

STRAND: PROBABILITY

Unit 4 *Risk*

TEXT

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4 Risk

4.1 Expected Values

Life is full of risks: some we cannot avoid and for which the likelihood of, for example, an accident is unknown (for example, travelling to school or work each day) and some we take because we choose to (for example, windsurfing, rock-climbing). There is a third category of risk-taking for which we can actually calculate the risk (for example, winning the National Lottery and other forms of gambling).

In this first section, we consider activities for which we can calculate the mathematical risk or the payoff when playing a game. This is based on the concept of expected value or expectation. (In this context, the word 'payoff' means profit or gain.)



Worked Example 1

You are taking a multi-choice test in which each question has five options. Scores are given as

- 1 for choosing the correct option,
- 0 for giving no answer,
- $-\frac{1}{4}$ for choosing an incorrect option.

If you know nothing about the topics being tested, is it in your interest to guess?



Solution

Label the options as A, B, C, D and E. Supposing that the correct answer is option A, we can summarise the payoffs as below.

Option	Leave blank	A	B	C	D	E
Payoff	0	1	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$

A guess will be correct with probability $\frac{1}{5}$, so the expected payoff is given by:

$$\begin{aligned}
 & 1 \times \frac{1}{5} + \left(-\frac{1}{4}\right) \times \frac{1}{5} + \left(-\frac{1}{4}\right) \times \frac{1}{5} + \left(-\frac{1}{4}\right) \times \frac{1}{5} + \left(-\frac{1}{4}\right) \times \frac{1}{5} \\
 &= \frac{1}{5} + 4 \times \left(-\frac{1}{20}\right) \\
 &= 0
 \end{aligned}$$

So guessing would be equivalent to leaving all options blank!



Note

If incorrect answers were marked with probability $= -\frac{1}{2}$ rather than $= -\frac{1}{4}$, then the payoff would be

$$\begin{aligned} & 1 \times \frac{1}{5} + 4 \times \left(-\frac{1}{2}\right) \times \frac{1}{5} \\ &= \frac{1}{5} - \frac{2}{5} \\ &= -\frac{1}{5} \end{aligned}$$

In this case, it would not be in your interest to guess.



Worked Example 2

In a game, you roll a single unbiased dice. If an ODD number comes up, you win the amount shown on the dice in £s but if an EVEN number comes up, you have to pay that amount in £s.

Should you play this game?



Solution

We can illustrate the payoff in a table.

<i>Dice</i>	1	2	3	4	5	6
<i>Payoff</i>	£1	-£2	£3	-£4	£5	-£6
<i>Probability</i>	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

So the expectation $= \frac{1}{6} \times (1 - 2 + 3 - 4 + 5 - 6) = £ -0.50$

So don't risk playing this game!



Worked Example 3

In a different game, you throw two dice. If the difference between the scores shown is 0, 1 or 2, I win. If the difference is 3 or more, you win.

Do you want to play?



Solution

Consider all the possible outcomes before deciding.

The following matrix shows the differences.

		<i>Dice 1</i>					
		<i>1</i>	<i>2</i>	<i>3</i>	<i>4</i>	<i>5</i>	<i>6</i>
<i>Dice 2</i>	<i>1</i>	0	1	2	3	4	5
	<i>2</i>	1	0	1	2	3	4
	<i>3</i>	2	1	0	1	2	3
	<i>4</i>	3	2	1	0	1	2
	<i>5</i>	4	3	2	1	0	1
	<i>6</i>	5	4	3	2	1	0

Since each of these outcomes is equally likely to occur, you win on 12 occasions and I win on 24.

So $p(\text{you win}) = \frac{12}{36} = \frac{1}{3}$ so you'll probably decide not to play!



Exercises

1. A multi-choice test consists of 10 questions, each with 4 different options for the answer. Marks are given as follows:

1 mark for a correct answer

0 marks for a blank answer

$-\frac{1}{2}$ for any incorrect answer.

- (a) How many questions would you expect to answer correctly if you guess the answers?
 - (b) If the pass mark is 6, what is the probability that you pass if you guess all the answers?
2. This game is to roll two dice and add the numbers shown to give your score. If the score is odd, you win £1; if it is even, you pay your opponent £1. Should you play this game?
 3. This game entails rolling two dice.

Before starting to play you have to agree to one of these two options:

Option 1: You win £5 if a 2 or a 5 (or both) appear.

Option 2: You lose £5 if there is no 2 or 5 uppermost on either of the dice.

- (a) Which option should you choose?
- (b) What are your expected winnings if you play the game 10 times?

4.2 Gambling

Whilst many gamblers think (whenever they bet) that "Today is my lucky day", the reality is that you never see poor bookmakers or unprofitable casinos as the odds are stacked against the punters. Yes, sometimes you can be lucky and, for example, win £34 million on the National Lottery (on Saturday 9th January 2016) but the great majority of lottery players will *never* win a big prize!

In this section, we consider some more complex gambling problems to assess the risk of losing (or winning!).



Worked Example 1

In about 1620, the Grand Duke of Tuscany thought it was better to bet on a total of 9 occurring when 3 dice are thrown, rather than 10, but in the long run, he lost.

Can you explain why he lost?



Solution

Think first of just 2 dice being thrown and the numbers added. The matrix below shows the possible outcomes.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Since each outcome is equally possible, the probability of any event is $\frac{1}{36}$ and

Score	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Now consider throwing the third dice.

You can score 9, with the following combinations for the first 2 dice and the third dice:

<i>First 2 dice</i>	<i>Third dice</i>	<i>Probability</i>
3	6	$\frac{2}{36} \times \frac{1}{6}$
4	5	$\frac{3}{36} \times \frac{1}{6}$
5	4	$\frac{4}{36} \times \frac{1}{6}$
6	3	$\frac{5}{36} \times \frac{1}{6}$
7	2	$\frac{6}{36} \times \frac{1}{6}$
8	1	$\frac{5}{36} \times \frac{1}{6}$
Total		$p = \frac{25}{216}$

You score 10 though with the combinations below:

<i>First 2 dice</i>	<i>Third dice</i>	<i>Probability</i>
4	6	$\frac{3}{36} \times \frac{1}{6}$
5	5	$\frac{4}{36} \times \frac{1}{6}$
6	4	$\frac{5}{36} \times \frac{1}{6}$
7	3	$\frac{6}{36} \times \frac{1}{6}$
8	2	$\frac{5}{36} \times \frac{1}{6}$
9	1	$\frac{4}{36} \times \frac{1}{6}$
Total		$p = \frac{27}{216}$

$$p(\text{scoring 9}) = \frac{25}{216}$$

$$p(\text{scoring 10}) = \frac{27}{216}$$

So the Duke was incorrect in thinking that, in the long run, he was more likely to win when betting on a total of 9 than 10.



Worked Example 2 (Birthday Problem)

You are in a group of 40 people in a club one evening. The waiter says to you,

"I think that at least 2 people here have their birthdays on the same day.

If I'm wrong, you can have free drinks all evening; if I'm right, you owe me £10."

Should you agree to the waiter's bet?



Solution

Instinctively, it seems unlikely that the waiter will win but we can analyse the risk by starting with a smaller group and then trying to generalise.

So, for just 2 people, the probability of them having birthdays on the same day is $\frac{1}{365}$, giving

$$p(\text{having birthdays different days}) = \frac{364}{365}$$

Now consider 3 people. The probability of different birthdays = $\frac{364}{365} \times \frac{363}{365}$ giving

$$\begin{aligned} p(\text{at least 2 with same birthday}) &= 1 - \frac{364}{365} \times \frac{363}{365} \\ &\approx 0.008 \end{aligned}$$

We can continue in this way: for example, with 4 people,

$$p(\text{at least 2 with same birthday}) = 1 - \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365}$$

and eventually, for 40 people,

$$\begin{aligned} p(\text{at least 2 with same birthday}) &= 1 - \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \dots \times \frac{324}{365} \\ &\approx 1 - 0.086 \\ &\approx 0.914 \end{aligned}$$

So the advice is *not* to take the waiter's bet! He is very likely to win.



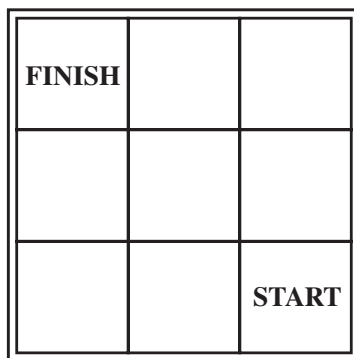
Note

Even with a group of 30 people, the probability of at least 2 people have their birthday on the same day is about 0.706.



Exercises

1. This is a simple game where you throw a dice which controls the position of your counter on a 3×3 board.



Place your counter at the START square and throw a non-biased dice.

If you get an EVEN number, move your counter one square *upwards*.

If you get an ODD number, move your counter one square to the *left*.

If your counter moves off any side of the board, you lose!

If your counter reaches the FINISH square, you have won.

- (a) What is the probability of winning?
 - (b) Extend your analysis to the same game on a 4×4 board.
What is the probability of winning using this board?
2. For the Birthday Problem in Worked Example 2, with a group of 20 people, what is the probability that at least 2 of them have their birthdays on the same day?
 3. Are you more likely to win by betting on
Option A: obtaining at least one 6 when 6 dice are tossed,
 or
Option B: obtaining at least 2 sixes when 12 dice are tossed?

4. A fruit machine with 3 DIALS and 20 SYMBOLS (not all different) on each dial is illustrated opposite. Each dial can stop on any one of its 20 symbols, and each of the 20 symbols on a dial is equally likely to occur.
- So, for example, the *pineapples* on DIAL 1 are likely to occur on average 7 times out of 20.
- You insert 10p, press a button and the three dials spin round.
- You then press more buttons to stop each dial at random.
- The three symbols highlighted determine how much, if anything, is won.
- Suppose the machine makes the payouts shown in the table below.

Combination	Payout (in 10p's)
3 BARS	40
3 ORANGES	5
3 PINEAPPLES	5
3 BELLS	5
2 BARS	20
2 CHERRIES	5

- (a) Complete the frequency chart below for each dial.

Symbol	Dial 1	Dial 2	Dial 3
BAR	2	1	1
ORANGE	1	8	..
PINEAPPLE	7	..	..
BELL	..	..	..
CHERRIES	..	..	..
SEVEN	..	..	..

We need to find the probability of each of the combinations above to see if it is worth playing.

We first consider the 3 BARS combination.

- (b) (i) In how many ways can you obtain 3 BARS?
 (ii) How many possible combinations (including repeats) are there?
 (iii) What is the probability of obtaining 3 BARS?
- (c) Find the probabilities of obtaining all the other winning combinations.
- (d) What is the expected gain or loss for each go?

