

STRAND: DISCRETE MATHS

Unit 1 *Algorithmic Approach*

TEXT

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1 Algorithmic Approach

1.1 Examples of Algorithms

Algorithms play an important part in mathematics problem solving. We define an algorithm as an iterative (repeating a process) set of procedures that uniquely solves a certain type of problem.

An algorithm to solve a certain class of problem will be characterised by having

- input data to start the algorithm
- iterative procedures that are uniquely defined
- output data which is unique and dependent on the input data.

A simple example is the procedure for solving quadratic equations of the form

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

Here

- input data is a, b, c
- the procedure is to calculate

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- the output is dependent on whether $b^2 > 4ac$ and is summarised by

if $b^2 > 4ac$, the output is x_1 and x_2

if $b^2 = 4ac$, the output is $x_1 \left(= x_2 = \frac{-b}{2a} \right)$

if $b^2 < 4ac$, there is no output.

So the output is precisely defined by the input data and the formula, giving a unique solution for specified values of $a, (a \neq 0), b$ and c .

The algorithm described above is a rather special example as it is not iterative, that is, the formula is applied only once.

The next example is based on finding an approximation to $\sqrt{2}$ (it is known as *Hero's method*).



Worked Example 1

- (a) Show that $x^2 = 2$ can be rewritten as

$$x = \frac{1}{2} \left(x + \frac{2}{x} \right)$$

1.1

- (b) Use the iterative formula

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{2}{x_n} \right)$$

for $n = 0, 1, 2, \dots$ to find $\sqrt{2}$ to 4 decimal places, given that $x_0 = 1$.

**Solution**

- (a) If
- $x^2 = 2$
- , then
- $x \times x = 2$
- or
- $x = \frac{2}{x}$
- .

Add x to both sides of the equation, so

$$2x = x + \frac{2}{x}$$

or

$$x = \frac{1}{2} \left(x + \frac{2}{x} \right)$$

- (b) You can see the derivation of the iterative formula by noting that if

$x_{n+1} = x_n$ ($= x$), then

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{2}{x_n} \right) \Rightarrow x = \frac{1}{2} \left(x + \frac{2}{x} \right)$$

which is the equation in part (a). So if this iterative formula converges as n increases, it must converge to $\sqrt{2}$.

n	x_n
0	1
1	$x_1 = \frac{1}{2} \left(1 + \frac{2}{1} \right) = 1.5$
2	$x_2 = \frac{1}{2} \left(1.5 + \frac{2}{1.5} \right) = 1.41667$
3	$x_3 = \frac{1}{2} \left(1.41667 + \frac{2}{1.41667} \right) = 1.41422$
4	$x_4 = \frac{1}{2} \left(1.41422 + \frac{2}{1.41422} \right) = 1.41421$

It is clear that the sequence $x_1, x_2, x_3, \dots$ converges, and to 4 decimal places, the value is

$$x = 1.4142$$

(Check, using your calculator: $\sqrt{2} = 1.414213562 \dots$)

1.1

We complete this section with a rather different problem, namely that of determining the date of Easter Sunday.

This depends on both the Gregorian calendar (in which there is a leap year every four years, except in century years where only one in every four is a leap year) and the Jewish calendar (which follows a solar cycle for years and a lunar cycle for months - it has a 19-year cycle in which the 3rd, 6th, 8th, 11th, 14th, 17th and 19th years have a thirteenth month).

In the Bible (Luke, Ch. 22) there is a description of how Jesus ate the Passover supper with his disciples before going to Gethsemane. The Last Supper, the Agony in the Garden and the Crucifixion were at the time of the full moon following March 21st (Spring Equinox). The hurry to remove the body from the cross before the beginning of the Jewish Sabbath (Saturday) fixes the Crucifixion on a Friday.

The Church has sought to preserve this sequence. In 1960, T H O'Beirne published in the New Scientist an algorithm, called 'Ten Divisions to Easter' for finding the date of Easter Sunday for any year.

Ten Divisions to Easter

Step	Number	Divide by	Answer	Remainder (if needed)
1.	$x = \text{year}$	100	$b =$	$c =$
2.	$5b + c$	19	-	$a =$
3.	$3(b + 25)$	4	$r =$	$s =$
4.	$8(b + 11)$	25	$t =$	-
5.	$19a + r - t$	30	-	$h =$
6.	$a + 11h$	319	$g =$	-
7.	$60(5 - s) + c$	4	$j =$	$k =$
8.	$2j - k - h + g$	7	-	$m =$
9.	$h - g + m + 110$	30	$n =$	$q =$
10.	$q + 5 - n$	32	-	$p =$

Where a dash (-) is shown, it means that the number is not needed.

Easter Sunday is on the p th day of the n th month in the year x .

1.1



Worked Example 2

Use the algorithm 'Ten Divisions to Easter' to find the date of Easter Sunday, 2015.



Solution

Ten Divisions to Easter

Step	Number	Divide by	Answer	Remainder (if needed)
1.	$x = \text{year} = 2015$	100	$b = 20$	$c = 15$
2.	$5b + c = 115$	19	(6)	$a = 1$
3.	$3(b + 25) = 135$	4	$r = 33$	$s = 3$
4.	$8(b + 11) = 248$	25	$t = 9$	(23)
5.	$19a + r - t = 43$	30	(1)	$h = 13$
6.	$a + 11h = 144$	319	$g = 0$	(144)
7.	$60(5 - s) + c = 135$	4	$j = 33$	$k = 3$
8.	$2j - k - h + g = 50$	7	(7)	$m = 1$
9.	$h - g + m + 110 = 124$	30	$n = 4$	$q = 4$
10.	$q + 5 - n = 5$	32	(0)	$p = 5$

Thus Easter Sunday in 2015 is 5th April.



Exercises

1. Solve the quadratic equation

$$x^2 + kx + 1 = 0$$

How many distinct solutions are there when

- (a) $k = 1$, (b) $k = -1$?
2. How many iterations are needed to find $\sqrt{2}$ to 4 decimal places by Hero's method when
- (a) $x_0 = 1.4$ (b) $x_0 = 10$?

1.1

3. Use a similar algorithm for finding $\sqrt{3}$. Check its accuracy.
4. Use the algorithm 'Ten Divisions of Easter' to find the date of Easter Sunday in 2016.
5. Use a spreadsheet approach (or computer program) to find the date of Easter Sunday for the next 100 years.
6. What are the earliest and latest possible dates for Easter Sunday?

1.2 Using Algorithms for Problem Solving

Here two contrasting problems are considered to show how an algorithmic approach can be developed to find an optimal solution.

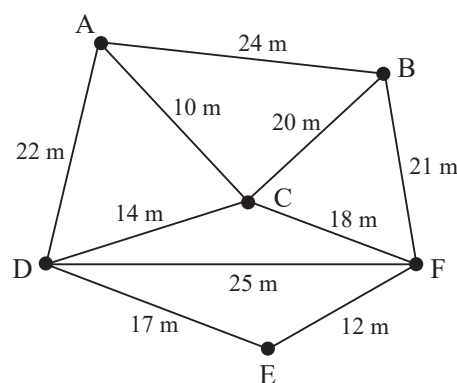
Minimum connector

This is the problem of finding the shortest total distance (or time, etc.) that connects a number of nodes (vertices) together in a network, given the individual distances (times) between them. A typical case would be in finding the minimum connector for cable TV. The following simple example illustrates this class of problem



Worked Example 1

Gary and Alison plan to hang lights between trees in their large garden for a children's party. They have 75 metres of cable with lights. Will this be enough to connect, directly or indirectly, the five trees (nodes) at B, C, D, E and F and the power supply at node A?



Solution

You can use 'guesswork' to find a possible solution but a more intelligent method is to develop an algorithmic approach, for example, with the trees, power supply and lengths represented by letters/numbers on a network,

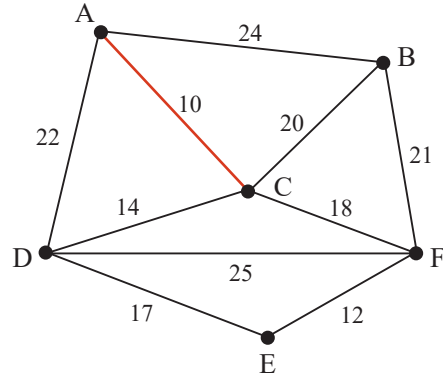
1.2

Prim's algorithm:

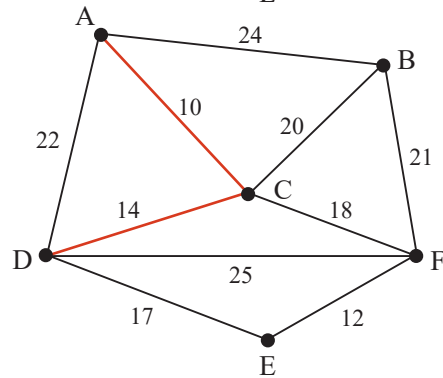
Start with the smallest number from any node and keep joining in the numbers between more nodes that are not already in the solution.

This is how it works.

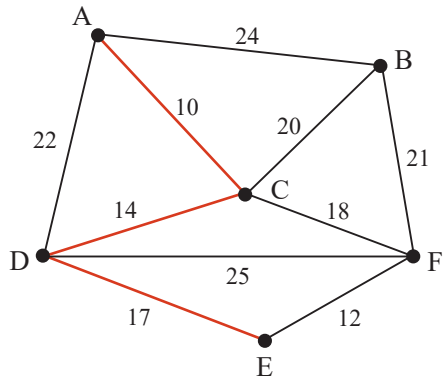
1. If we start with node A, the smallest number from here is $AC = 10$ so this is in the solution.



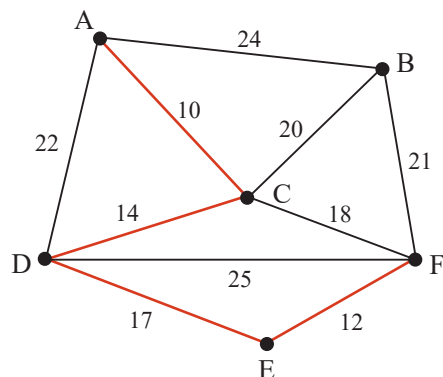
2. The next smallest number (from A or C) to any other node in the solution is $DC = 14$.



3. The next smallest number to be included is $DE = 17$.

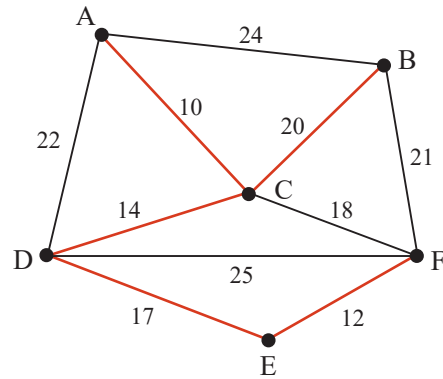


4. Now $EF = 12$ is added.



1.2

5. $CB = 20$ is added. All nodes are now included in the solution:
 $10 + 14 + 17 + 12 + 20 = 73$



All the nodes are now connected, directly or indirectly, to A. (We could have chosen any of the nodes to start the process, as they are all included in the solution.)

This is the optimal solution but not the only way to achieve it, as shown in Worked Example 2.



Worked Example 2

Solve the problem in Worked Example 1 using

Kruskal's algorithm:

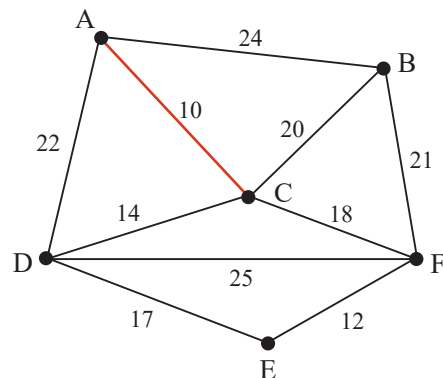
Find the smallest number between any 2 nodes and keep adding into the solution the next smallest number, provided it is not redundant (already included) in the solution.

(Kruskal's algorithm is sometimes referred to as the 'greedy algorithm'.)

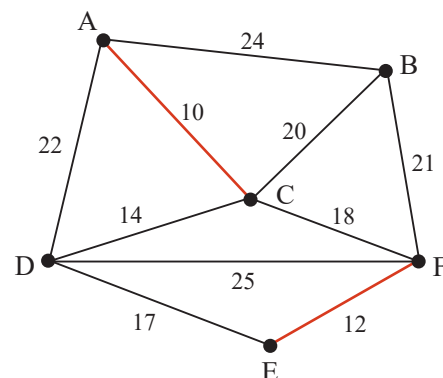


Solution

1. The smallest number is $AC = 10$ so this is in the solution.

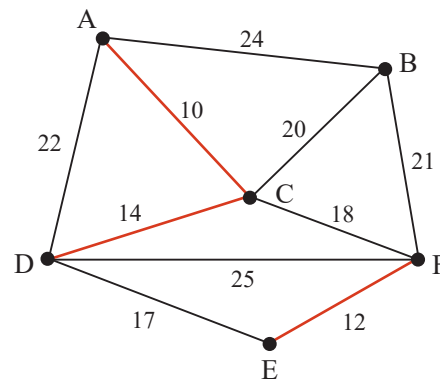


2. Looking at the network, the next smallest number is $EF = 12$, so this is included in the solution.

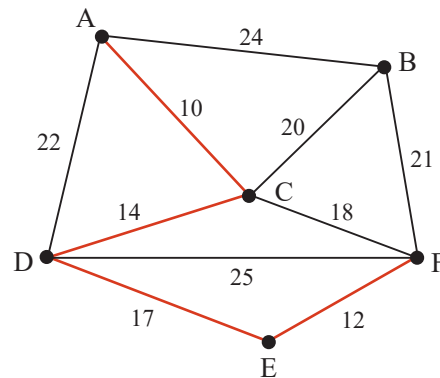


1.2

3. Now $CD = 14$ is added to the solution.



4. The next smallest number is $DE = 17$, so this is added to the solution.

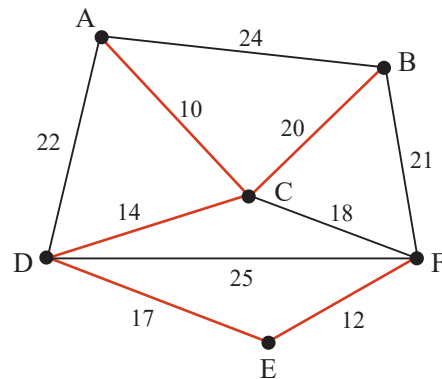


5. The next smallest is $CF = 18$ but this is redundant as both C and F are already in the solution, so this number is discarded.

6. The next smallest is $CB = 20$ and this enters the solution, which is now complete.

The total length is

$$10 + 12 + 14 + 17 + 20 = 73$$



All the nodes (trees) are now connected, directly or indirectly, to node A (the power point) so the 75 metre length of cable with lights will be sufficient to decorate the garden.



Note

This is an identical result to that given in the previous Worked Example; in fact both Prim's and Kruskal's algorithms produce the optimum solution.

1.2

Bin packing

Packing is an everyday event for many people and it is usually a relatively easy activity. However, when preparing for travel on low-cost airlines using only hand luggage, packing can be problematic. For example, Wizz Air's free hand luggage allowance is one piece of baggage measuring not more than

$$42 \text{ cm} \times 32 \text{ cm} \times 35 \text{ cm}$$

This is often strictly enforced, with a substantial charge for bags exceeding these dimensions.

In business and commerce, packing problems abound and often require mathematical solutions rather than just common sense. For example, efficient packing of cargo containers that are carried between ports by ship, and then transferred to rail or road for final delivery, is crucial. This is a 3-dimensional bin packing problem but for simplicity, we will consider here one-dimensional problems.

**Worked Example 3**

Estimate the minimum number of bins of height 1.6 m needed to pack the following items:

Item	A	B	C	D	E	F	G	H	I	J	K	L
Height (m)	0.5	0.7	1.0	0.8	0.9	1.1	0.7	0.4	0.2	0.6	0.8	0.3

**Solution**

To obtain an estimate of the number of bins needed we add up the total item heights and divide by the height of the bin, in this case 1.6 m. So

$$\begin{aligned} \text{total height} \\ &= 8 \text{ m} \end{aligned}$$

giving an estimate of

$$\text{no. of bins} \approx \frac{8}{1.6} = 5$$

**Note**

So our estimate is for 5 bins but this does not show how we can actually achieve this.

**Worked Example 4**

Use the data from Worked Example 3 to develop a systematic method for finding the number of bins needed.

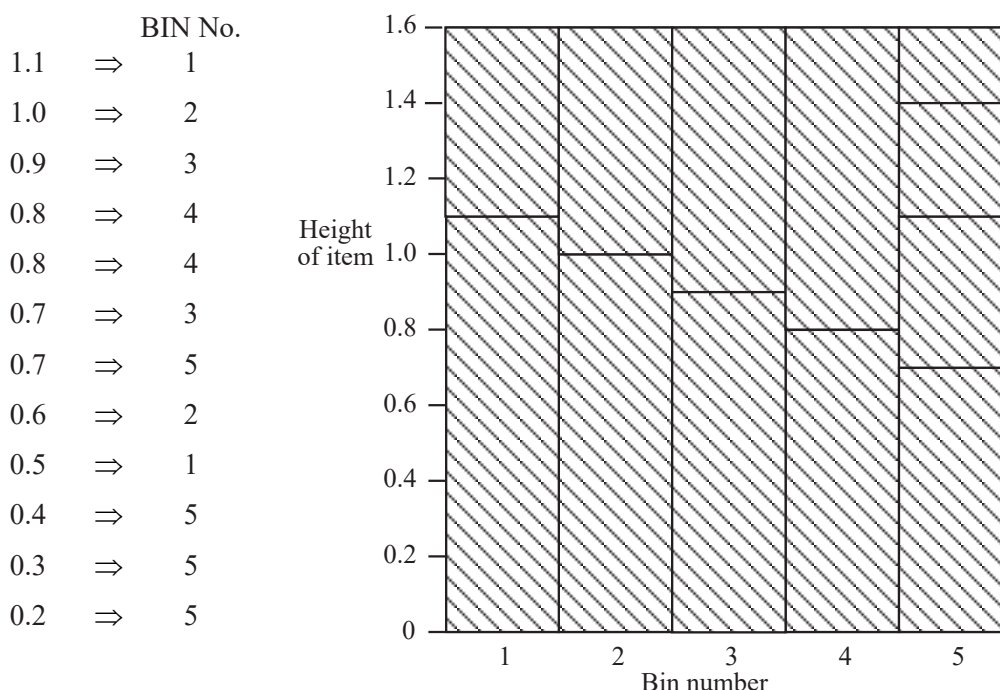
**Solution**

Clearly, the largest items will be the key ones for efficient packing, so we will reorder them, starting with the largest. This gives

$$1.1, 1.0, 0.9, 0.8, 0.8, 0.7, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2$$

The method we use is to pack each item, in order, into the first available bin.

1.2

**Note**

This method is called the 'first-fit decreasing' method but it does not always provide the optimum solution, as is shown in the next example.

**Worked Example 5**

A builder has metal piping in a standard length of 12 metres. He requires the following lengths for various tasks.

For task	A	B	C	D	E	F	G	H	I	J	K	L
Pipe length (m)	2	2	3	3	3	3	4	4	4	6	7	7

Use the first-fit decreasing method to determine how many 12 m lengths of piping are needed.

Is this the optimal solution?

**Solution**

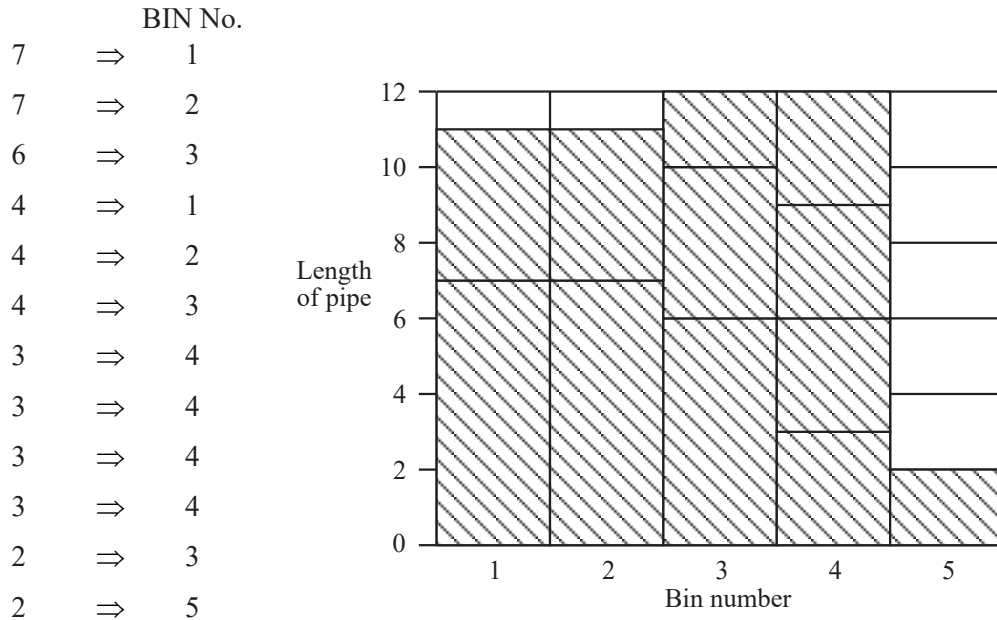
Before starting on the method, we can obtain an estimate (lower bound) for the number of 12 m lengths of piping needed by calculating

$$\begin{aligned}
 \frac{\text{total length}}{12} &= \frac{(2 + 2 + 3 + 3 + 3 + 3 + 4 + 4 + 4 + 6 + 7 + 7)}{12} \\
 &= \frac{48}{12} \\
 &= 4
 \end{aligned}$$

1.2

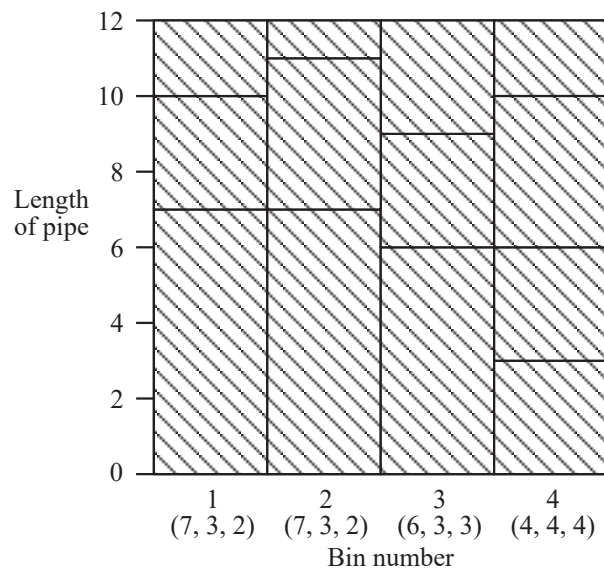
This does not imply that it can be achieved with just 4 lengths but that the number needed is ≥ 4 .

Using the first-fit decreasing method, we obtain the following:



So, using this method, 5 lengths of 12 m pipe are needed.

We know, though, that it might be feasible to obtain a more efficient solution. This can be achieved by looking for 'full' bins as shown in the following diagram.



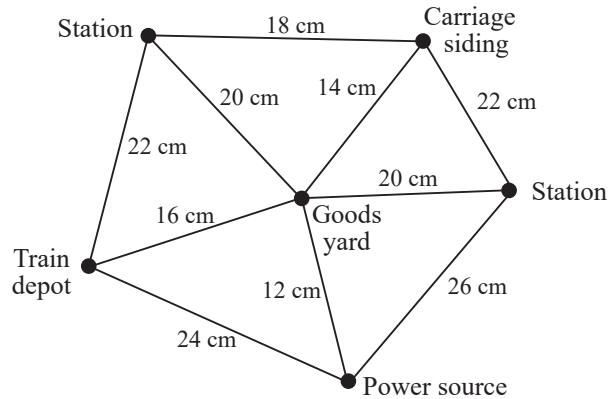
Note

Whilst it is not difficult to find the optimal solution, the method we have used, namely first-fit decreasing, is not technically an algorithm, as it does not always produce the optimum solution.

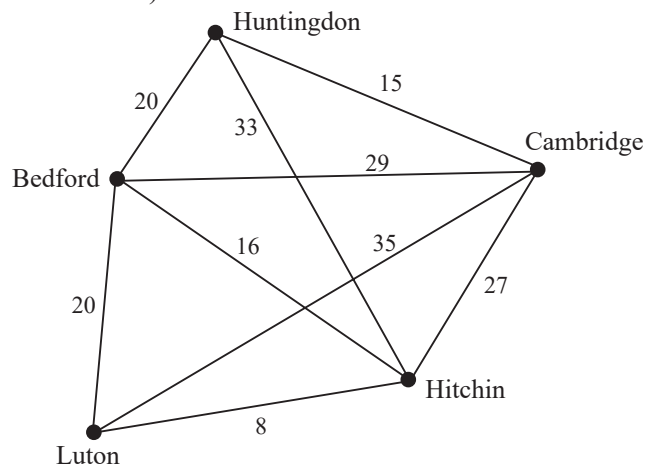


Exercises

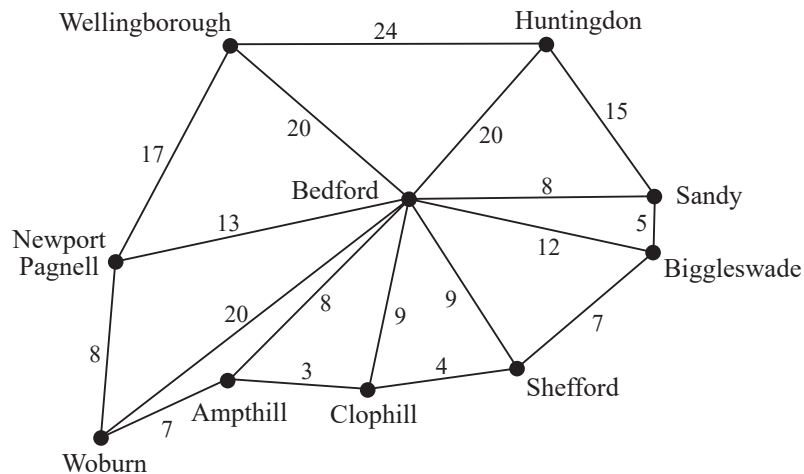
1. Shebena has a new model railway set for Christmas and the layout includes two stations, goods yard, train depot and carriage siding, fixed on a board. She wants to light them all so they need to be connected to a power source. The distances are shown in the diagram. Can Shebena wire up the system with 80 cm of wire?



2. Find the shortest route connecting all these towns using only the roads shown. (Distances are in miles.)

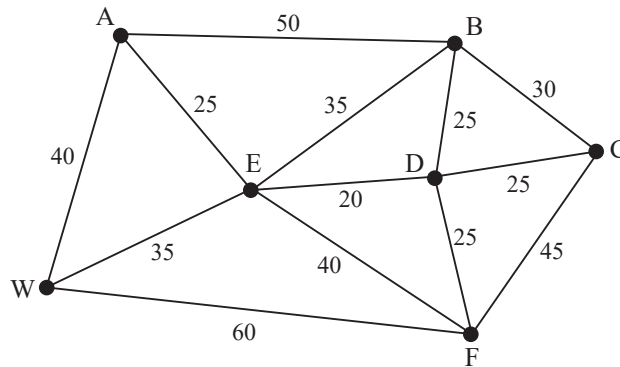


3. Cable TV is being piped from Bedford to the towns shown below. Find the shortest length of cable needed to connect all the towns to the system.

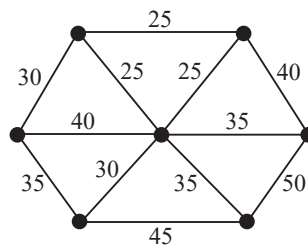


1.2

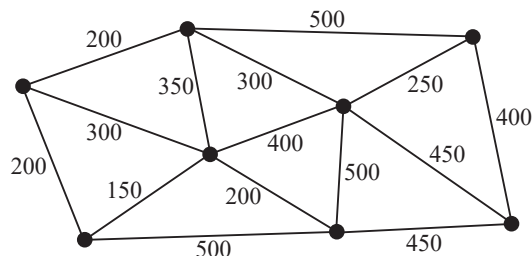
4. The owner of a caravan site has caravans positioned as shown on the diagram, with distances in metres between them. He wants to lay on a water supply to each of the caravans. Use Kruskal's algorithm to determine how the caravans should be connected so that the total length of pipe required is a minimum.



5. The warden of an outdoor studies centre wants to set up a public address system linking all the huts. The distances in metres between the huts are shown in the diagram. How should the huts be linked to minimise the total distance?



6. A group of friends wants to set up a message system so that any one of them can communicate with any of the others either directly or via others in the group. If their homes are situated as shown below, with distances in metres between them as marked, use Prim's algorithm to decide where they should make the links so that the total length of the system will be as small as possible.



1.2

7. A building society based in Bristol with offices throughout the region wants to link its branches in a private computer network. The distances in miles between the branches are as shown in the table.

Find a way of connecting the offices so that the total length of cable required is a minimum.

	BA	BR	CS	CL	KE	KI	LU	PA	PO	RA	WE
Bath		13	12	23	7	9	16	16	22	8	29
Bristol			11	12	6	5	8	7	9	14	21
Chipping Sodbury				23	11	7	19	8	19	20	32
Clevedon					17	17	8	16	5	22	8
Keynsham						4	10	11	15	9	23
Kingswood							13	7	14	13	26
Lulsgate								15	8	14	13
Patchway									11	20	24
Portishead										21	13
Radstock											24
Weston-super-Mare											

8. A project consists of eight activities whose durations are as follows. There are no precedence relations (i.e. the activities can be carried out in any order).

Activity	A	B	C	D	E	F	G	H
Duration (hours)	1	2	3	4	4	3	2	1

It is required to find the minimum number of workers needed to finish the project in 5 hours.

Find the answers to this problem given by

- (a) the first-fit packing method;
 (b) the first-fit decreasing method.
9. A plumber uses pipes of standard length 10 m and wishes to cut out the following lengths:

Length (m)	10	9	8	7	6	5	4	3	2	1
Number	0	0	2	3	1	1	0	2	3	0

Use the first-fit decreasing method to find how many standard lengths are needed to meet his order. Does this method give an optimum solution? If not, find an optimum solution.

1.2

10. Determine the minimum number of sheets of metal required, 10 m by 10 m, to meet the following order, and how they should be cut. (Assume no wastage in cutting.)

Size	Number
$3 \times 1 \text{ m}^2$	60
$4 \times 2 \text{ m}^2$	49
$7 \times 5 \text{ m}^2$	12

Develop a **general** method of solving 2-dimensional packing problems of this type.

1.3 Summary

Algorithms are an increasingly important branch of mathematics. Their power is that they can solve not just one particular problem but a whole class of problems; that is, the data can change but the method remains the same.

In an ever-increasingly technological world, where we can use computer technology to solve ever more complex problems, the use of algorithms is crucial.

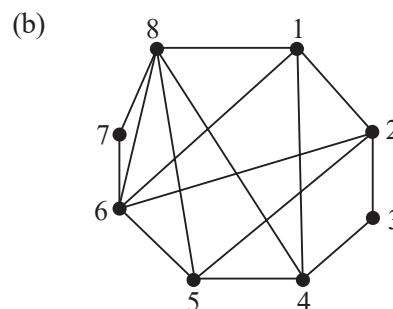
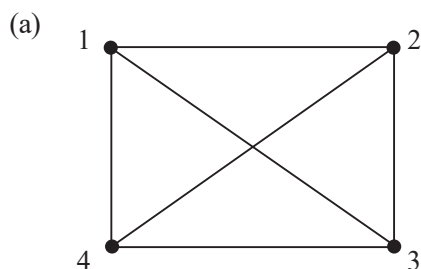
Note though the difference between an 'algorithm' and a 'method'. By definition, an algorithm will produce an optimal solution (if one exists) whereas a method will produce a solution, but not necessarily the best one. This is well illustrated in the bin packing problem in Section A2.2. The 'first-fit decreasing' method is a method, not an algorithm as it does not guarantee giving the best solution, that is, a solution with the minimum number of bins. This is still an unsolved problem but its solution for 2- and particularly 3-dimensional problems is one that is very important for efficient packing of 3-dimensional storage containers.

We complete this section with one additional example, the development of an algorithm to determine what is called a *planar graph*. Here a graph is not of the usual form $y = f(x)$ but is a network of nodes (vertices) and the edges (connections) between them. We have seen examples of these with the minimum connector problem in Section 1.1 above.



Worked Example 1

Redraw the networks below so that there are no crossovers.

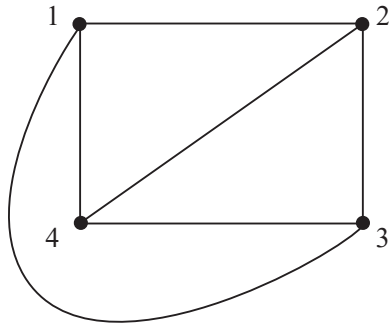


1.3



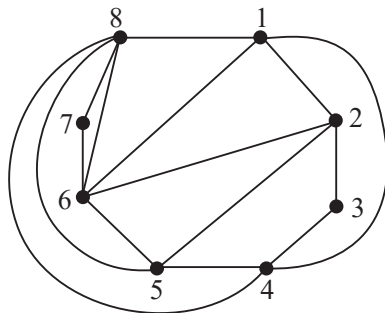
Solution

- (a) This is straightforward; you can, for example, move the 1-3 edge outside the rectangle to give the graph shown below.



This is now known as a *planar graph*.

- (b) This is not quite so obvious but you can put edges 8-5, 8-4 and 1-4 outside the octagon to obtain this graph, which is now planar.

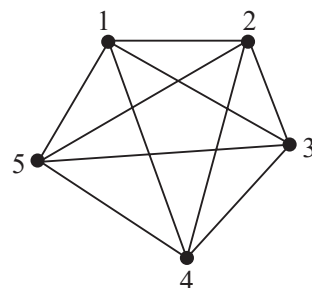


Of course, not all graphs are planar, as shown in the next example.



Worked Example 2

Show that this graph is non-planar.



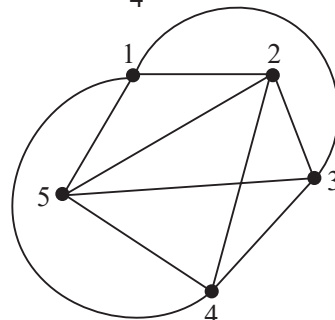
Solution

If you leave 2-5 inside the pentagon, then 1-3 and 1-4 must go outside (see opposite).

As arcs 2-4 and 3-5 cross, one of these will have to go outside the pentagon shape.

In either case the arc will cross other arcs.

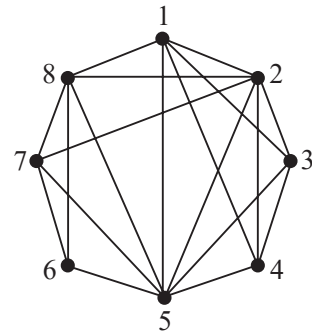
This is an example of a *non-planar graph*.



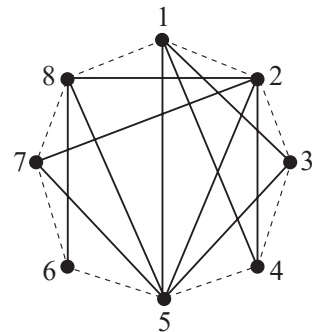
1.3

This idea of an arc being inside or outside provides a way of developing a planarity algorithm but it works only if the graph has a Hamiltonian cycle; that is, where there is a cycle which includes every vertex of the graph. The algorithm will either redraw the graph as a planar graph or show that it is non-planar.

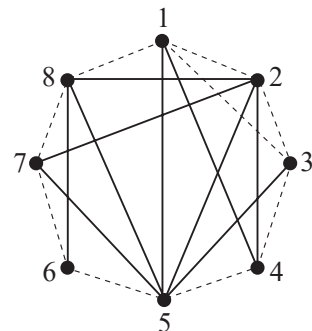
We'll look at how the algorithm works for this graph.



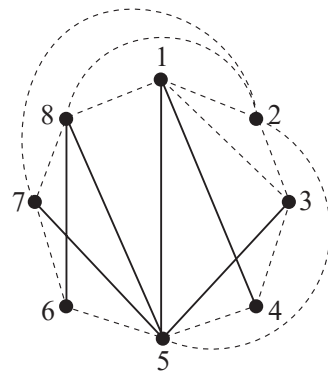
We first put the cycle (1-2-3-4-5-6-7-8-1) into the solution, shown by the dashed line on the graph.



The next stage is to choose any edge, say 1-3, and decide whether this is to go inside or outside. Let's choose inside, as illustrated in this graph.

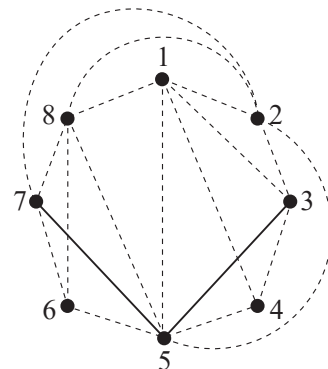


Since 1-3 crosses 2-8, 2-7 and 2-5, all these edges must go outside, as shown.



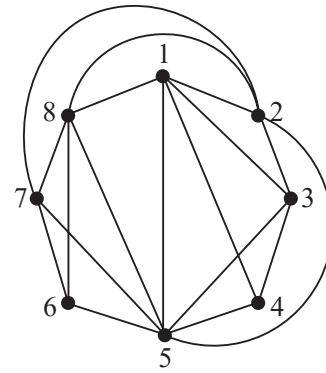
Originally edge 2-7 crossed 1-4, 1-5, 8-5 and 8-6, so all these edges must now remain inside (or they would cross 2-7 outside).

Finally, because 1-4 stays inside, 3-5 must go outside, and since 8-6 stays inside, 7-5 must go outside, as shown.



1.3

This is now a planar graph, as shown opposite, where the dotted lines have been redrawn as solid lines.

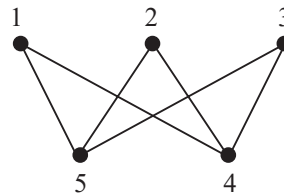
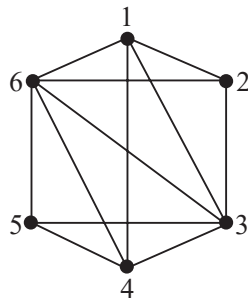


This is a very different sort of algorithm and it has been shown to be useful in a modern-day application, that is, the efficient design of electrical networks in silicon chips, where crossovers need to be avoided.

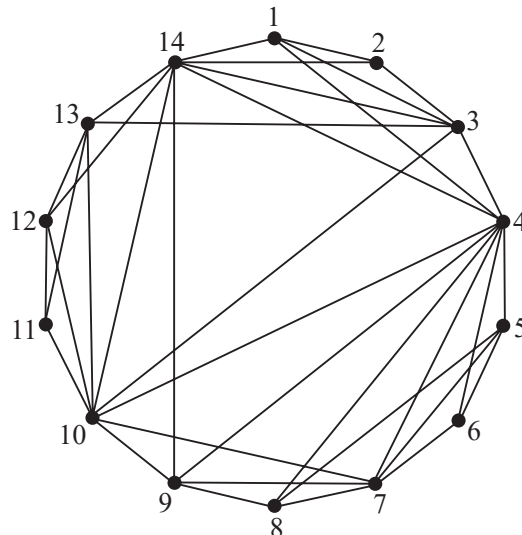


Exercises

- Is it possible to make plane drawings of the following two graphs?

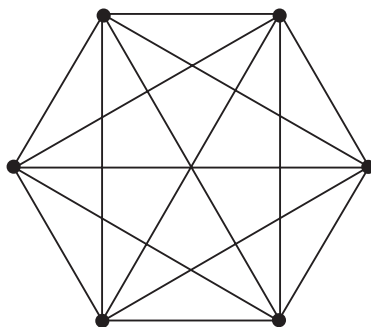


- Use the planarity algorithm to find a plane drawing of the graph below.



1.3

3. Use the planarity algorithm to show that this graph is non-planar.



4. The graph below represents connections in an electrical circuit. Use a planarity algorithm to decide whether or not it is possible to redraw the connections so that the graph is planar.

