PROJECTILES

Text

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Projectiles

1 Projectiles and 2D Problems

This Unit applies the material learnt in the previous Units to projectiles. A projectile is an object which has been given an initial velocity and travels under the influence of gravity along a curved path; examples are a cannonball, or a stone that has been thrown into the air. The *suvat* equations can be applied to this scenario, which becomes a problem based in two dimensions.

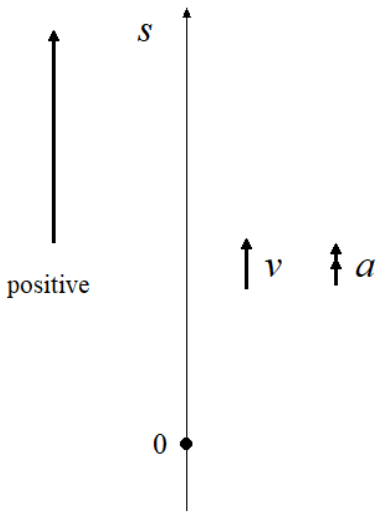
Assumptions

A number of assumptions will be used to model projectile motion:

- The projectile is treated as a particle: its size and spin are negligible.
- The effect of gravity can be treated as a constant downwards acceleration on the object. The value $a = 9.81\ \text{ms}^{-2}$ is fairly accurate and has been found by theory and experimentation, but $a = 10\ \text{ms}^{-2}$ will be used for convenience, which is sufficiently accurate. Note that in reality the effect of gravity varies depending on where you are on the Earth's surface and how far you are from the centre of the Earth, as discussed in the first Unit.
- Air resistance is negligible, the only force acting on the projectile is gravity.

Motion under Gravity

To begin, consider the motion of an object under gravity which has been thrown directly upwards. This motion is addressed in the Unit *One-Dimensional Motion*. If an object is given an initial velocity upwards, it travels up with gravity acting downwards, directly against it.



A reminder of the sign convention for this scenario is shown on the left, with positive being the 'up' direction. Using this convention, *s*, *v* and *a* will all have positive signs when directed upwards and a negative sign when directed downwards.

For acceleration, *a*, a positive sign indicates acceleration upwards. A negative sign indicates deceleration, or a 'pull' downwards. This will be gravity, so in this scenario,

$$\boldsymbol{a = g = -10\ \text{ms}^{-2}}$$ will be used.

For *s*, a position of zero is set as the starting position, then positive *s* is above this position (height) and negative *s* is below it.

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For v, a positive sign indicates that the direction of travel is upwards, a negative sign indicates that the direction of travel is downwards.

For this scenario the *suvat* equations can be easily applied, with gravity as the constant acceleration acting upon the motion of the object.



Worked Example 1

A ball is thrown vertically upwards with an initial speed of 15 ms^{-1}.

Calculate the height reached after 2 seconds.



Solution

Using $s = ut + \frac{1}{2}at^2$ with $a = -10$ ms^{-2}, $u = 15$ ms^{-1} and $t = 2$ seconds

$$s = 15 \times 2 + \frac{1}{2} \times -10 \times 2^2$$

$$s = 10 \text{ metres}$$



Worked Example 2

A ball is thrown vertically upwards with a velocity of 12 ms^{-1}.

Calculate the times on its path when the ball is at a height of 4 metres.



Solution

Using $s = ut + \frac{1}{2}at^2$ with $a = -10$ ms^{-2}, $u = 12$ ms^{-1} and $s = 4$ m gives

$$4 = 12t + 0.5 \times -10 \times t^2$$

This rearranges to

$$5t^2 - 12t + 4 = 0$$

$$t^2 - 2.4t + 0.8 = 0$$

Using the quadratic formula

$$t = \frac{2.4 \pm \sqrt{2.56}}{2} = 0.4 \text{ or } 2$$

The ball is 4 m above the point of release twice, at $t = 0.4$ s (on way up) and at $t = 2$ s (on way down).

Working in Two Dimensions

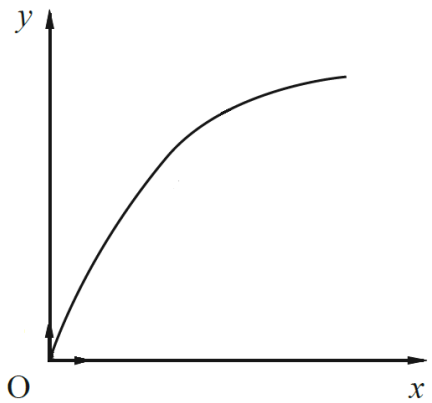
Following the recap on motion of an object under the influence of gravity in one dimension, the extension can be made to two dimensions, which allows for the general motion of a projectile to be modelled.

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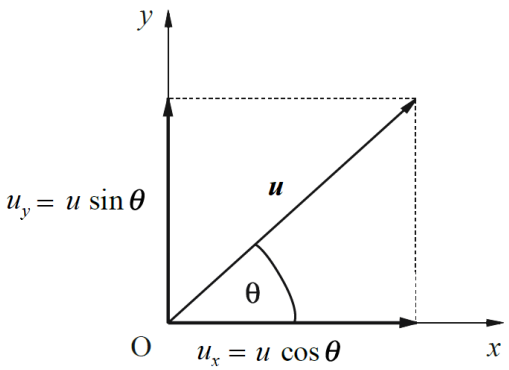
It will be demonstrated that projectiles, under the assumptions used here, will travel along a parabolic path as shown below.



To show this the progression needs to be made to two dimensions. The position in the upwards direction, the *vertical position* described earlier, will be labelled as *y* and the *horizontal position* will be labelled as *x*. This is shown in the diagram below.



The horizontal position *x* and the vertical position *y* can be handled independently from one another. So for a given time the horizontal position and the vertical position can be calculated using the *suvat* equations. For a given initial velocity $\boldsymbol{u}$ of magnitude *u* at an angle of θ to the horizontal, the horizontal and vertical components can be calculated using some basic trigonometry. This is illustrated in the diagram below.



The horizontal component of initial velocity will be called u_x and the vertical component of initial velocity will be called u_y. They can be calculated as follows:

Initial horizontal velocity $\quad u_x = u\cos\theta$

Initial vertical velocity $\quad u_y = u\sin\theta$

The *suvat* equations can now be applied to horizontal and vertical directions for a projectile which has initial speed u at an angle of θ to the horizontal and the position of the projectile can be found at a given time.

The vertical component was effectively discussed at the beginning of this section: the *suvat* equations are applied with the upwards direction as positive; gravity acts against the motion so a = -g, and the initial speed is u_y. The vertical position y can be found using the *suvat* equation $s = ut + \frac{1}{2}at^2$:

$$y = u_y t + \frac{1}{2}at^2$$

With a set as -10, this equation becomes:

vertical position	$y = u_y t - 5t^2$

If you consider the horizontal component of motion, the situation is actually much simpler. Gravity acts in a vertical direction only, so horizontally there are no forces acting on the projectile, hence no acceleration. The initial speed horizontally is u_x and this does not change! Any calculations for position, or time, need only use the relationship *distance* = *speed* × *time*. Therefore the following equation for horizontal position holds:

horizontal position	$x = u_x t$

So given the initial velocity in terms of speed u and angle to the horizontal θ, problems involving the position both vertically and horizontally are easily tackled. The *suvat* equations can also provide information on the final velocity of the projectile.

Note that the above equations need not necessarily be memorised as a separate exercise, they are just applications of the *suvat* equations to the projectile scenario. Be aware that g is often given the more accurate value of 9.81, or can be left in solutions as 'g', and other resources will vary on this matter.



Worked Example 3

A stone is thrown at a speed of 15 ms^{-1} at an angle of 60° to the horizontal. Find its position horizontally and vertically at the following times:

a) 0.5 seconds
b) 1 second
c) 2 seconds



Solutions

a) First the horizontal and vertical components of initial velocity need to be found.

Horizontal speed is, $u_x = 15\cos 60 = 7.5\ \text{ms}^{-1}$

Vertical speed is, $u_y = 15\sin 60 = 13.0\ \text{ms}^{-1}$

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Then the horizontal position, *x*, after 0.5 seconds is

$$x = u_x \times t = 7.5 \times 0.5 = 3.75 \text{ m}$$

The vertical position, *y*, after 0.5 seconds is

$$y = 13.0 \times 0.5 + \frac{1}{2} \times -10 \times 0.5^2 = 5.25 \text{ m}$$

So after 0.5 seconds the stone has travelled 3.75 metres horizontally and 5.25 metres vertically (up!). The positions for the other two times can be calculated in the same way.

b) The horizontal position, *x*, after 1 second is

$$x = u_x \times t = 7.5 \times 1 = 7.5 \text{ m}$$

The vertical position, *y*, after 1 second is

$$y = 13.0 \times 1 + \frac{1}{2} \times -10 \times 1^2 = 8 \text{ m}$$

So after 1 second the stone has travelled 7.5 metres horizontally and 8 metres vertically. Comparing this to the previous answer, it continues to travel upwards, but is slowing down.

c) The horizontal position, *x*, after 2 seconds is

$$x = u_x \times t = 7.5 \times 2 = 15 \text{ m}$$

The vertical position, *y*, after 0.5 seconds is

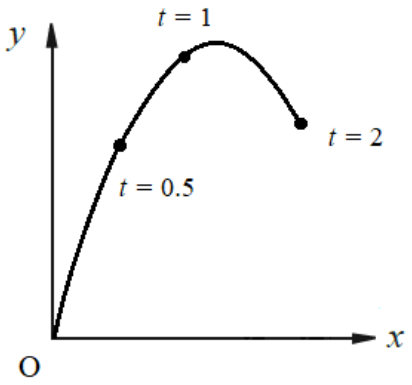
$$y = 13.0 \times 2 + \frac{1}{2} \times -10 \times 2^2 = 6.0 \text{ m}$$

So after 2 seconds the stone has travelled 15 metres horizontally and 6 metres vertically. Comparing this to the previous answer tells you that the stone is now travelling back towards the ground.

The diagram on the following page shows the position of the stone horizontally and vertically at these three times.

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The stone in this example can be seen to follow a parabolic path for the positional values of *t* used, as expected.

Worked Example 4

A ball is thrown with a speed of 8 ms^{-1} at an angle of 30° to the horizontal. How high above its point of projection is it when it has travelled 2 m horizontally?

Solutions

First the horizontal and vertical components of initial velocity need to be found.

Horizontal speed is, $u_x = 8\cos 30 = 6.928\ \text{ms}^{-1}$

Vertical speed is, $u_y = 8\sin 30 = 4\ \text{ms}^{-1}$

The time for the ball to travel 2 m horizontally can be found, then used to find the height, i.e. the vertical position *y*.

The time is found using $x = u_x t$:

$$2 = 6.928t$$

This solves to give *t* = 0.289 seconds

Then this value of *t* can be used to find the height *y*:

$$y = u_y t + \frac{1}{2}at^2 = 4 \times 0.289 + \frac{1}{2} \times -10 \times 0.289^2 = 0.74\ \text{m}$$

So when the ball has travelled 2 metres horizontally it is at a height of 0.74 metres.

Final Velocity and Speed

The *suvat* equations can also be applied to find the horizontal speed and vertical speed at a given time, for a projectile which has initial speed *u* at an angle of θ to the horizontal.

In keeping with the components of initial velocity, the following notation can be used for the components of the final velocity:

Final horizontal velocity	v_x
Final vertical velocity	v_y

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Once again, for the horizontal component, the situation is very simple. Gravity acts in a vertical direction only, so horizontally there are no forces acting on the projectile, hence no acceleration. The initial speed horizontally is u_x and this does not change.

So

$$\boxed{v_x = u_x}$$

The vertical component of final velocity v_y is calculated by applying the *suvat* equations, again with the upwards direction as positive; gravity acts against the motion so a = -g, and the initial speed is u_y. The final vertical velocity is found using the equation $v = u + at$:

$$v_y = u_y + (-g)t$$

So

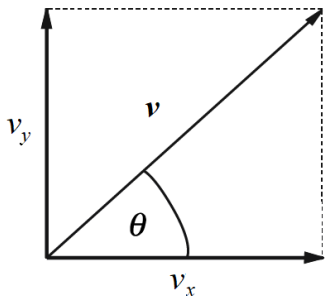
$$v_y = u_y - gt$$

With a set as -10, this equation becomes:

$$\boxed{v_y = u_y - 10t}$$

So given the initial velocity in terms of speed u and angle to the horizontal θ, the horizontal and vertical components of final velocity are easily found.

The speed of the projectile and its angle to the horizontal can be found from these two components. The diagram below shows the final horizontal and vertical components of velocity as a vector $\boldsymbol{v}$.



From the diagram some basic trigonometry tells you that the final speed v is calculated by finding the magnitude of the vector $\boldsymbol{v}$:

$$v = \sqrt{v_x^2 + v_y^2}$$

The angle with the horizontal is also found using basic trigonometry:

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

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Now the scenario from Worked Example 3 is returned to and demonstrates these results.

Worked Example 5

A stone is thrown at a speed of 15 ms^{-1} at an angle of 60˚ to the horizontal. Find the horizontal and vertical components of velocity at the following times along with the speed and the angle to the horizontal:

a) 0.5 seconds
b) 1 second
c) 2 seconds



Solutions

a) The horizontal and vertical components of initial velocity were found previously:

Horizontal speed is, $u_x = 15\cos 60 = 7.5\ \text{ms}^{-1}$

Vertical speed is, $u_y = 15\sin 60 = 13.0\ \text{ms}^{-1}$

Then the horizontal component of final velocity after 0.5 seconds is

$$v_x = u_x = 7.5\ \text{ms}^{-1}$$

The vertical component of final velocity after 0.5 seconds is

$$v_y = u_y + (-10)t = 13.0 - 10 \times 0.5 = 8\ \text{ms}^{-1}$$

The speed at 0.5 seconds can then be calculated from the horizontal and vertical components of final velocity:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{7.5^2 + 8^2} = 11.0\ \text{ms}^{-1}$$

The angle to the horizontal is also calculated:

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{8}{7.5}\right) = 46.8°$$

b) The horizontal component of final velocity after 1 second is

$$v_x = u_x = 7.5\ \text{ms}^{-1}$$

The vertical component of final velocity after 1 second is

$$v_y = u_y + (-10)t = 13.0 - 10 \times 1 = 3\ \text{ms}^{-1}$$

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The speed at 1 second can then be calculated from the horizontal and vertical components of final velocity:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{7.5^2 + 3^2} = 8.1 \text{ ms}^{-1}$$

The angle to the horizontal is also calculated:

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{3}{7.5}\right) = 21.8°$$

Note that this angle is less than the angle at 0.5 seconds.

c) The horizontal component of final velocity after 2 seconds is

$$v_x = u_x = 7.5 \text{ ms}^{-1}$$

The vertical component of final velocity after 2 seconds is

$$v_y = u_y + (-10)t = 13.0 - 10 \times 2 = -7 \text{ ms}^{-1}$$

The speed at 2 seconds can then be calculated from the horizontal and vertical components of final velocity.

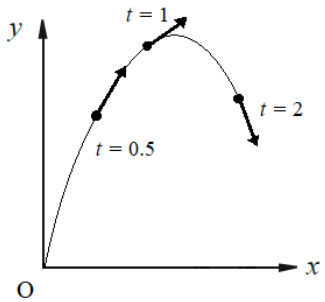
$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{7.5^2 + (-7)^2} = 10.3 \text{ ms}^{-1}$$

The angle to the horizontal is also calculated:

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{-7}{7.5}\right) = -43.0°$$

Comparing this to the previous answer, the angle has changed sign, it is now below the horizontal, pointing down.

The diagram below shows the velocity of the stone at these three times. The angle of the velocity θ can be seen to reduce from $t = 0.5$ to $t = 1$, then at $t = 2$ it has become negative and the angle is below the horizontal, pointing down.



The stone in this example can again be seen to follow a parabolic path for the positional values of t used, as expected. The velocities illustrate the direction of travel.

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Exercises

1. A ball is kicked with speed 25 ms^{-1} at an angle of projection of $45°$. How high above the ground is it when it has travelled 10 m horizontally?

2. A stone is thrown at a speed of 20 ms^{-1} at an angle of 60˚ to the horizontal. Find its position horizontally and vertically after the following times:

   a) 0.8 seconds
   b) 1 second
   c) 3 seconds

3. An arrow is fired from a bow with a speed of 50 ms^{-1} at an angle of 5˚ to the horizontal. What is its speed and the angle its velocity makes with the horizontal after 0.6 s?

2 Further Projectile Problems

The first section introduced a technique for solving problems based around projectiles, by simply applying the *suvat* equations. This section continues to use this approach to find significant attributes in the motion of a projectile such as the maximum height and the range, then addresses the possibility of starting the motion from a different height.

Maximum Height

The maximum height of a projectile can easily be found by considering its vertical motion. At the maximum height the vertical component of the velocity is momentarily zero, as the projectile slows to a halt before changing direction and travelling back down.

The maximum height and the time taken to reach maximum height can each be calculated by letting $v_y = 0$.



Worked Example 1

A ball is thrown with initial speed 20 ms^{-1} at an angle of 60˚ to the horizontal.

a) How high does it rise?
b) How long does it take to reach this maximum height?
c) How far has it travelled horizontally at its maximum height?



Solutions

a) The maximum height is found by considering the vertical motion. First u_y can be found

$$u_y = u \sin\theta = 20 \times \sin 60 = 17.32 \text{ ms}^{-1}$$

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Then $v^2 = u^2 + 2as$ can be used by letting $v_y = v = 0$ and solving to find *s*:

$$0^2 = 17.32^2 + 2 \times -10 \times s$$

Re-arrange

$$-300 = -20s$$

Then solve to obtain *s* = 15 metres as the maximum height.

b) The equation $v = u + at$ can be used by letting $v_y = v = 0$ and solving to find *t* at the maximum height:

$$0 = 17.32 + -10 \times t$$

Re-arrange

$$-17.32 = -10t$$

Then solve to obtain *t* = 1.732 seconds as the time at maximum height.

c) First find the horizontal component of initial velocity:

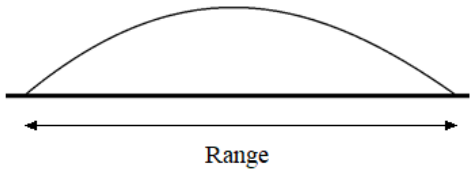
$$u_x = u\cos\theta = 20\cos 60 = 10 \text{ ms}^{-1}$$

The time at maximum height can be applied to the formula for horizontal position *x*:

$$x = u_x t = 10 \times 1.732 = 17.32 \text{ m}$$

Range of a Projectile

The range of a projectile is the total distance it travels horizontally as shown below.



It is found by first calculating the total time the projectile spends in flight. Given this time the horizontal distance travelled can easily be calculated by substituting this time into the formula for horizontal component of velocity, $x = u_x t$.

The total flight time can be found by calculating the time to maximum height and doubling this time; this takes advantage of the fact that the path of a projectile is symmetric: it takes the same amount of time to reach a given height as it does to return from that height back to the original position.

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Worked Example 2

A stunt motorcyclist takes off at a speed of 35 ms^{-1} up a ramp of 30˚ to the horizontal to clear a river 50 m wide. Does the cyclist succeed in doing this?



Solution

The cyclist clears the river if the horizontal distance travelled, the range, is greater than 50 m. To find this distance it is necessary to first find the time of flight. This can be done by doubling the time to the maximum height, as found in the previous example.

The horizontal and vertical components of initial velocity can first be found:

Horizontal speed is, $u_x = 35 \cos 30 = 30.3 \text{ ms}^{-1}$

Vertical speed is, $u_y = 35 \sin 30 = 17.5 \text{ ms}^{-1}$

The equation $v = u + at$ can be used by letting $v_y = v = 0$ and solving to find *t* at the maximum height:

$$0 = 17.5 + -10 \times t$$

Re-arrange

$$-17.5 = -10t$$

Then solve to obtain *t* = 1.75 seconds as the time at maximum height.

The time taken to return back to the vertical starting position is equal to this time due to the symmetry of projectile motion, so doubling this time will give you the total flight time.

Total flight time is 2 × 1.75 = 3.5 seconds.

Now the horizontal distance travelled, the range, can be found using this time.

$$x = u_x t = 30.3 \times 3.5 = 106 \text{ m}$$

So the cyclist easily clears the 50 metre river.

Initial Height

Some problems require more consideration of the initial height of the projectile. The same equations used so far are still applicable, with the understanding that they are based around the position of the projectile as relative to its starting position.



Worked Example 3

A tennis player makes a return at a speed of 15 ms^{-1} and at a height of 3 m to land in the opposite court. If she hits the ball at an angle of 1˚ above the horizontal, how far does the ball travel horizontally?

2

Solution

The horizontal and vertical components of initial velocity must first be found:

$$\text{Horizontal speed is, } u_x = 15\cos 1 = 15.0 \text{ ms}^{-1}$$

$$\text{Vertical speed is, } u_y = 15\sin 1 = 0.262 \text{ ms}^{-1}$$

The time taken to reach the ground height can be found by letting s = -3 metres. Notice how the correct use of the minus sign is essential, as described in the convention established in the first section. The equation $s = ut + \frac{1}{2}at^2$ is used to find the time.

$$-3 = 0.262t + \frac{1}{2}(-10)t^2$$

This rearranges to

$$5t^2 - 0.262t - 3 = 0$$

$$t^2 - 0.0873t - 0.6 = 0$$

Using the quadratic formula

$$t = \frac{0.0873 \pm\sqrt{2.41}}{2} = 0.4 \text{ or } 2$$

This gives solutions of t = 0.801 and t = -0.749.

The negative value is not valid, so t = 0.801 is used to find the horizontal distance travelled:

$$x = u_x t = 15.0 \times 0.801 = 12 \text{ m}$$

So the ball travels 12 metres horizontally, into the opposite court.

Vector Representation of Projectiles

The two dimensional scenario of a projectile given initial speed u at an angle of θ to the horizontal naturally calls upon the use of vectors, as a means to express the independent horizontal and vertical motions.

The components of initial velocity are

$$\boldsymbol{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} u\cos\theta \\ u\sin\theta \end{pmatrix}$$

The velocity at a later time t is given by

$$\boldsymbol{v}(t) = \begin{pmatrix} u_x \\ u_y - gt \end{pmatrix}$$

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The position relative to the initial position is

$$\boldsymbol{s}(t) = \begin{pmatrix} u_x t \\ u_y t - \frac{1}{2} g t^2 \end{pmatrix}$$

The acceleration is simply

$$\boldsymbol{a}(t) = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

Notice that this acceleration can be confirmed by differentiating $\boldsymbol{v}(t)$. Similarly, differentiating the position $\boldsymbol{s}(t)$ gives you $\boldsymbol{v}(t)$. This is in keeping with the material in the previous Unit on the relationships between *s*, *v* and *a*.

The techniques used here can be applied to other scenarios in mechanics, other than projectiles. They extend easily to three dimensions, providing a modelling technique which is extremely powerful and has allowed a deep understanding of the world around us.



Exercises

1. David kicks a ball with a speed of 20 ms^{-1} at an angle of 30˚ to the horizontal. How far away from him does the ball land?

2. In the Pony Club gymkhana Carol wants to release a ball to drop into a box. The height above the box from which she drops the ball is 1.5 m and the pony's speed is 12 ms^{-1}. How far from the box should Carol drop the ball?

3. A bowler releases a cricket ball from a height of 2.25 m above the ground so that initially its path is level. Find the speed of delivery if it is to hit the ground a horizontal distance of 16 m from the point of release.

4. Karen is standing 4 m away from a wall which is 2.5 m high. She throws a ball at 10 ms^{-1} at an angle of 40˚ to the horizontal at a height of 1 m above the ground. Will the ball pass over the wall?

5. A bushbaby makes hops with a take-off speed of 6 ms^{-1} and angle of 30 . How far does it go in each hop?

6. A stone is thrown up at an angle of 30˚ to the horizontal with a speed of 20 ms^{-1} from the edge of a cliff 15 m above sea level so that the stone lands in the sea. Find how long the stone is in the air and how far from the base of the cliff it lands. What are the speed and direction of the stone as it hits the water?

7. A stone is thrown with speed 10 ms^{-1} at an angle of projection of 30˚ from the top of a cliff and hits the sea 2.5 s later. How high is the cliff? How far from the base of the cliff does the stone hit the water?

8. A ball is thrown with a speed of 12 ms^{-1} at an angle of 30˚ to the horizontal. Find the maximum height to which it rises, the time of flight and the range. Find also the speed and direction of flight of the ball after 0.5 s and 1.0 s.

9. The initial speed of a projectile is 20 ms^{-1} . Find the two angles of projection which give a range of 30 m and the times of flight for each of these angles. What is the maximum range that can be achieved?

10. A projectile has range 100 m and reaches a maximum height of 20 m. What is its initial speed and angle of projection?

11. What is the least speed of projection with which a projectile can achieve a range of 90 metres? What is the time of flight for this speed?

12. Robin Hood shoots an arrow with a speed of 60 ms^{-1} to hit a mark on a tree 60 m from him and at the same level as the arrow is released from. What are his possible angles of projection and which one is he likely to choose?

13. A locust can make long jumps of 0.7 m at a takeoff angle of 55˚. Use the projectile model to find its take-off speed and the maximum height it reaches. (The take-off speed of locusts is observed to be about 3.4 ms^{-1}, which is higher than the value found using the projectile model. Why do you think this should be?).

14. A ball is thrown so that it goes as high as it goes forward. At what angle is it thrown?

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