F4 *Solving Quadratic Equations* Activities

Activities

F4.1.1 Quadratic Equation Formula 1
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Notes and Solutions (2 pages)

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ACTIVITY F4.1.1 *Quadratic Equation Formula 1*

The method of completing the square can be used to find a formula for solving quadratic equations.

1. Consider first the quadratic equation

$$x^2 + bx + c = 0$$

(a) By completing the square, show that,

$$\left(x + \frac{b}{2}\right)^2 + q = 0$$

and state the value of q in terms of b and c.

(b) Now show that,

$$\left(x + \frac{b}{2}\right)^2 = \frac{b^2 - 4c}{4}$$

and hence show that ,

$$x = \frac{-b + \sqrt{b^2 - 4c}}{2}$$

(c) Use your formula to solve the equation

$$x^2 + 6x - 7 = 0$$

Check that you get the same answers by factorising.

(d) Solve these equations using your formula:

(i) $x^2 - 7x + 2 = 0$

(ii) $x^2 + 3x + 1 = 0$

(iii) $x^2 - 6x + 5 = 0$

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ACTIVITY F4.1.2 *Quadratic Equation Formula 2*

2. Now consider equations of the form

$$ax^2 + bx + c = 0$$

(a) Dividing by a and completing the square gives,

$$\left(x + \frac{b}{2a}\right)^2 + q = 0$$

Express q in terms of a, b and c.

(b) Show that

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

and hence express x in terms of a, b and c.

(c) Use your formula to find the solutions of this equation.

$$12x^2 - 11x + 2 = 0$$

Check your answers by substituting them back intot he equation.

(d) Use your formula to solve these quadratic equations:

(i) $3x^2 + 6x - 1 = 0$

(ii) $2x^2 + 3x - 5 = 0$

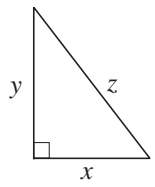
(iii) $2x^2 - 6x + 3 = 0$

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ACTIVITY F4.2 *Diophantine Equations*

Equations which have many *integer* (positive/negative whole numbers) solutions are known as *Diophantine equations,* after the Greek mathematician and philosopher, *Diophantos* of Alexandria (c. 250 A.D.). He is credited with being the founder of modern algebra. The use of symbols to represent numbers was found in hs published document, *Arithmetic*.

1. One example of a *Diophantine equation* could be Pythagoras' result that, for any right-angled triangle,



$$x^2 + y^2 = z^2$$

Find 3 different integer solutions to $x^2 + y^2 = z^2$.

2. Another example of a *Diophantine equation* is:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

for some number, n, and where x and y are integers.

(a) For example, when $n = 6$,

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{6}$$

(i) One possible solution to this equation is $x = 8$, $y = 24$.
Check that these values do indeed give a solution for the equation.

(ii) A second solution is obviously $x = 24$, $y = 8$, but there are many more.
Find in total 17 solutions of this equation.
[*Hint*: Remember that x or y can be a negative integer.]

(b) How many integer solutions can you find for the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{8}$?

Extension

1. Find two solutions, using small integers, to the equation $x^3 + y^3 + z^3 = 3$.

2. An unsolved conjecture is that the *Diophantine equation*,

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{4}{n}$$

has at least *one* solution for x, y and z for any integer, $n > 1$.

Show that this conjecture is true for $n = 2$, 3 and 4.

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ACTIVITY F4.1 *Notes and Solutions*

Notes and solutions given only where appropriate.

F4.1 1. (a) $q = c - \frac{b^2}{4}$

(b) $\left(x + \frac{b}{2}\right)^2 = \frac{b^2}{4} - c$

$= \frac{b^2 - 4c}{4}$

$x + \frac{b}{2} = \pm\sqrt{\frac{b^2 - 4c}{4}}$

$x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$

$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

(c) $x = 1$ or $x = -7$

(d) (i) $x = 6.702$ or $x = 0.298$ (ii) $x = -0.382$ or $x = -2.618$

(iii) $x = 1$ or $x = 5$

2. (a) $q = \frac{c}{a} - \frac{b^2}{4a^2}$

(b) $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$

$= \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$

$= \frac{b^2 - 4ac}{4a^2}$

$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

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ACTIVITY F4.2 *Notes and Solutions*

Notes and solutions given only where appropriate.

F4.2 1. There is an infinite number of solutions;
e.g. $9, 40, 41$ or $6, 8, 10$ (although this is really $3, 4, 5$).

3.

x	12	8	24	3	–6	2	–3	4	–12	9	18	10	15	5	–30	7	42
y	12	24	8	–6	3	–3	2	–12	4	18	9	15	10	–30	5	42	7

4.

x	16	4	–8	12	24	6	–24	10	40	7	–56	9	72
y	16	–8	4	24	12	–24	6	40	10	–56	7	72	9

Extensions

1. $x = y = z = 1$ is an obvious solution;
$x = y = 4,\ z = -5$ provides another solution
(or $x = z = 4,\ y = -5$; or $x = -5,\ y = z = 4$)

2. For example, if $n = 3$, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{4}{3}$ has solution $x = 3,\ y = z = 2$.

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