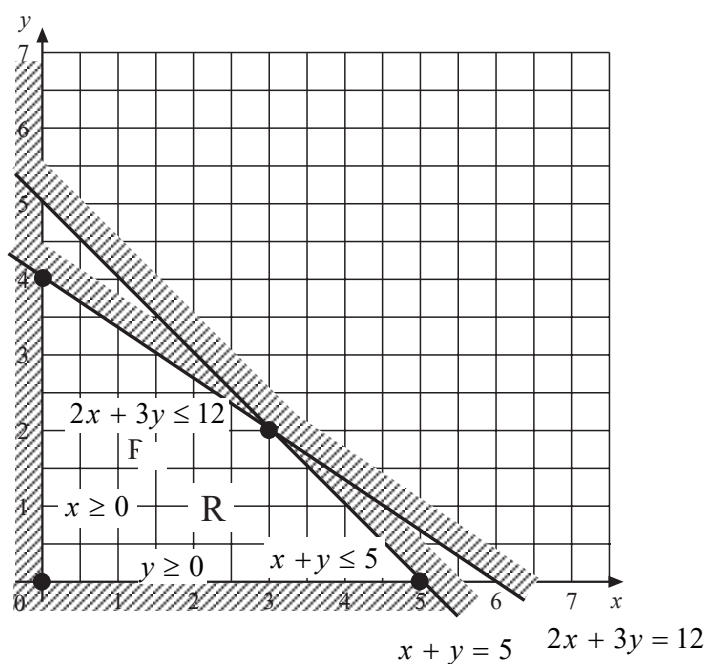
$$2x + 3y \leq 12$$

$$x \geq 0, y \geq 0$$

Feasible region

This is the region in which all the inequalities are satisfied.

For the example above, the feasible region is labelled R; it is the region not shaded.



(**Note:** the maximum of $P = 2x + y$ will occur at one of the vertices marked by ● on the diagram; in fact, it will occur at (5, 0) and has value $P = 10$.)