

STRAND: ALGEBRA

Unit 6 *Binomial Theorem*

TEXT

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6 Binomial Theorem

6.1 Binomial Expansions

'Binomial' is a word meaning 'two terms', and is used in algebra to mean expressions such as $a + 2$ and $2x - y$. (Compare with the word 'polynomial' - 'an expression of more than two algebraic terms'.)

You are probably familiar with expressions such as

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b) \\ &= a^2 + ab + ba + b^2 \\ &= a^2 + 2ab + b^2\end{aligned}$$



Worked Example 1

Here are six statements about binomials. Which of these are true, and which false?

- | | |
|--|--|
| (a) $2(a + b) = 2a + 2b$ | (b) $(a + b) \div 2 = (a \div 2) + (b \div 2)$ |
| (c) $(a + b)^2 = a^2 + b^2$ | (d) $(a + b)^3 = a^3 + b^3$ |
| (e) $\sqrt{(a + b)} = \sqrt{a} + \sqrt{b}$ | (f) $2^{(a+b)} = 2^a + 2^b$ |



Solution

- (a) $2(a + b) = 2 \times a + 2 \times b$
 $= 2a + 2b$ so this statement is true.
- (b) $(a + b) \div 2 = a \div 2 + b \div 2$ so this statement is true.
- (c) $(a + b)^2 = a^2 + 2ab + b^2$
 $\neq a^2 + b^2$ (unless a or b equals 0) so statement (c) is false.
- (d) $(a + b)^3 = (a + b)(a + b)^2$
 $= (a + b)(a^2 + 2ab + b^2)$
 $= (a^3 + 2a^2b + ab^2) + (ba^2 + 2ab^2 + b^3)$
 $= a^3 + 3a^2b + 3ab^2 + b^3$ so this statement, too, is false (unless a or b equals zero).

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(e) $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

as $(\sqrt{a+b})^2 = a+b$

but $(\sqrt{a} + \sqrt{b})^2 = (\sqrt{a})^2 + 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2$

$$= a + 2\sqrt{ab} + b \text{ so this statement is false (unless } a \text{ or } b \text{ equals zero).}$$

(f) This statement is false as can easily be shown by substituting $a = b = 2$.

Then $2^{a+b} = 2^{2+2} = 2^4 = 16$

and $2^a + 2^b = 2^2 + 2^2 = 4 + 4 = 8$ So statement (f) is not true in general.

Note that:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

These are simple examples of **binomial expansions**.



Worked Example 2

Expand $(1+x)^4$.



Solution

$$\begin{aligned} (1+x)^4 &= (1+x)^2(1+x)^2 \\ &= (1+2x+x^2)(1+2x+x^2) \\ &= 1(1+2x+x^2) + 2x(1+2x+x^2) + x^2(1+2x+x^2) \\ &= 1+2x+x^2 + 2x+4x^2+2x^3 + x^2+2x^3+x^4 \\ &= 1+4x+6x^2+4x^3+x^4 \end{aligned}$$



Exercises

1. Write out the expansion of these:

(a) $(1+x)^2$ (b) $(1+x)^3$ (c) $(1+x)^4$ (d) $(1+x)^5$

2. Repeat for these :

(a) $(a+x)^2$ (b) $(a+x)^3$ (c) $(a+x)^4$ (d) $(a+x)^5$

3. Write out the expansions of

(a) $(3-2x)^2$ (b) $(2+5p)^3$ (c) $(\frac{1}{2}m-5)^4$

6.2 Binomial Coefficients

You will be thankful that the Exercises in Section 6.1 were neither lengthy nor involved powers higher than the 5^{th} power. As the index rises, so the expansions become more tedious. Expanding $(1+x)^{50}$ by this method, for example, would be somewhat daunting!



Worked Example 1

- (a) Using your answers to Question 1 in Section 6.1, complete this table :

	Coefficients of							
	1	x	x^2	x^3	x^4	x^5	x^6	x^7
$(1+x)^1$	1	1						
$(1+x)^2$	1	2	1					
$(1+x)^3$	1	3	3	1				
$(1+x)^4$								
$(1+x)^5$								

- (b) Describe the pattern in this table.
 (c) Write out the expansions of $(1+x)^6$ and $(1+x)^7$.
 (d) What is the **sum** of the numbers in the n^{th} row?



Solution

- (a) The extra rows are

$(1+x)^4$	1	4	6	4	1			
$(1+x)^5$	1	5	10	10	5	1		

- (b) Apart from the initial 1, you add two consecutive terms together; that is,
 $1 + 4 = 5$, $4 + 6 = 10$, etc.

$(1+x)^6$	1	6	15	20	15	6	1	
$(1+x)^7$	1	7	21	35	35	21	7	1

The numbers in the table are called the **binomial coefficients**. It is evident that they obey a pattern, but finding the numbers in a particular row depends on knowing the numbers in the row before. To expand $(1+x)^{50}$ you would need to know all the coefficients up to and including the 49^{th} row first.

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In a similar way,

$$(a + x)^5 = (a + x)(a + x)(a + x)(a + x)(a + x)$$

and the expansion is

$$a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$$

In each term the two index numbers add up to 5 (noting that $x = x^1$, $a = a^1$). Moreover, the sum of the coefficients is 32 ($1 + 5 + 10 + 10 + 5 + 1$). These two facts reflect what is going on here: each term is the product of a mixture of 'a's and 'x's, one letter from each of the 5 brackets. Each bracket can supply one of two letters, so the number of different combinations of 'a's and 'x's is $2 \times 2 \times 2 \times 2 \times 2$.

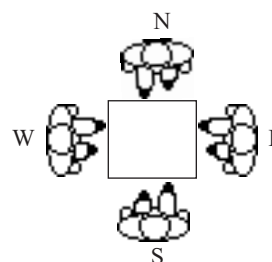
Only one such combination is all 'a's, hence the expansion starts with a single a^5 term. Five of the 32 possibilities give a^4x (that is, $aaaax$, $aaaxa$, $aaxaa$, $axaaa$ and $xaaaa$), 10 of them give a^3x^2 , and so on. It is the general method of finding the binomial coefficients 1, 5, 10, 10, 5, 1 that is being sought. That is the aim of the next section.

Bridge is one of the best of all card games. In particular, it is one in which the skills of the players can overcome the luck of the deal. We use a situation often faced by bridge players in our next Worked Example.



Worked Example 2

Suppose you are sitting at S (south); your partner is facing you at N. The opposing players are sitting either side of you at W (west) and E (east). You know that between them the opposing players have four cards in the heart suit. It is important to know the relative likelihood of various distributions of these four cards between W and E.



Each of the four cards can be in one of two places. Hence there are $2^4 = 16$ different arrangements.

- How many of these arrangements involve all four hearts being with W and none with E?
- How many of them involve E having only **one** heart?
- Complete the table below.

W	E	No. of ways
4	0	
3	1	
2	2	
1	3	
0	4	

Total 16

- Bridge players sometimes claim that in these circumstances a '1-3 split' (either way) is more likely than an even split. Is this true? Explain.
- Extend this to the case where S knows that W and E have 5 hearts between them.

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**Solution**

(a) 1 (no hearts)

(b) 4 (could be any of the 4 hearts)

(c)	W	E	No. of ways
	4	0	1
	3	1	4
	2	2	6
	1	3	4
	0	4	1
			Total 16

(d) This is true as a 1-3 split or a 3-1 split has probability of $\frac{8}{16}$ while a 2-2 split has probability $\frac{6}{16}$.

(e)	W	E	No. of ways
	5	0	1
	4	1	5
	3	2	10
	2	3	10
	1	4	5
	0	5	1
			Total 32

You should by now have realised that it could be dangerous to play bridge with a mathematician!

Many US cities have a street layout based on a rectangular pattern. On Manhattan Island, for instance, more than 200 streets run across the island and ten avenues run at right angles to them.

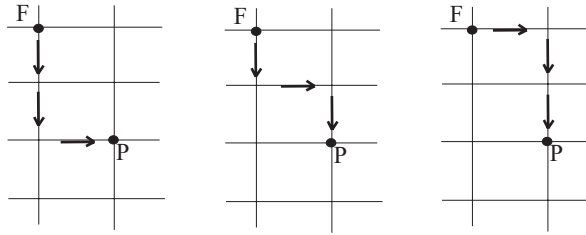
**Worked Example 3**

In the following diagrams, F represents a fire station. A fire is reported at junction P and the fire services need to take the shortest route from F to P. In fact, there are three equally short routes, as the three diagrams show.

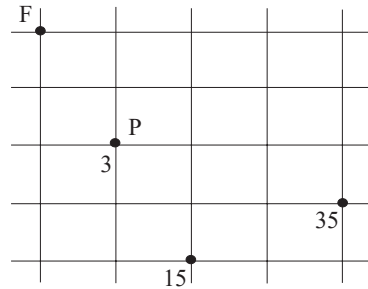
Copy the street plan shown and, by each crossroad, write the number of shortest routes there are from F. Some have been written in to help you.

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Three equally short routes:



Street plan:



Solution

Here is the solution!

F					
	1		1		1
1	2		3		4
	3	P	6		10
1	4		10		20
	5		15		35
1					70

The 'bridge' example and the 'Manhattan Island streets' example give the same pattern as that of the binomial coefficients. This is because all three situations can be reduced essentially to the same thing. Here are specific questions and solutions from all three.

Bridge problem

In how many ways can 5 cards be divided so that 3 are with E and 2 with W?

Solution

Imagine all five cards in a row. Each can be marked either E or W. There must be 3 Es and 2 Ws in any order.

The answer to the question is, therefore, the number of different combinations possible with 3 Es and 2 Ws.

As the list opposite shows, that number is 10.

EEEW	EWEE
EEWE	WEEW
EEWE	WEEWE
EWEE	WEWEE
EWEE	WEEEE

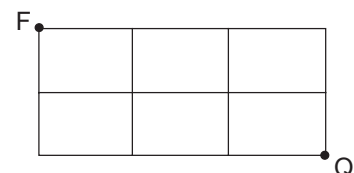
Streets problem

In this plan, how many shortest routes are there from F to Q?

Solution

F and Q are 5 blocks away. To get to Q by the shortest route you must go across (A) 3 and down (D) 2, in any order. The number of such routes is therefore the number of different combinations of 3 As and 2 Ds.

The answer is 10.



AAADD	ADDAA
AADAD	DAAAD
AADDA	DADAA
ADADA	DAADA
ADAAD	DDAAA

6.2

Binomial coefficients

In the expansion of $(a + x)^5$, what is the coefficient of the term a^3x^2 ?

Solution

$(a + x)^5$ is short for $(a + x)(a + x)(a + x)(a + x)(a + x)$.

To get a^3x^2 three brackets must supply an 'a' and two of them an 'x'. The coefficient will be the number of ways this can be done. A list of the possible ways is shown on the right.

<i>aaaxx</i>	<i>axxaa</i>
<i>aaxax</i>	<i>xaaxa</i>
<i>aaxxa</i>	<i>xaaxa</i>
<i>axaax</i>	<i>xaxaa</i>
<i>axaxa</i>	<i>xxaaa</i>

It should now be clear why three apparently unrelated problems give identical answers. The number of ways of arranging 3 As and 2 Ds is denoted

$$\binom{5}{3} \quad \text{or} \quad {}^5C_3 \quad (= 10)$$

The 'C' stands for COMBINATION.

To take another example.

$$\binom{7}{2} \quad \text{or} \quad {}^7C_2$$

means the number of different combinations of 7 objects, 2 of one type and 5 of another.

**Exercises**

- From what you have done in this section, write down the values of

(a) $\binom{4}{2}$	(b) $\binom{5}{1}$	(c) $\binom{5}{2}$
(d) $\binom{6}{3}$	(e) $\binom{3}{1}$	
- How many different ways are there of arranging the letters of the words BOB and ANNA? Express your answers in terms of combinations, i.e. $\binom{p}{q}$.

6.3 **Factorials**

First note that the notation $n! = (n - 1)(n - 2) \dots 3 \times 2 \times 1$ so that, for example,

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

We call this ' n factorial'.

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**Worked Example 1**

- (a) In the 8-letter word DOMINATE all the letters are different. Including the one given, how many arrangements are there of these letters?
- (b) The word NOMINATE also has 8 letters, but two of them are the same. How many arrangements are there now?
- (c) The word ADDITION has eight letters with two pairs of identical ones. How many different arrangements are there?
- (d) How many arrangements are there of the word CALCULUS?
- (e) The word DIVISION has one letter that appears three times. How many arrangements of these eight letters are there?

**Solution**

- (a) You have 8 choices for the first letter, 7 for the next one, etc, giving

$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8! \\ = 40\,320 \text{ arrangements.}$$

- (b) As two of the letters are the same, each arrangement will be repeated. Thus there are

$$\frac{8!}{2} = \frac{40\,320}{2} \\ = 20\,160 \text{ arrangements}$$

- (c) Now we have two pairs of letters repeated, so there are

$$\frac{8!}{2 \times 2} = 10\,080 \text{ arrangements}$$

- (d) Here 3 letters are repeated; the number of unique arrangements is

$$\frac{8!}{2 \times 2 \times 2} = 5040$$

- (e) There are 6 ($= 3!$) arrangements of three letters so we are left with

$$\frac{8!}{3!} = 6720 \text{ unique arrangements.}$$

In general, the number of **arrangements** of an n -letter word with one letter repeated p times is given by the formula

$$\frac{n!}{p!}$$

Here, $n!$, read as ' n factorial', is short for the product of all the natural numbers up to and including n ; so for example

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

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The number of different arrangements of letters in the word **SIMILAR** is therefore

$$\frac{7!}{2!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 2520$$

This rule can be extended to cover the repetition of more than one letter. The number of arrangements of letters in the word **SENSES** is

$$\frac{6!}{3!2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 60$$

This automatically provides a way of finding binomial coefficients.

$\binom{5}{3}$ is the number of arrangements of the letters AAADD

$$\text{so } \binom{5}{3} = \frac{5!}{2!3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 3 \times 2 \times 1} = \frac{5 \times 4}{2 \times 1} = 10$$

Similarly the binomial coefficient

$$\begin{aligned} \binom{10}{7} &= \frac{10!}{3!7!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1) \times (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)} \\ &= \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120 \end{aligned}$$

and

$$\binom{10}{3} = \frac{10!}{3!7!} = 120$$

In general,

$$\binom{n}{n-r} = \frac{n!}{(n-(n-r))!(n-r)!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Thus the general formula for binomial coefficients is given by

$$\boxed{\binom{n}{r} = \frac{n!}{(n-r)!r!}}$$

Your calculator probably has a factorial function on it. It may also have a function for working out binomial coefficients; consult your manual if you are not certain. In any case, it is often possible to work them out without a calculator; though the formula looks difficult there is always a good deal of cancelling that can be done, as the examples so far have shown.



Worked Example 2

How many ways are there of selecting a tennis team of 5 from a squad of 9 players?



Solution

Imagine all 9 players in a line, 5 of them are in the team: label them T; 4 are not: label them N. The answer to the question is the number of arrangements of 5 Ts and 4 Ns, since

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each different arrangement gives a different team.

$$\binom{9}{5} = \frac{9!}{4!5!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1)(5 \times 4 \times 3 \times 2 \times 1)} = 126$$



Exercises

- How many 'words' can be made from
 - 6 As and 3 Xs;
 - 7 Us and 5 Ws;
 - 10 Ys and 13 Zs?
- How many different ways are there of arranging the letters in these names?

(a) SADIA	(b) WILLIAM
(c) BARBARA	(d) CHRISTOPHER
- A committee of four is to be selected from a club of 25 people. How many different possibilities are there?
- A florist puts ten pot plants in the window. Six of the plants are red, four are green. How many different arrangements of the colours can be made?
- Eleven hockey players are to be selected from a squad of 16. How many possible selections are there?
- 12 people are to be split up into
 - group A - four people;
 - group B - six people;
 - group C - two people.
 How many different arrangements are there?

6.4 Binomial Theorem

First, there are some results concerning binomial coefficients which will be very useful later.



Worked Example 1

- Show that $\binom{n}{0} = 1$, $\binom{n}{1} = n$ and $\binom{n}{2} = \frac{n(n-1)}{2}$.
- Write down corresponding formulae for $\binom{n}{3}$ and $\binom{n}{4}$.
- What is $\binom{n}{r}$?



Solution

$$(a) \quad \binom{n}{0} = \frac{n!}{(n-0)!0!} = \frac{n!}{n!0!} = 1 \quad (\text{as } 0! = 1)$$

$$\binom{n}{1} = \frac{n!}{(n-1)!1!} = \frac{n(n-1)!}{(n-1)!1!} = \frac{n}{1!} = n$$

$$\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)(n-2)!}{(n-2)!2!} = \frac{n(n-1)}{2!} = \frac{n(n-1)}{2}$$

$$(b) \quad \binom{n}{3} = \frac{n(n-1)(n-2)}{3!}$$

$$\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{4!}$$

$$(c) \quad \binom{n}{r} = \frac{n(n-1)(n-2) \dots (n-r)}{r!}$$

The expansion of $(1+x)^5$ can now be written as follows :

$$\begin{aligned} 1 + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + x^5 \\ = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \end{aligned}$$

Similarly the expansion of $(a+x)^8$, where a is any number, can be found thus :

$$\begin{aligned} a^8 + \binom{8}{1}a^7x + \binom{8}{2}a^6x^2 + \binom{8}{3}a^5x^3 + \binom{8}{4}a^4x^4 + \binom{8}{5}a^3x^5 + \binom{8}{6}a^2x^6 + \binom{8}{7}ax^7 + x^8 \\ = a^8 + 8a^7x + 28a^6x^2 + 56a^5x^3 + 70a^4x^4 + 56a^3x^5 + 28a^2x^6 + 8ax^7 + x^8 \end{aligned}$$

The general result is known as the **binomial theorem** for a positive integer index.

For $a, x \in \mathbb{R}$ and $n \in \mathbb{N}$

$$(a+x)^n = a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \dots + \binom{n}{r}a^{n-r}x^r + \dots + x^n$$

Note the way this result is set out. Since n is unknown it is impossible to show the whole expansion. So the formula shows how the expansion starts and finishes, and gives a formula for the general term as

$$\binom{n}{r}a^{n-r}x^r$$

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One use of the binomial theorem is to give rough approximations to numbers like $(1.01)^8$. You can write

$$1.01 = 1 + 0.01$$

so that

$$\begin{aligned}(1.01)^8 &= 1 + \binom{8}{1}(0.01) + \binom{8}{2}(0.01)^2 + \binom{8}{3}(0.01)^3 + \dots \\ &= 1 + 8 \times 0.01 + 28 \times 0.0001 + 56 \times 0.000001 + \dots\end{aligned}$$

If 3 s.f. accuracy is required, clearly only the first three terms are needed at most:

$$(1.01)^8 = 1.0828\dots = 1.08 \text{ to 3 s.f.}$$

The first four terms give an accuracy of at least 6 s.f.

$$\begin{aligned}(1.01)^8 &= 1 + 0.08 + 0.0028 + 0.000056 \\ &= 1.08286 \text{ to 6 s.f.}\end{aligned}$$

And all this without a calculator!

One of the really important uses of the Binomial Theorem is with probability where we develop what is called the Binomial Distribution (Probability, Unit 5).



Worked Example 2

Expand $(2x + 3y)^4$.



Solution

$$\begin{aligned}(2x + 3y)^4 &= (2x)^4 + \binom{4}{1}(2x)^3(3y) + \binom{4}{2}(2x)^2(3y)^2 + \binom{4}{3}(2x)(3y)^3 + (3y)^4 \\ &= 16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4\end{aligned}$$



Worked Example 3

Expand $(3 - \frac{1}{2}p)^5$.



Solution

$$\begin{aligned}(3 - \frac{1}{2}p)^5 &= 3^5 + \binom{5}{1}3^4(-\frac{1}{2}p) + \binom{5}{2}3^3(-\frac{1}{2}p)^2 + \binom{5}{3}3^2(-\frac{1}{2}p)^3 + \binom{5}{4}3(-\frac{1}{2}p)^4 + (-\frac{1}{2}p)^5 \\ &= 243 - \frac{405}{2}p + \frac{135}{2}p^2 - \frac{45}{4}p^3 + \frac{15}{16}p^4 - \frac{1}{32}p^5\end{aligned}$$



Worked Example 4

Find the coefficient of x^5 in the expansion of $(4 - 3x)^9$.

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**Solution**

$$\begin{aligned}
 \text{The } x^5 \text{ term must be } & \binom{9}{5} 4^4 (-3x)^5 \\
 &= 126 \times 256 \times (-243)x^5 \\
 &= -7\,838\,208x^5
 \end{aligned}$$

So the coefficient of x^5 is $-7\,838\,208$.

**Exercises**

1. Expand in full :

$$\begin{array}{lll}
 \text{(a)} & (1+2x)^4 & \text{(b)} \quad (5-2p)^3 \quad \text{(c)} \quad (6-\frac{1}{2}a)^6 \\
 \text{(d)} & (2m+3n)^5 & \text{(e)} \quad (2-\frac{1}{2}r)^4
 \end{array}$$

2. Find the coefficients of

$$\begin{array}{lll}
 \text{(a)} & x^3 \text{ in } (2+x)^{10} & \text{(b)} \quad a^5 \text{ in } (5-2a)^7 \quad \text{(c)} \quad q^6 \text{ in } (2p+5q)^9 \\
 \text{(d)} & k^4 \text{ in } (3k-2l)^{12} & \text{(e)} \quad n^7 \text{ in } (5-\frac{5}{6}n)^{10}
 \end{array}$$

3. Find these to 3 s.f. without using a calculator.

$$\begin{array}{lll}
 \text{(a)} & (1.01)^5 & \text{(b)} \quad (1.02)^{10} \quad \text{(c)} \quad (0.99)^7 \\
 \text{(d)} & (2.01)^3 & \text{(e)} \quad (99.5)^4 \quad \text{(f)} \quad 11^6
 \end{array}$$