

STRAND: ALGEBRA

Unit 5 *Sequences and Series*

TEXT

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5 Sequences and Series

5.1 Geometrical Sequences

Suppose you take part in a sponsored walk. In the first hour you walk 3 miles, in the second hour 2 miles and in each succeeding hour $\frac{2}{3}$ of the distance the hour before. How far would you walk in 10 hours? How far would you go if you kept on like this for ever?

This gives a sequence of numbers: 3, 2, $1\frac{1}{3}$, .. etc. This unit is about how to tackle problems that involve sequences like this and gives further examples of where they might arise. It also examines sequences and series in general, quick methods of writing them down, and techniques for investigating their behaviour.

Legend has it that the inventor of the game called **chess** was told to name his own reward. His reply was along these lines:

"Imagine a chessboard.

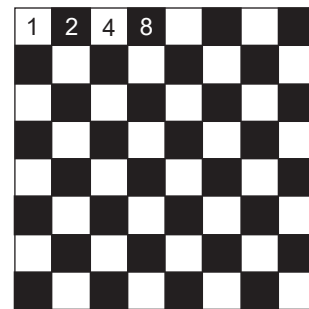
Suppose 1 grain of corn is placed on the first square,

2 grains on the second,

4 grains on the third,

8 grains on the fourth,

and so on, doubling each time up to and including the 64th square. I would like as many grains of corn as the chessboard now carries."

Number of grains
of corn shown

It took his patron a little time to appreciate the enormity of this request, but not as long as the inventor would have taken to use up all the corn.



Worked Example 1

- How many grains would there be on the 64th square?
- How many would there be on the n^{th} square?
- Work out the numerical values of the first 10 terms of the sequence.

$$2^0, 2^0 + 2^1, 2^0 + 2^1 + 2^2 \text{ etc.}$$

- (d) How many grains are there on the chessboard?



Solution

- (a) $1 + 2 + 2^2 + 2^3 + \dots + 2^{63}$
 (b) $1 + 2 + 2^2 + \dots + 2^{n-1}$
 (c) 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023
 (d) Noting that the total number on the first 10 squares is

$$511 + 512 = 2^9 + (2^9 - 1) = 2 \times 2^9 - 1 = 2^{10} - 1$$

the total number on all 64 squares is

$$2^{64} - 1 \approx 1.8 \times 10^{19}$$

5.1

The series of numbers 1, 2, 4, 8, 16 ... is an example of a **geometric sequence**, sometimes called a **geometric progression (GP)**. Each term in the progression is found by **multiplying** the previous number by 2.

Such sequences occur in many situations; the multiplying factor does not have to be 2. For example, if you invested £2000 in an account with a fixed interest rate of 8% p.a. then the amounts of money in the account after 1 year, 2 years, 3 years etc. would be as shown in the table. The first number in the sequence is 2000 and each successive number is found by multiplying by 1.08 each time.

Number of years	Money in account (£)
0	2000
1	2160
2	2332.80
3	2519.42
4	2720.98

Accountants often work out the residual value of a piece of equipment by assuming a fixed depreciation rate. Suppose a piece of equipment was originally worth £35 000 and depreciates in value by 10% each year. Then the values at the beginning of each succeeding year are as shown in the table opposite. Notice that they too form a geometric progression.

Year	Value (£)
0	35 000
1	31 500
2	28 350
3	25 515
4	22 963.50

The chessboard problem in the previous section involved adding up

$$2^0 + 2^1 + 2^2 + \dots + 2^{63}$$

The sum of several terms of a sequence is called a **series**. Hence the sum $2^0 + 2^1 + 2^2 + \dots + 2^{63}$ is called a **geometric series**.



Worked Example 2

- (a) Work out the values of

$$3^0, 3^0 + 3^1, 3^0 + 3^1 + 3^2, 3^0 + 3^1 + 3^2 + 3^3$$

- (b) Find a formula for

$$1 + 3 + 3^2 + \dots + 3^{n-1}$$

- (c) Find a formula for

$$1 + 4 + 4^2 + \dots + 4^{n-1}$$



Solution

- (a) 1, 4, 13, 40

- (b) Note that (using the form in Worked Example 1 above),

$$3^1 - 1 = 2, 3^2 - 1 = 8, 3^3 - 1 = 26, 3^4 - 1 = 80$$

and comparing this sequence, 2, 8, 26, 80 with (a), we see that

$$1 + 3 + 3^2 + \dots + 3^{n-1} = \frac{3^n - 1}{2}$$

- (c) Using the method above, the sequence is

$$1, 5, 21, 85, \dots$$

whilst 4^{n-1} gives the sequence

$$3, 15, 63, 255, \dots$$

5.1

and each term is 3 times the original sequence.

Hence

$$1 + 4 + 4^2 + \dots + 4^{n-1} = \frac{4^n - 1}{3}$$

The general form of a geometric sequence with n terms is

$$a, ar, ar^2, \dots, ar^{n-1}$$

The ratio r of consecutive terms, is known as the **common ratio**. Notice that the n^{th} term of the sequence is ar^{n-1} .

In the chessboard problem the solution involved adding up the first 64 terms. The sum of the first n terms of a series is often denoted by S_n , and there is a formula for S_n which you may have found in Worked Example 2. Here is a way of proving the formula, when $r \neq 1$.

$$S_n = a + ar + ar^2 + \dots + ar^{n-1} \quad (1)$$

Multiply both sides by r :

$$rS_n = ar + ar^2 + \dots + ar^{n-1} + ar^n \quad (2)$$

Notice that the expressions for S_n and rS_n are identical, with the exception of the terms a and ar^n . Subtracting equation (1) from equation (2) gives

$$rS_n - S_n = ar^n - a$$

$$\Rightarrow S_n(r-1) = a(r^n - 1)$$

$$\Rightarrow S_n = \frac{a(r^n - 1)}{r - 1}$$

If $r < 1$, it is easier to write

$$S_n = \frac{a(1 - r^n)}{1 - r}$$



Worked Example 3

Find

(a) $4 + 6 + 9 + \dots + 4 \times (1.5)^{10}$

(b) $8 + 6 + 4.5 + \dots + 8 \times (0.75)^{25}$



Solution

(a) First term $a = 4$, common ratio $r = 1.5$ and number of terms $n = 11$;

$$S_{11} = \frac{4(1.5^{11} - 1)}{1.5 - 1} = 684.0 \text{ to 4 s.f.}$$

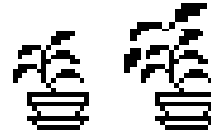
(b) First term $a = 8$, common ratio $r = 0.75$ and number of terms $n = 26$;

$$S_{26} = \frac{8(1 - 0.75^{26})}{1 - 0.75} = 31.98 \text{ to 4 s.f.}$$

5.1

**Worked Example 4**

A plant grows 1.67 cm in its first week. Each subsequent week it grows by 4% more than in the previous week. How much does it grow in total in the first nine weeks?

**Solution**

The growths in the first 9 weeks are as follows :

$$1.67, 1.67 \times 1.04, 1.67 \times 1.04^2, \dots$$

Total growth in first nine weeks is

$$S_9 = \frac{1.67(1.04^9 - 1)}{1.04 - 1} = 17.67 \text{ cm to 4 s.f.}$$

**Worked Example 5**

After how many complete years will a starting capital of £5000 first exceed £10 000 if it grows at 6% per annum?

£5000



£10 000

How many years later?

**Solution**

After n years, the capital sum has grown to

$$5000 \times (1.06)^n$$

When is this first greater than 10 000, n being a natural number? In other words, the smallest value of n is required so that

$$5000 \times (1.06)^n > 10\,000, \quad n \in \mathbb{N}$$

$$\Rightarrow (1.06)^n > 2$$

You can either use your calculator to find the value of n , or use logs if you are familiar with this concept. Now take logs of both sides:

$$n \ln 1.06 > \ln 2$$

$$\Rightarrow n > \frac{\ln 2}{\ln 1.06}$$

$$\Rightarrow n > 11.9$$

After 12 years, the investment has doubled in value.

Check: $5000 \times (1.06)^{11} = 9481.49$

$$5000 \times (1.06)^{12} = 10\,000.98$$

5.1



Exercises

1. Write down formulae for the n^{th} term of these sequences:
 - (a) 3, 6, 12, 24, ...
 - (b) 36, 18, 9, 4.5, ...
 - (c) 2, -6, 18, -54, ...
 - (d) 90, -30, 10, $-3\frac{1}{3}$, ...
 - (e) 10, 100, 1000, ...
 - (f) 6, -6, 6, -6, ...
 - (g) $\frac{1}{4}$, $\frac{1}{12}$, $\frac{1}{36}$, $\frac{1}{108}$, ...

2. Use the formula for S_n to calculate to 4 s.f.
 - (a) $5 + 10 + 20 + \dots$ to 6 terms
 - (b) $4 + 12 + 36 + \dots$ to 10 terms
 - (c) $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \dots$ to 8 terms
 - (d) $100 - 20 + 4 - \dots$ to 20 terms
 - (e) $16 + 17.6 + 19.36 + \dots$ to 50 terms
 - (f) $26 - 16.25 + 10.15625 \dots$ to 15 terms.

3. Give the number (e.g. 12th term) of the earliest term for which
 - (a) the sequence 1, 1.5, 2.25, ... exceeds 50;
 - (b) the sequence 6, 8, $10\frac{2}{3}$, ... exceeds 250;
 - (c) the sequence $\frac{2}{5}$, $\frac{1}{5}$, $\frac{1}{10}$, ... goes below $\frac{1}{1000}$

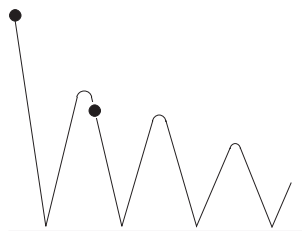
4.
 - (a) For what value of n does the sum $50 + 60 + 72 + \dots + 50 \times (1.2)^{n-1}$ first exceed 1000?
 - (b) To how many terms can the following series be summed before it exceeds 2 000 000?

$$2 + 2.01 + 2.02005 + \dots$$

5. Dave invests £500 in a building society account at the start of each year. The interest rate in the account is 7.2% p.a. Immediately after he invests his 12th instalment he calculates how much money the account should contain. Show this calculation as the sum of a GP and use the formula for S_n to evaluate it.

5.2 Never-Ending Sums

Many of the ideas used so far to illustrate geometric series have been to do with money. Here is one example that is not. If you drop a tennis ball, or any elastic object, onto a horizontal floor it will bounce back up part of the way. If left to its own devices it will continue to bounce, the height of the bounces decreasing each time.



The ratio between the heights of consecutive bounces is constant, hence these heights follow a geometric progression. The same thing is true of the times between bounces.



Worked Example 1

- (a) A tennis ball is dropped from a height of 1 metre onto a concrete floor. After its first bounce, it rises to a height of 49 cm. Call the height after the n^{th} bounce h_n . Find a formula for h_n and say what happens to h_n as n gets larger.
- (b) Under these circumstances the time between the first and second bounces is 0.6321 seconds. Call this t_1 . The next time, t_2 , is $0.7t_1$, and each successive time is 0.7 times the previous one. Find a formula for t_n .



Solution

- (a) The sequence, working in metres, is

$$1, 0.49, (0.49)^2, (0.49)^3, \dots$$

Hence, working in cm,

$$h_n = 100(0.49)^n$$

As $n \rightarrow \infty$, $h_n \rightarrow 0$.

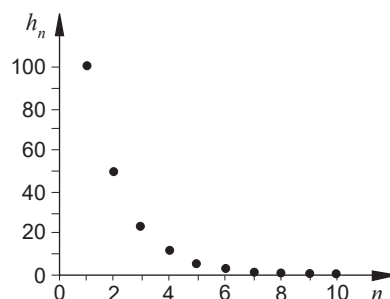
- (b) $t_n = (0.7)^{n-1} 0.6321$

Worked Example 1 gave an example of a **convergent** sequence. Convergence, in this context, means that the further along the sequence you go, the closer you get to a specific value. For example, in part (a) the sequence to the nearest 0.1 cm is

$$100, 49.0, 24.0, 11.8, 5.8, 2.8, 1.4, \dots$$

and the numbers get closer and closer to zero. Zero is said to be the **limit** of the sequence.

The series $1, 2, 4, 8, \dots$ is a **divergent** sequence. It grows without limit as the number of terms increases. The same is true, in a slightly different sense, of the sequence $1, -2, 4, -8, \dots$. Any sequence that does **not** converge is said to be **divergent**.



5.2

**Worked Example 2**

For each of these sequences,

- (i) write a formula for the n^{th} term;
 - (ii) find whether the sequence converges;
 - (iii) find whether the sum S_n converges.
- (a) $6, 2, \frac{2}{3}, \frac{2}{9}, \dots$
- (b) $1, 1.5, 2.25, 3.375, \dots$
- (c) $4, -3, \frac{9}{4}, -\frac{27}{16}, \dots$

**Solution**

- (a) (i) As each term is $\frac{1}{3}$ of the previous term, the n^{th} term $= \frac{6}{3^{n-1}}$.
 (Check: $\frac{6}{3^0}, \frac{6}{3}, \frac{6}{3^2}, \frac{6}{3^3}, \dots \Rightarrow 6, 2, \frac{2}{3}, \frac{2}{9}, \dots$)
- (ii) n^{th} term $\rightarrow 0$ as $n \rightarrow \infty$ so the sequence converges.
- (iii) The sum, S_n , will also converge (to 9, see below).
- (b) (i) Here each term is 1.5 times the previous term, so
 n^{th} term $= 1.5^{n-1}$
 (Check: $1.5^0, 1.5^1, 1.5^2, \dots \Rightarrow 1, 1.5, 2.25, \dots$)
- (ii) This diverges as $n \rightarrow \infty$.
- (iii) Clearly the sum, S_n , also diverges.
- (c) (i) Here each term is $\left(-\frac{3}{4}\right)$ times the previous term, giving
 n^{th} term $= 4\left(-\frac{3}{4}\right)^{n-1}$
- (ii) n^{th} term $\rightarrow 0$ as $n \rightarrow \infty$, so it converges.
- (iii) The sum, S_n , also converges $\left(\text{to } \frac{16}{7}, \text{ see below}\right)$

We know that the sum of the first n terms of a geometric series $a, ar, ar^2, \dots, ar^{n-1}$ is given by

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

5.2

If $r^n \rightarrow 0$ as $n \rightarrow \infty$, then $S_n \rightarrow \frac{a}{(1-r)}$ as this is the sum of all the terms in the series.

The common ratio, r , is such that when $|r| < 1$, then $r^n \rightarrow 0$. Hence we have the following important result.

A geometric series, $a + ar + ar^2 + \dots + ar^{n-1}$ converges when $|r| < 1$; i.e. for $-1 < r < 1$. Since if $|r| < 1$, $r^n \rightarrow 0$ as $n \rightarrow \infty$ and

$$S_n \rightarrow \frac{a}{1-r} \text{ as } n \rightarrow \infty$$

The limit $\frac{a}{1-r}$ is known as the 'sum to infinity' and is denoted S_∞ .



Worked Example 3

Find

- (a) $8 + 4 + 2 + 1 + \dots$
- (b) $20 - 16 + 12.8 - 10.24 + \dots$



Solution

- (a) This is a geometric series with first term 8 and common ratio $\frac{1}{2}$, so

$$S_\infty = \frac{8}{1 - \frac{1}{2}} = 16$$

- (b) This is a geometric series with first term 20 and common ratio -0.8 , so

$$S_\infty = \frac{20}{1 - (-0.8)} = \frac{20}{1.8} = \frac{100}{9} = 11.1 \text{ (to 3 s.f.)}$$



Exercises

1. Find these sums to infinity, where they exist.

- (a) $80 + 20 + 5 + 1.25 + \dots$
- (b) $180 - 60 + 20 - \frac{20}{3} + \dots$
- (c) $2 + 1.98 + 1.9602 + \dots$
- (d) $-100 + 110 - 121 + \dots$
- (e) $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots$

5.2

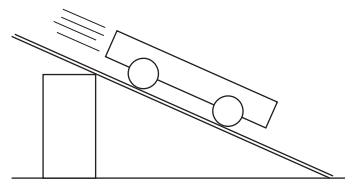
2. (a) What is $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots$ as a recurring decimal?
 (b) Express 0.37373737 ... as an infinite geometric series and find the fraction it represents.
3. What fractions do these decimals represent?
 (a) 0.52525252 ...
 (b) 0.358358358...
 (c) 0.194949494 ...
4. (a) A GP has a common ratio of 0.65. Its sum to infinity is 120. What is the first term?
 (b) Another GP has 2.8 as its first term and its sum to infinity is 3.2 Find its common ratio.
5. Rosita is using a device to extract air from a bottle of wine. This helps to preserve the wine left in the bottle.
 The pump she uses can extract a maximum of 46 cm^3 . In practice what happens is that the first attempt extracts 46 cm^3 and subsequent extractions follow a geometric sequence.
 Rosita's second attempt extracts 36 cm^3 . What is the maximum amount of air she can remove in total?
6. A rubber ball is dropped from a height of 6 metres and after the first bounce rises to a height of 4.7 m. It is left to continue bouncing until it stops.
 (a) A computerized timer is started when it first hits the ground. The second contact with the ground occurs after 1.958 seconds and the third after 3.690 seconds. Given that the times between consecutive contacts with the ground follow a geometric sequence, how long does the ball take to stop bouncing?
 (b) The heights to which the ball rises after each impact also follow a geometric sequence. Between the release of the ball and the second bounce the ball travels $6 + 2 \times 4.7 = 15.4 \text{ m}$. How far does the ball travel altogether?



5.3 Arithmetic Sequences

Geometric sequences involve a constant ratio between consecutive terms. Another important type of sequence involves a constant **difference** between consecutive terms; such a sequence is called an **arithmetic** sequence.

In an experiment to measure the descent of a trolley rolling down a slope a 'tickertape timer' is used to measure the distance travelled in each second. The results are shown in the table.



5.3

The sequence 3, 5, 7, 9, 11, 13 is an example of an arithmetic sequence. The sequence starts with 3 and thereafter each term is 2 more than the previous one. The difference of 2 is known as the **common difference**.

It would be useful to find the total distance travelled in the first 6 seconds by adding the numbers together. A quick numerical trick for doing this is to imagine writing the numbers out twice, once forwards once backwards, as shown below

Second	cm travelled in second
1	3
2	5
3	7
4	9
5	11
6	13

3	5	7	9	11	13
13	11	9	7	5	3

Each pair of vertical numbers adds up to 16. So adding the two sequences, you have 6×16 between them. Hence the sum of the original series is

$$\frac{1}{2} \times (6 \times 16) = 48$$

The sum of terms of an arithmetic sequence is called an **arithmetic series** or **progression**, often called AP for short.



Worked Example 1

Use the example above of a trolley rolling down a slope to answer these questions.

- Work out the distance travelled in the 20th second.
- Calculate S_{20} , the distance travelled in the first 20 seconds, using the above method.
- What is the distance travelled in the n^{th} second?
- Show that the trolley travels a distance of $n(n + 2)$ cm in the first n seconds.



Solution

- (a) The sequence is

3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41
and the 20th term is 41.

- (b) The distance travelled in 20 seconds can be found by rewriting the sequence as

3	5	7	9	11	13	15	17	19	21
41	39	37	35	33	31	29	27	25	23

Each pair of the vertical numbers adds to 44 and there are 10 pairs.

$$S_{20} = 10 \times 44 = 440 \text{ cm}$$

- (c) Distance travelled in n^{th} second is $(2n + 1)\text{cm}$.

5.3

- (d) Total distance travelled in
- n
- seconds can be found in the same way.

$$\begin{array}{cccccc}
 3 & 5 & 7 & 9 & 11 & \dots \\
 2n+1 & 2n-1 & 2n-3 & 2n-5 & 2n-7 & \dots
 \end{array}$$

Each pair now adds up to $2n+4$ and there are $\frac{n}{2}$ pairs.

Thus

$$\begin{aligned}
 S_n &= \frac{n}{2} \times (3 + (2n+1)) \\
 &= \frac{n}{2} \times (2n+4) \\
 &= n(n+2)
 \end{aligned}$$



Worked Example 2

Consider the arithmetic sequence 8, 12, 16, 20 ...

Find expressions

- (a) for u_n , (the n^{th} term) (b) for S_n .



Solution

In this AP the first term is 8 and the common difference 4.

- (a) $u_1 = 8$

$$u_2 = 8 + 4$$

$$u_3 = 8 + 2 \times 4$$

$$u_4 = 8 + 3 \times 4 \quad \text{etc.}$$

u_n is obtained by adding on the common difference $(n-1)$ times.

$$\begin{aligned}
 \Rightarrow u_n &= 8 + 4(n-1) \\
 &= 4n + 4
 \end{aligned}$$

- (b) To find S_n , follow the procedure explained previously:

$$\begin{array}{cccccc}
 8 & 12 & \dots & 4n & 4n+4 \\
 4n+4 & 4n & \dots & 12 & 8
 \end{array}$$

Each pair adds up to $4n+12$ There are n pairs.

$$\begin{aligned}
 \text{So} \quad 2S_n &= n(4n+12) \\
 &= 4n(n+3)
 \end{aligned}$$

$$\text{giving} \quad S_n = 2n(n+3)$$

5.3

The general arithmetic sequence is often denoted by

$$a, a + d, a + 2d, a + 3d, \text{ etc.}$$

To sum the series of the first n terms of the sequence,

$$S_n = a + (a + d) + (a + 2d) + \dots + (a + (n - 1)d)$$

Note that the order can be reversed to give

$$S_n = (a + (n - 1)d) + (a + (n - 2)d) + \dots + a$$

Adding the two expressions for S_n gives

$$\begin{aligned} 2S_n &= (2a + (n - 1)d) + (2a + (n - 1)d) + \dots + (2a + (n - 1)d) \\ &= n[2a + (n - 1)d] \end{aligned}$$

So

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

An alternative form for S_n is given in terms of its first and last term, a and l , where

$$l = a + (n - 1)d$$

since the n^{th} term of the sequence is given by

$$u_n = a + (n - 1)d$$

Thus

$$S_n = \frac{n}{2}(a + l)$$



Worked Example 3

Sum the series $5 + 9 + 13 + \dots$ to 20 terms.



Solution

This is an arithmetic sequence with first term 5 and common difference 4; so

$$S_{20} = \frac{20}{2}(2 \times 5 + 19 \times 4) = 860$$



Worked Example 4

The sum of the series $1 + 8 + 15 + \dots$ is 396. How many terms does the series contain?



Solution

This is an arithmetic sequence with first term 1 and common difference 7. Let the number of terms in the sequence be n .

5.3

$$S_n = 396$$

$$\Rightarrow \frac{n}{2}(2 + 7(n - 1)) = 396$$

$$\Rightarrow n(7n - 5) = 792$$

$$\Rightarrow 7n^2 - 5n - 792 = 0$$

$$\Rightarrow (7n + 72)(n - 11) = 0$$

$$\Rightarrow n = 11 \text{ since } -\frac{72}{7} \text{ is not an integer.}$$

The number of terms is 11.



Exercises

- Use the 'numerical trick' to calculate
 - $3 + 7 + 11 + \dots + 27$
 - $52 + 46 + 40 + \dots + 4$
 - the sum of all the numbers on a traditional clock face
 - the sum of all the odd numbers between 1 and 99.
- Find formulae for u_n and S_n in these sequences.
 - 1, 4, 7, 10, ...
 - 12, 21, 30, 39, ...
 - 60, 55, 50, 45, ...
 - $1, 2\frac{1}{2}, 4, 5\frac{1}{2}, \dots$
- A model railway manufacturer makes pieces of track of lengths 8 cm, 10 cm, 12 cm, etc. up to and including 38 cm. An enthusiast buys 5 pieces of each length. What total length of track can be made?
- Find the sum of
 - $11 + 14 + 17 + \dots$ to 16 terms
 - $27 + 22 + 17 + \dots$ to 10 terms
 - $5 + 17 + 29 + \dots + 161$
 - $7.2 + 7.8 + 8.4 + \dots$ to 21 terms
 - $90 + 79 + 68 + \dots - 20$
 - $0.12 + 0.155 + 0.19 + \dots$ to 150 terms
- The last three terms of an arithmetical sequence with 18 terms is as follows : ..., 67, 72, 77. Find the first term and the sum of the series.
- How many terms are there if
 - $52 + 49 + 46 + \dots = 385$
 - $0.35 + 0.52 + 0.69 + \dots = 35.72$?

5.3

7. The first term of an arithmetic series is 16 and the last is 60. The sum of the arithmetic series is 342. Find the common difference.
8. New employees joining a firm in the clerical grade receive an annual salary of £8500. Every year they stay with the firm they have a salary increase of £800, up to a maximum of £13300 p.a. How much does a new employee earn in total, up to and including the year on maximum salary?
9. Ten brothers receive 100 shekels between them. Each brother receives a constant amount more than the next oldest. The seventh oldest brother receives 7 shekels. How much does each brother receive?

5.4 Sigma Notation

Repeatedly having to write out terms in a series is time consuming. Mathematicians have developed a form of notation which both shortens the process and is easy to use. It involves the use of the Greek capital letter Σ (sigma), the equivalent of the letter *S*, for sum.

The series $2 + 4 + 8 + \dots + 2^{12}$ can be shortened to $\sum_{r=1}^{12} 2^r$.

This is because every term in the series is of the form 2^r , and all the values of 2^r , from $r = 1$ to $r = 12$ are added up. In this example the ' 2^r ' is called the **general term**; 12 and 1 are the top and bottom **limits** of the sum.

Similarly, the series

$$60 + 60 \times (0.95) + \dots + 60 \times (0.95)^{30}$$

can be abbreviated to

$$\sum_{r=0}^{30} 60 \times (0.95)^r$$

Often there is more than one way to use the notation. The series

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots + \frac{99}{100}$$

has a general term that could be thought of as either $\frac{r}{r+1}$ or $\frac{r-1}{r}$. Hence the series can be written as either

$$\sum_{r=1}^{99} \frac{r}{r+1} \text{ or } \sum_{r=2}^{100} \frac{r-1}{r}$$

5.4

**Worked Example 1**

Write out what $\sum_{r=1}^9 (10-r)^2$ means and write down another way of expressing the same series, using Σ notation.

**Solution**

$$\begin{aligned}\sum_{r=1}^9 (10-r)^2 &= (10-1)^2 + (10-2)^2 + \dots + (10-9)^2 \\ &= 9^2 + 8^2 + \dots + 1^2\end{aligned}$$

An alternative way of writing the same series is to think of it in reverse:

$$1^2 + 2^2 + \dots + 8^2 + 9^2 = \sum_{r=1}^9 r^2$$

**Worked Example 2**

Express in Σ notation 'the sum of all multiples of 5 between 1 and 100 inclusive'.

**Solution**

All multiples of 5 are of the form $5r$, $r \in \mathbb{N}$.

$100 = 5 \times 20$ so the top limit is 20. The lowest multiple of 5 to be included is 5×1 . The sum is therefore

$$5 + 10 + 15 + \dots + 100 = \sum_{r=1}^{20} 5r$$

**Worked Example 3**

Express in Σ notation 'the sum of the first n positive integers ending in 3'.

**Solution**

Numbers ending in 3 have the form $10r + 3$ $r \in \mathbb{N}$. The first number required is 3 itself, so the bottom limit must be $r = 0$. This means that the top limit must be $n - 1$. Hence the answer is

$$\sum_{r=0}^{n-1} (10r + 3) \quad (= 3 + 13 + \dots + (10n - 7))$$

[An alternative answer is $\sum_{r=1}^n (10r - 7)$]

5.4



Exercises

1. Write out the first three and last terms of:

$$(a) \sum_{r=5}^{15} r^2 \quad (b) \sum_{r=1}^{10} (2r - 1) \quad (c) \sum_{r=1}^n r$$

$$(d) \sum_{r=3}^{10} \frac{r-2}{r} \quad (e) \sum_{r=6}^{100} (r-2)^2$$

2. Shorten these expressions using Σ notation.

$$(a) 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{25}$$

$$(b) 10 + 11 + 12 + \dots + 50$$

$$(c) 1 + 8 + 27 + \dots + n^3$$

$$(d) 1 + 3 + 9 + 27 + \dots + 3^{12}$$

$$(e) 6 + 11 + 16 + \dots + (5n + 1)$$

$$(f) 14 + 17 + 20 + \dots + 62$$

$$(g) 5 + 50 + 500 + \dots + 5 \times 10^n$$

$$(h) \frac{1}{6} + \frac{2}{12} + \frac{3}{20} + \dots + \frac{20}{21 \times 22}$$

3. Use Σ notation to write:

- (a) the sum of all natural numbers with two digits;
- (b) the sum of the first 60 odd numbers;
- (c) the sum of all the square numbers from 100 to 400 inclusive;
- (d) the sum of all numbers between 1 and 100 inclusive that leave remainder 1 when divided by 7.

4. Find alternative ways, using Σ notation, of writing these:

$$(a) \sum_{r=1}^{19} (20 - r) \quad (b) \sum_{r=2}^{41} \frac{1}{r-1} \quad (c) \sum_{r=-3}^3 r^2$$

5.5 More Series



Worked Example 1

- (a) Write down the values of $(-1)^0, (-1)^1, (-1)^2, (-1)^3$ etc.
Generalise your answers.
- (b) Write down the first three terms and the last term of
- (i) $\sum_{r=0}^{10} (-1)^r \frac{1}{2^r}$ (ii) $\sum_{r=0}^{10} (-1)^{r+1} \left(\frac{r^2}{r^2 + 1} \right)$
- (c) How can you write the series
 $100 - 100 \times (0.8) + 100 \times (0.8)^2 \dots$ to n terms using Σ notation?



Solution

- (a) 1, -1, 1, -1; clearly the sequence continues to alternate in this way.

(b) (i) $(-1)^0 \frac{1}{2^0} + (-1)^1 \frac{1}{2^1} + (-1)^2 \frac{1}{2^2} = 1 - \frac{1}{2} + \frac{1}{4}$

and last term is $(-1)^{10} \frac{1}{2^{10}} = \frac{1}{2^{10}} \left(= \frac{1}{1024} \right)$

(ii) $(-1)^1 \frac{0^2}{(0^2 + 1)} + (-1)^2 \left(\frac{1^2}{1^2 + 1} \right) + (-1)^3 \left(\frac{2^2}{2^2 + 1} \right) = -\frac{1}{2} + \frac{1}{2} - \frac{4}{5}$

and last term is $(-1)^{11} \left(\frac{10^2}{10^2 + 1} \right) = -\frac{100}{101}$

(c) $\sum_{r=0}^n 100(-8)^r$ or $\sum_{r=0}^n (-1)^r 100 \times (0.8)^r$

It is also useful to know these three important results, the first of which is the sum of the arithmetic progression,

$$1, 2, 3, \dots, n$$

$$\begin{aligned} \sum_{r=1}^n r &= \frac{1}{2} n(n+1) \\ \sum_{r=1}^n r^2 &= \frac{1}{6} n(n+1)(2n+1) \\ \sum_{r=1}^n r^3 &= \frac{1}{4} n^2(n+1)^2 \end{aligned}$$

The useful fact that $\sum r^3 = (\sum r)^2$ is a coincidence (if there is such a thing in maths). It is not possible to extend this to find $\sum r^4, \sum r^5$ etc. Formulae do exist for sums of higher powers, but they are somewhat cumbersome and seldom useful.



Exercises

1. Work out the numerical value of

$$(a) \sum_{r=1}^{10} r \quad (b) \sum_{r=1}^{25} 4r \quad (c) \sum_{r=0}^{16} (3 + 5r)$$

2. Use Σ notation to write these:

$$(a) 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{6}$$

$$(b) -1 + 4 - 9 + 16 - \dots + 144$$

$$(c) 12 - 12 \times 0.2 + 12 \times 0.04 - \dots + 12 \times (0.2)^{50}$$

3. If you know that

$$\sum_{r=1}^n u_r = 20 \text{ and } \sum_{r=1}^n v_r = 64$$

calculate where possible:

$$(a) \sum_{r=1}^n (u_r + v_r) \quad (b) \sum_{r=1}^n u_r v_r \quad (c) \sum_{r=1}^n u_r^2$$

$$(d) \sum_{r=1}^n \frac{1}{2} v_r \quad (e) \sum_{r=1}^n (v_r - u_r) \quad (f) \sum_{r=1}^n (5u_r - v_r)$$

$$(g) \sum_{r=1}^{2n} u_r \quad (h) \sum_{r=1}^n (-1)^r v_r$$

4. Write down the general term, and hence evaluate:

$$(a) 1 + 2 + 3 + \dots + 20$$

$$(b) 1^2 + 2^2 + 3^2 + \dots + 10^2$$

$$(c) 2 + 8 + 18 + \dots + (2 \times 15^2)$$

$$(d) 2 + 4 + 6 + \dots + 100$$

$$(e) 1 + 3 + 5 + \dots + 25$$

$$(f) 1 + 8 + 27 + \dots + 1000$$

5. Work out $\sum_{r=10}^{20} r$. Use the fact that

$$\sum_{r=10}^{20} r = \sum_{r=0}^{20} r - \sum_{r=0}^9 r$$

5.5

6. Use techniques similar to that in Question 5 to calculate
- (a) $11^2 + 12^2 + \dots + 24^2$
 - (b) $7^3 + 8^3 + \dots + 15^3$
 - (c) $21 + 23 + \dots + 61$
7. Calculate $21 + 23 + 25 \dots + 161$
8. Prove that $\sum_{r=0}^{2n} r = n(2n + 1)$ and hence that $\sum_{r=n+1}^{2n} r = \frac{1}{2}n(3n + 1)$.