

STRAND: DISCRETE MATHS

Unit 2 *Critical Path Analysis*

TEXT

Contents

Section

- | | |
|-----|-------------------|
| 2.1 | Activity Networks |
| 2.2 | Critical Path |
| 2.3 | Scheduling |

2 Critical Path Analysis

2.1 Activity Networks

In order to be able to use critical path analysis (CPA), you first need to be able to form what is called an **activity network**. This is essentially a way of illustrating the given project data concerning the tasks to be completed, how long each task takes and the constraints on the order in which the tasks are to be completed. As an example, consider the activities shown below for the construction of a garage.

Activity	Duration (in days)
A prepare foundations	7
B make and position door frame	2
C lay drains, floor base and screed	15
D install services and fittings	8
E erect walls	10
F plaster ceiling	2
G erect roof	5
H install door and windows	8
I fit gutters and pipes	2
J paint outside	3

Clearly, some of these activities cannot be started until other activities have been completed. For example

activity G - erect roof

cannot begin until

activity E - erect walls

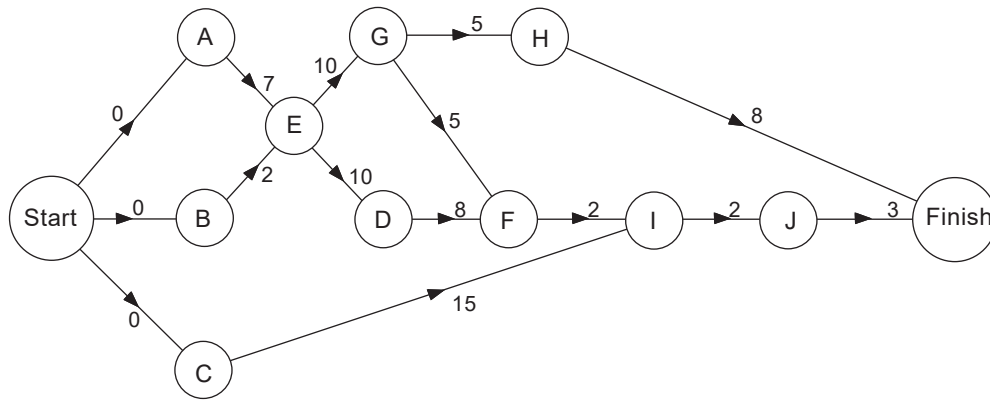
has been completed. The following table shows which activities must precede which.

D	must follow	E
E	must follow	A and B
F	must follow	D and G
G	must follow	E
H	must follow	G
I	must follow	C and F
J	must follow	I

We call these the **precedence relations**.

All this information can be represented by the following network.

2.1



In this network

each **activity** is represented by a **vertex**;

joining vertex X to vertex Y shows that activity X must be **completed** before activity Y can be started;

the **number** marked on each arc shows the **duration** of the activity from which the arc starts.

Note the use of 'arc' here to mean a directed edge. Sometimes we can easily form the activity network, but not always, so we need to have a formal method. First try the following activity.



Worked Example 1

A furniture maker is going to produce a new wooden framed settee with cloth-covered foam cushions. These are the tasks that have to be done by the furniture maker and his assistants and the times they will take:

Activity	Time in days
A make wooden arms and legs	3
B make wooden back	1
C make wooden base	2
D cut foam for back and base	1
E make covers	3
F fit covers	1
G put everything together	1

Each activity can only be undertaken by one individual.

The following list gives the order in which the jobs must be done:

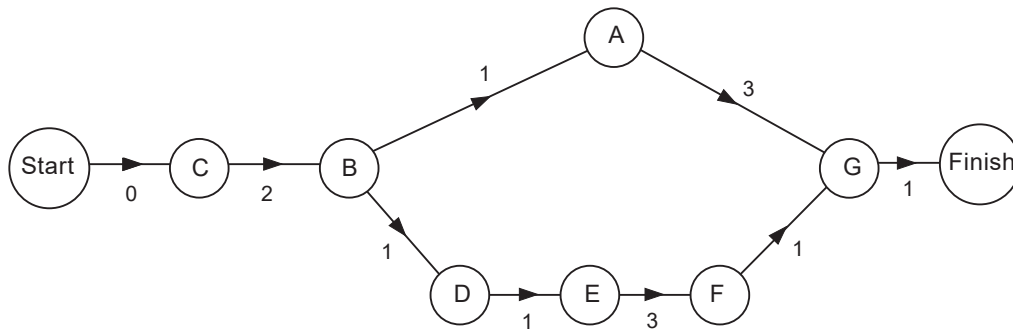
B	must be after	C
A	must be after	B and C
D	must be after	B and C
E	must be after	D
F	must be after	E
G	must be after	A, B, C, D, E and F

Construct an appropriate activity network to illustrate this information.

2.1



Solution



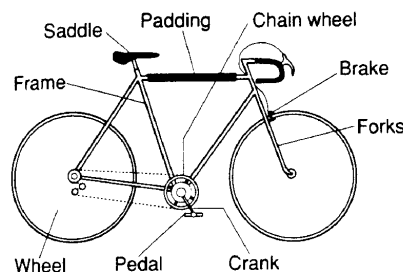
Exercises

- Suppose you want to redecorate a room and put in new self-assembly units. These are the jobs that need to be done, together with the time each takes:

Activity	Time (in hrs)	Preceded by
paint woodwork (A)	8	-
assemble units (B)	4	-
fit carpet (C)	5	hang wallpaper paint woodwork
hang wallpaper (D)	12	paint woodwork
hang curtains (E)	2	hang wallpaper paint woodwork

Complete an activity network for this problem.

- The Spodleigh Bicycle Company is getting its assembly section ready for putting together as many bicycles as possible for the Christmas market. This diagram shows the basic components of a bicycle.



Putting together a bicycle is split up into small jobs which can be done by different people.

2.1

The activities are:

	Activity	Time (mins)
A	preparation of the frame	9
B	mounting and aligning the front wheel	7
C	mounting and aligning the back wheel	7
D	attaching the chain wheel to the crank	2
E	attaching the chain wheel and crank to the frame	2
F	mounting the right pedal	8
G	mounting the left pedal	7
H	final attachments such as saddle, chain, stickers	21

The following chart shows the order of doing the jobs.

B	must be after	A
C	must be after	A
D	must be after	A
E	must be after	D
F	must be after	D and E
G	must be after	D and E
H	must be after	A, B, C, D, E, F and G

Draw an activity network to show this information.

3. An extension is to be built to a sports hall. Details of the activities are given below.

	Activity	Time (mins)
A	lay foundations	7
B	build walls	10
C	lay drains and floor	15
D	install fittings	8
E	make and fit door frames	2
F	erect roof	5
G	plaster ceiling	2
H	fit and paint doors and windows	8
I	fit gutters and pipes	2
J	paint outside	3

Some of these activities cannot be started until others have been completed.

2.1

The following chart shows the order of doing the jobs.

B	must be after	C
C	must be after	A
D	must be after	B
E	must be after	C
F	must be after	D and E
G	must be after	F
H	must be after	G
I	must be after	F
J	must be after	H

Draw an activity network to show this information.

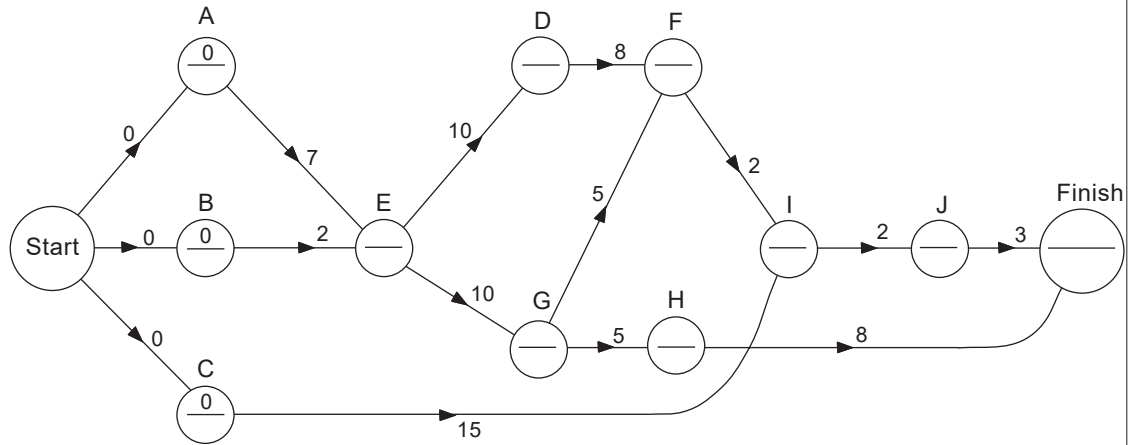
2.2 Critical Path

You have seen how to construct an activity network. In this section you will see how this can be used to find the **critical path**. This will first involve finding the **earliest** possible start for each activity, by going **forwards** through the network. Secondly, the **latest** possible start time for each activity is found by going **backwards** through the network. Activities which have **equal** earliest and latest start time are on the **critical path**. The technique will be illustrated using the 'garage construction' problem in Section 2.1.

Activity		Time in days
A	prepare foundations	7
B	make and position door frame	2
C	lay drains, floor base and screed	15
D	install services and fittings	8
E	erect walls	10
F	plaster ceiling	2
G	erect roof	5
H	install door and windows	8
I	fit gutters and pipes	2
J	paint outside	3

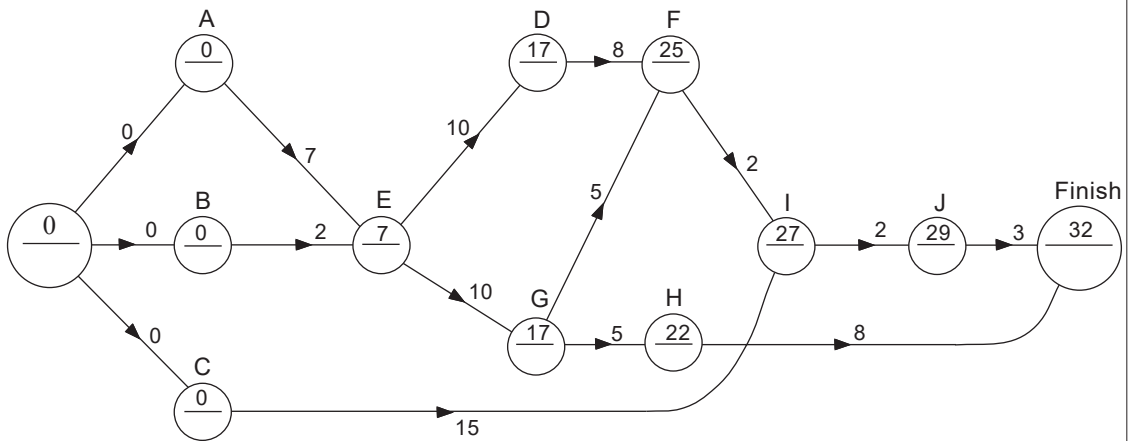
The activity network for this problem is shown below, where sufficient space is made at each activity node to insert two numbers.

2.2



The numbers in the top half of each circle will indicate the earliest possible starting time. So, for activities A, B and C, the number zero is inserted.

Moving forward through the network, the activity E is reached next. Since both A and B have to be completed before E can be started, the earliest start time for E is 7. This is put into the top half of the circle at E. The earliest times at D and G are then both 17, and for H, 22. Since F cannot be started until both D and G are completed, its earliest start time is 25, and consequently, 27 for I. The earliest start time for J is then 29, which gives an earliest completion time of 32.



Since 32 is the earliest possible completion time, it is also assumed to be the completion time in order to find the latest possible start times. So 32 is also put in the lower half of the 'finish' circle. Now working backwards through the network, the latest start times for each activity are as follows:

$$J \quad 32 - 3 = 29$$

$$I \quad 29 - 2 = 27$$

$$F \quad 27 - 2 = 25$$

$$H \quad 32 - 8 = 24$$

$$D \quad 25 - 8 = 17$$

$$G \quad \text{the minimum of } 25 - 5 = 20 \text{ and } 24 - 5 = 19$$

$$E \quad \text{the minimum of } 17 - 10 = 7 \text{ and } 19 - 10 = 9$$

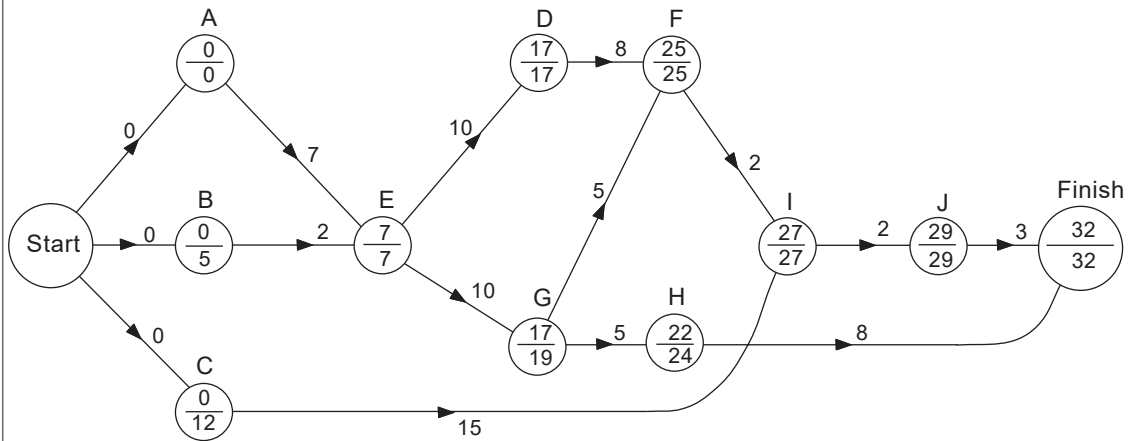
$$A \quad 7 - 7 = 0$$

$$B \quad 7 - 2 = 5$$

$$C \quad 27 - 15 = 12$$

2.2

This gives a completed network as shown below.



The vertices with equal earliest and latest starting times define the **critical path**. This is clearly seen to be

A E D F I J

Another way of identifying the critical path is to define the

float time = latest start time – earliest start time

The information for the activities can now be summarised in the table below.

Start times			
Activity	Earliest	Latest	Float
A	0	0	0 ←
B	0	5	5
C	0	12	12
E	7	7	0 ←
D	17	17	0 ←
G	17	19	2
F	25	25	0 ←
H	22	24	2
I	27	27	0 ←
J	29	29	0 ←

So now you know that if there are enough workers the job can be completed in 32 days. The activities on the critical path (i.e. those with zero float time) must be started punctually; for example, A must start immediately, E after 7 days, F after 25 days, etc. For activities with a non-zero float time there is scope for varying their start times; for example activity G can be started any time after 17, 18 or 19 days' work. Assuming that all the work is completed on time, you will see that this does indeed give a working schedule for the construction of the garage in the minimum time of 32 days. However it does mean, for example, that on the 18th day activities D and C will definitely be in

2.2

progress and G may be as well. The solution could well be affected if there was a limit to the number of workers available, but you will consider that sort of problem in the next section.



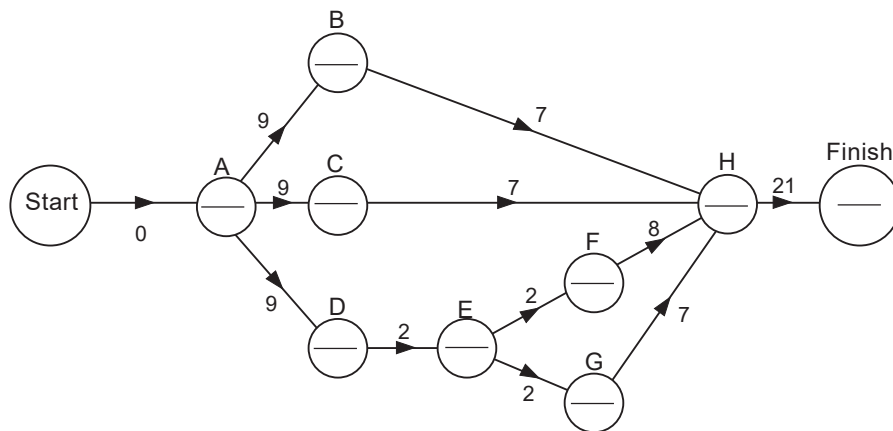
Worked Example 2

From the activity network for Question 2 in the Exercises in Section 2.1, find the critical path and the possible start times for all the activities in order to complete the job in the shortest possible time.

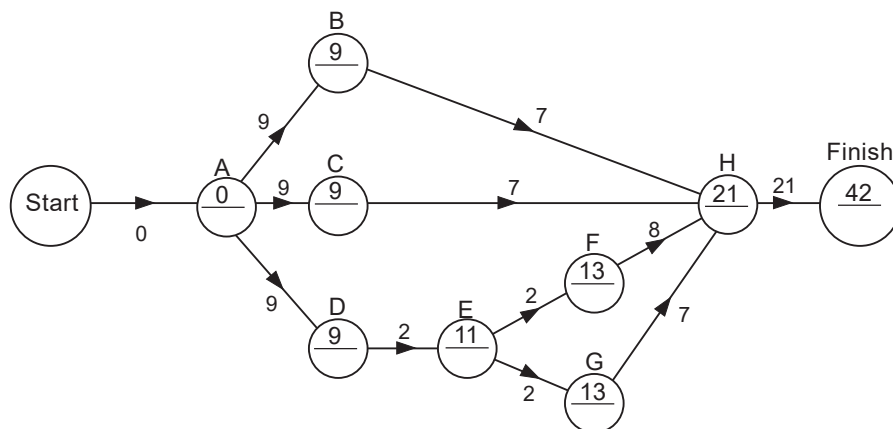


Solution

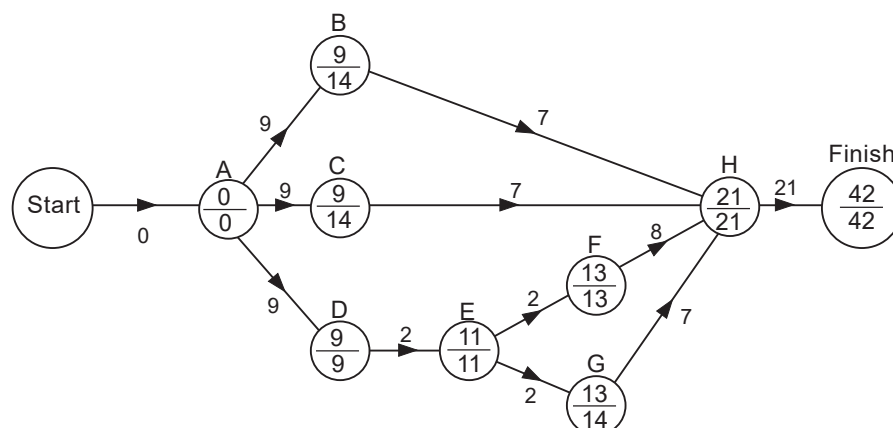
Here is an activity network for the problem:



With a forward scan, we obtain:



and with a backwards scan, we obtain:



2.2

Hence the critical path is

A D E F H

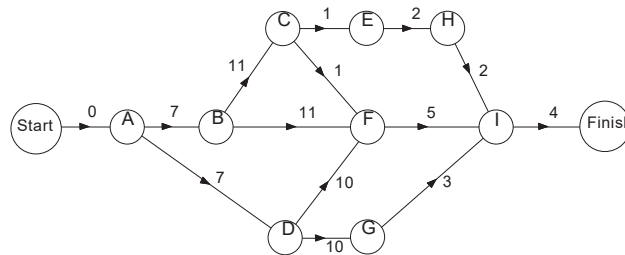
and the earliest and latest start times in order to finish in the minimum time of 42 minutes are as given in the final diagram.



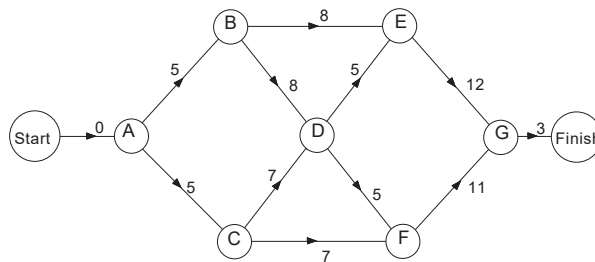
Exercises

1. Find the critical paths for each of the activity networks (*) shown below .

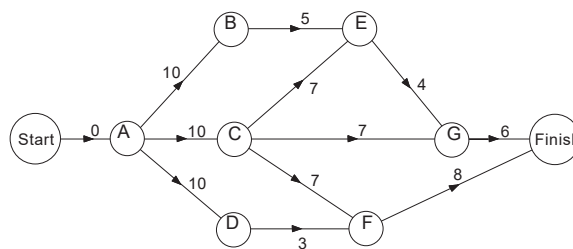
(a)



(b)



(c)



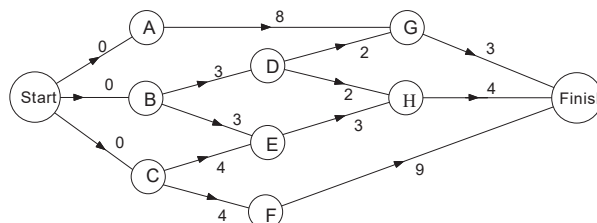
(* Enlarged versions of these networks, for you to work on, are given at the end of this unit.)

2.2

2. Your local school decides to put on a musical. These are the many jobs to be done by the organising committee, and the times they take:

A	make the costumes	6 weeks
B	rehearsals	12 weeks
C	get posters and tickets printed	3 weeks
D	get programmes printed	3 weeks
E	make scenery and props	7 weeks
F	get rights to perform the musical	2 weeks
G	choose cast	1 week
H	hire hall	1 week
I	arrange refreshments	1 week
J	organise make-up	1 week
K	decide on musical	1 week
L	organise lighting	1 week
M	dress rehearsals	2 days
N	invite local radio/press	1 day
P	choose stage hands	1 day
Q	choose programme sellers	1 day
R	choose performance dates	$\frac{1}{2}$ day
S	arrange seating	$\frac{1}{2}$ day
T	sell tickets	last 4 weeks
V	display posters	last 3 weeks

- (a) Decide on the precedence relationships.
 (b) Construct the activity network.
 (c) Find the critical path and minimum completion time.
3. Consider the following activity network, in which the vertices represent activities and the numbers next to the arcs represent time in days.



- (a) Assuming that an unlimited number of workers is available, write down:
 (i) the minimum completion time of the project;
 (ii) the corresponding critical path.
- (b) Find the float time of activity E.

2.2

4. A project consists of ten activities, A-J. The duration (in days) of each activity, and the activities preceding each of them, are as follows:

Activity	Duration	Preceding activities
A	10	-
B	4	-
C	8	B
D	6	C
E	8	I
F	5	-
G	10	A, D
H	2	G
I	4	-
J	10	D, E, F

- (a) construct an activity network for this project;
- (b) find a critical path in this activity network;
- (c) find the latest starting time for each activity.
5. A project consists of eight activities whose durations are as follows:

Activity	A	B	C	D	E	F	G	H
Duration	4	4	3	5	4	1	6	5

The precedence relations are as follows:

B	must follow	A
D	must follow	A and C
F	must follow	C and E
G	must follow	C and E
H	must follow	B and D

- (a) Draw an activity network in which the activities are represented by vertices.
- (b) Find a critical path by inspection, and write down the earliest and latest starting times for each activity.

2.2

6. The eleven activities A to K which make up a project are subject to the following precedence relations.

Preceding activities	Activity	Duration
C, F, J	A	7
E	B	6
-	C	9
B, H	D	7
C, J	E	3
-	F	8
A, I	G	4
J	H	9
E, F	I	9
-	J	7
B, H, I	K	5

- (a) Construct an activity network for the project.
- (b) Find:
- the earliest starting time of each activity in the network;
 - the latest starting time of each activity.
- (c) Calculate the float of each activity, and hence determine the critical path.
7. The activities needed to replace a broken window pane are given below.

Activity		Duration (in mins)	Preceding activities
A	order glass	10	-
B	collect glass	30	A
C	remove broken pane	15	B, D
D	buy putty	20	-
E	put putty in frame	3	C
F	put in new pane	2	E
G	putty outside and smooth	10	F
H	sweep up broken glass	5	C
I	clean up	5	all

2.2

- (a) Construct an activity network.
- (b) What is the minimum time to complete the replacement?
- (c) What is the critical path?

8. At 4.30 pm one day the BBC news team hear of a Government Minister resigning. They wish to prepare an item on the event for that evening's 6 o'clock news. The table below lists the jobs needed to prepare this news item, the time each job takes and the constraints on when the work can commence.

	Job	Time needed (in minutes)	Constraints
A	Interview the resigning Minister	15	Starts at 4.30 pm
B	Film Downing St.	20	None
C	Get reaction from regions	25	Cannot start until A and B are completed
D	Review possible replacements	40	Cannot start until B is completed
E	Review the Minister's career	25	Cannot start until A is completed
F	Prepare film for archives completed	20	Cannot start until C and E are
G	Edit	20	Cannot start until A, B, C, D, E and F are completed

- (a) Construct an activity network for this problem and, by finding the critical path in your network, show that the news item can be ready before 6.00 pm that day.
 - (b) If each of the jobs A, B, C, D, E and F needs a reporter, and once a reporter has started a job that same reporter alone must complete it; explain how three reporters can have the news item ready before 6.00 pm, but that two reporters cannot.
9. Write the major activities, duration time and precedence relationship for a real life project with which you are involved. Use the methods in this chapter to find the critical path for your project.

2.3 Scheduling

In the previous sections it was possible to find the critical path for complex planning problems, but no consideration was given to how many workers would be available to undertake the activities, or indeed to how many workers would be needed for each activity.

The following **operating rules** will be assumed:

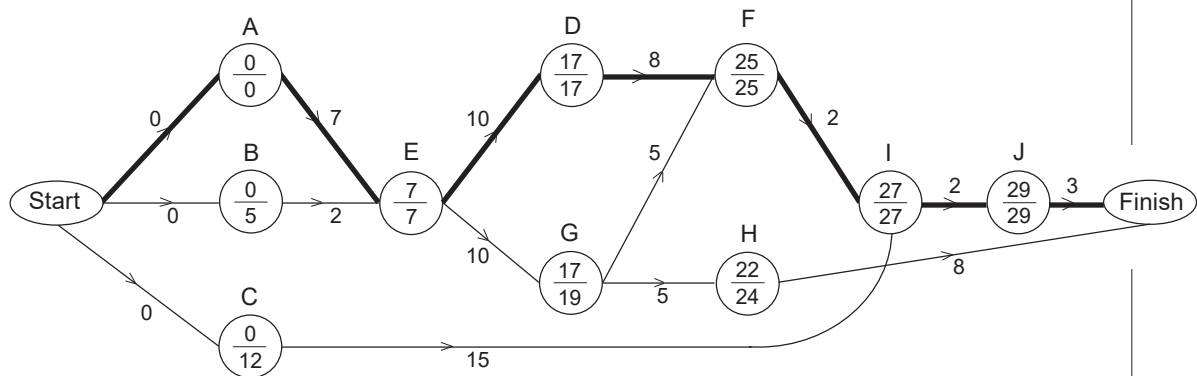
1. Each activity requires only **one** worker.
2. No worker may be idle if there is an activity that can be started.
3. Once a worker starts an activity, it must be continued by that worker until it is completed.

The **objective** will be to:

'Complete the project as soon as possible with the available number of workers.'

The main example from Section 2.2, which related to the construction of a garage, will be used to illustrate the problem.

The activity network, earliest and latest starting times, and the critical path (bold line), are shown below.



Suppose there are two workers available for the complete project. What is needed is a set procedure in order to decide who does what.

The method that will be adopted can be summarised as follows.

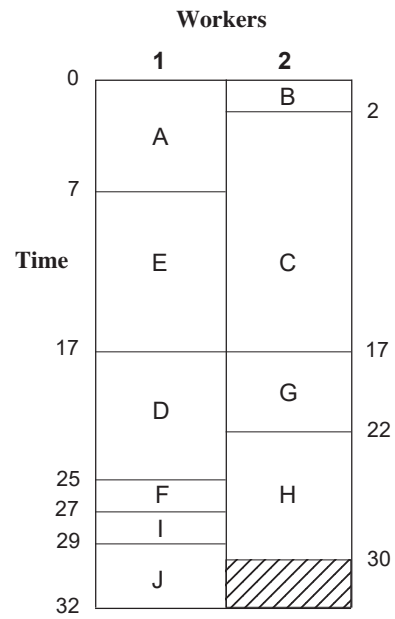
At any stage, when a worker becomes free, consider all the activities which have not yet been started but which can now be started. Assign to the worker the most 'critical' one of these (i.e. the one whose latest starting time is the smallest). If there are no activities which can be started at this stage you may have to wait until the worker can be assigned a job.

2.3

Using this as a basis for decisions, the solution shown opposite is obtained.

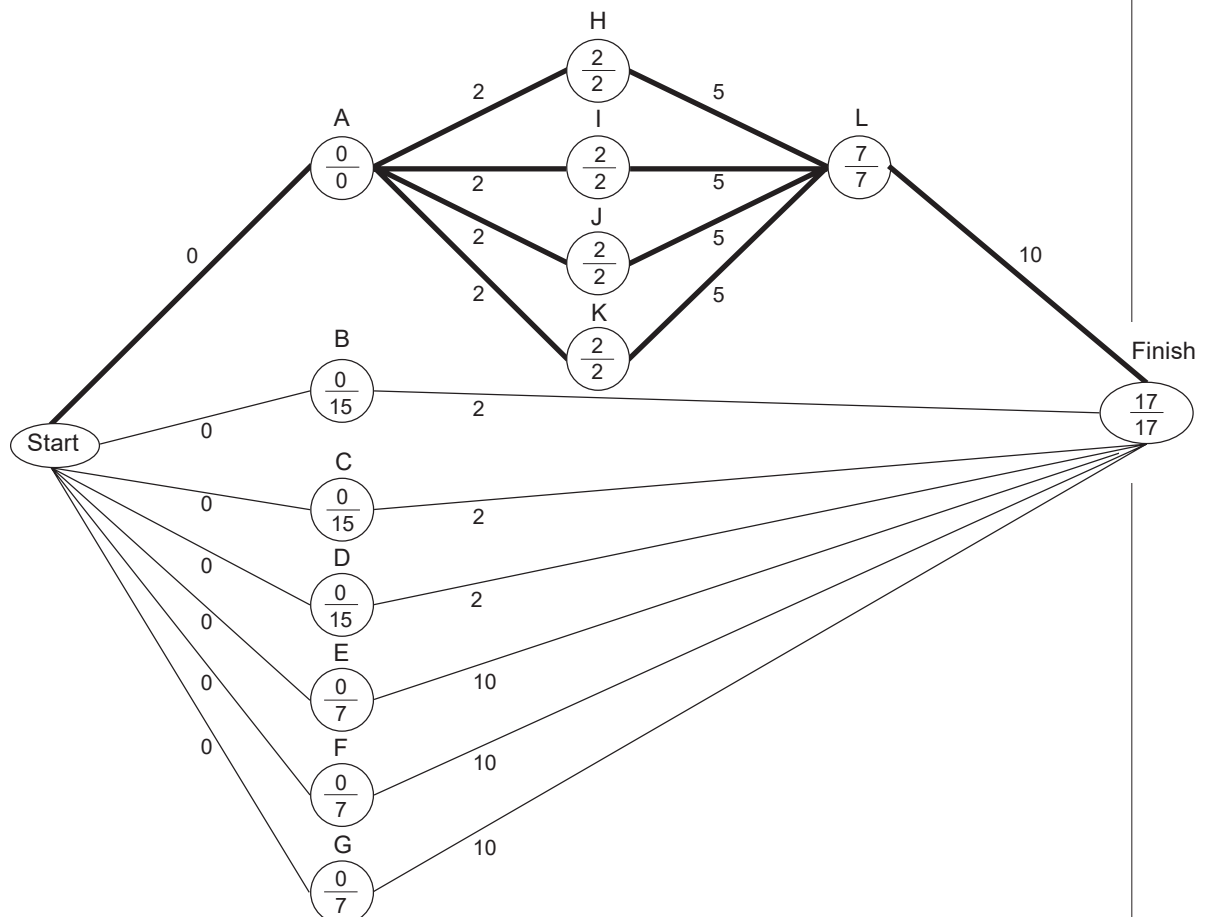
Note that worker 1 completes all the activities on the critical path, though, at time $t = 17$, workers 1 and 2 could have swapped over.

Since the whole project is completed in time 32 days, which you already know to be the minimum completion time, you can be assured that this method has produced an optimum solution. However, this is not always the case, as you will see in Worked Example 1.



Worked Example 1

For the activity network below, find the earliest and latest starting times for each activity, Use the method above to schedule our workers on this project.

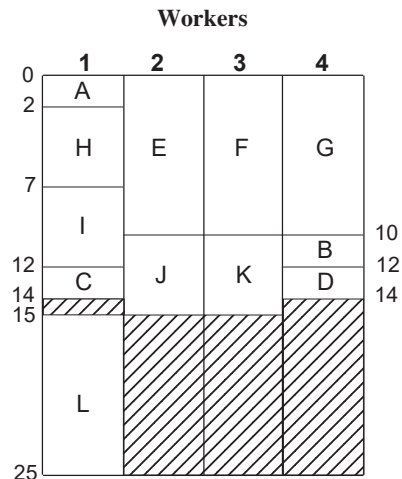


2.3



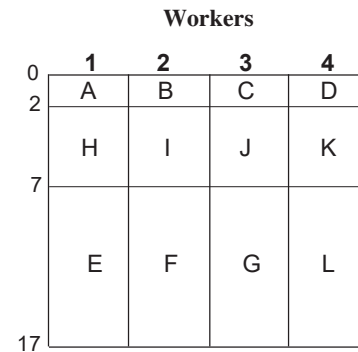
Solution

Applying the method as before results in the schedule shown in the first diagram opposite. This gives a completion time of $t = 25$.



Note

This schedule is not optimal; it should not take too long to find a schedule that completes the project in time 17. A possible solution is shown in the second diagram.



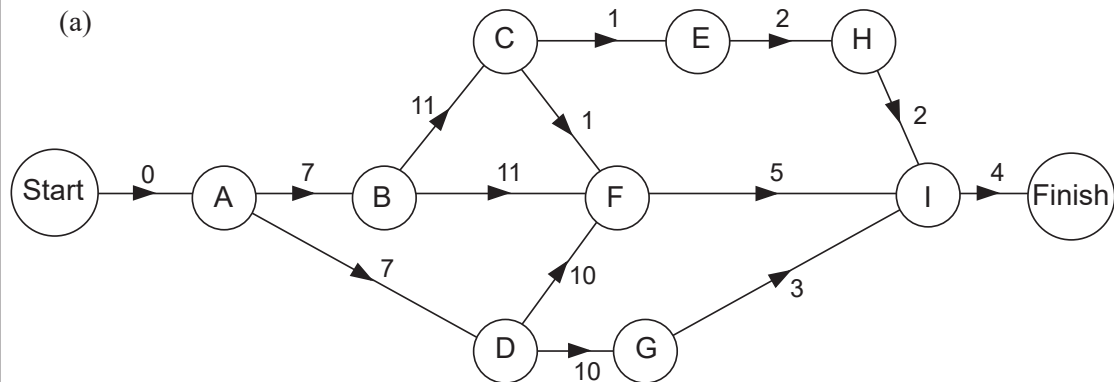
Exercises

1. Use the first scheduling method to find a solution to Question 5 in Section 2.2 using two workers. Does this produce an optimum schedule?
2. Find a schedule for the problem given in Question 2, in the Exercises in Section 2.1, using 2 workers and the two scheduling algorithms given in this section. Does either of these methods provide an optimal solution?
3. A possible modification to the method in this section is as follows :
 'Evaluate for each activity the **product** of the earliest and latest starting times, and rank the activities in ascending order according to these numbers. Assign activities using this ranking, taking the precedence relations into account.'
 Use this method to find possible schedules for the garage construction problem in this section. Does this method always give an optimum solution?

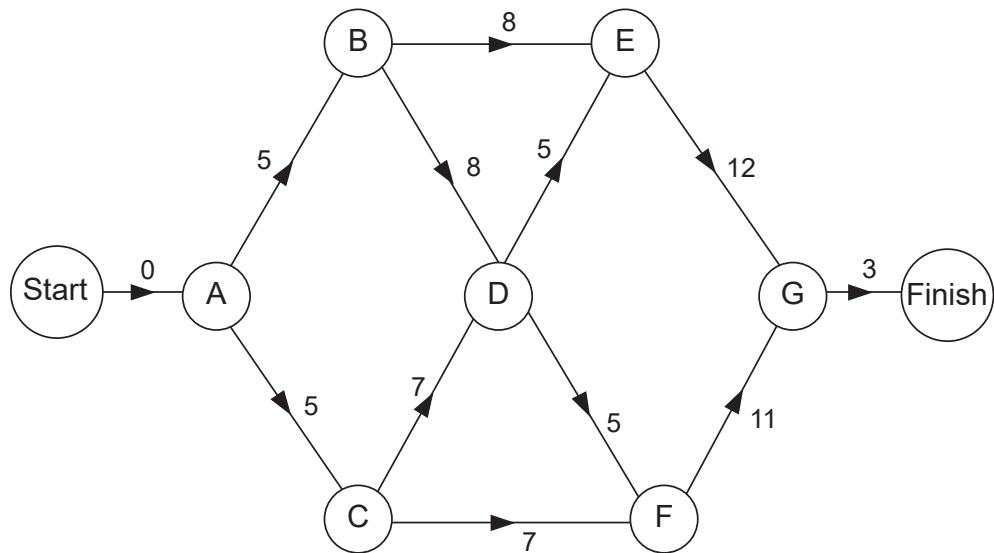
Unit 2 Exercises - networks enlarged

1.

(a)



(b)



(c)

